1 Finite-Volume Discretization of Heat Equation

Consider constructing a spatial semi-discretization for the diffusion equation with constant coefficients

\[ u_t = u_{xx}, \]

on the domain \( 0 < x < 1 \) with Dirichlet BCs at \( x = 0 \) and Neumann BCs at \( x = 1 \). Use the manufactured analytical solution

\[ u(x, t) = \exp^{-\pi^2 t/4} \cos\left(\frac{\pi x}{2}\right) \]

to obtain the specific forms of the initial and boundary conditions.

Write a finite-volume (flux based) second-ordered centered difference scheme and solve this equation up to time \( T = 1/4 \) with different grid spacings, and:

1. Find the numerical order of convergence in the \( L_1 \), \( L_2 \) and \( L_\infty \) norms.
2. Find the local truncation error at the left and right boundaries, and if possible, use that to prove second-order accuracy in some norm.
3. Optional: Prove stability in some norm.

2 Boundary Layers for Advection-Diffusion Equation

Consider constructing a spatial semi-discretization for the advection-diffusion equation with constant coefficients

\[ u_t + u_x = \epsilon u_{xx}, \]

on the domain \( 0 < x < 1 \), for initial condition \( u(x, 0) = 0 \) and boundary conditions

\[
\begin{align*}
    u(0, t) &= \sin(t) \\
    u_x(1, t) &= 0.
\end{align*}
\]

Write a code (choose the advective/diffusive stencils, the boundary condition treatment, number of grid points, etc., and explain your choices) to solve the equation up to time \( T = 2\pi \).

1. Show the spatially-discrete solution at this time for \( \epsilon = 1, \epsilon = 0.1 \) and \( \epsilon = 0.01 \) and comment on your observations and experiences.
2. Repeat with a Dirichlet condition on the right, \( u(1, t) = 0 \) and comment on the differences.
3. What is the spatial order of convergence for your discretization (numerically or theoretically)?