

THE LOCAL CIRCULAR LAW II: THE EDGE CASE

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Abstract

In the first part of this article [8], we proved a local version of the circular law up to the finest scale $N^{-1/2+\varepsilon}$ for non-Hermitian random matrices at any point $z \in \mathbb{C}$ with $||z| - 1| > c$ for any $c > 0$ independent of the size of the matrix. Under the main assumption that the first three moments of the matrix elements match those of a standard Gaussian random variable after proper rescaling, we extend this result to include the edge case $|z| - 1 = o(1)$. Without the vanishing third moment assumption, we prove that the circular law is valid near the spectral edge $|z| - 1 = o(1)$ up to scale $N^{-1/4+\varepsilon}$.

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1 INTRODUCTION

The circular law in random matrix theory describes the macroscopic limiting spectral measure of normalized non-Hermitian matrices with independent entries. Its origin goes back to the work of Ginibre [16], who found the joint density of the eigenvalues of such Gaussian matrices. More precisely, for an $N \times N$ matrix with independent entries $\frac{1}{\sqrt{N}}z_{ij}$ such that z_{ij} is identically distributed according to the measure $\mu_g = \frac{1}{\pi}e^{-|z|^2}dA(z)$ (dA denotes the Lebesgue measure on \mathbb{C}), its eigenvalues μ_1, \dots, μ_N have a probability density proportional to

$$\prod_{i < j} |\mu_i - \mu_j|^2 e^{-N \sum_k |\mu_k|^2} \quad (1.1)$$

with respect to the Lebesgue measure on \mathbb{C}^N . These random spectral measures define a determinantal point process with the explicit kernel (see [16])

$$K_N(z_1, z_2) = \frac{N}{\pi} e^{-\frac{N}{2}(|z_1|^2 + |z_2|^2)} \sum_{\ell=0}^{N-1} \frac{(N z_1 \bar{z}_2)^\ell}{\ell!} \quad (1.2)$$

with respect to the Lebesgue measure on \mathbb{C} . This integrability property allowed Ginibre to derive the circular law for the eigenvalues, i.e., $\frac{1}{N}\rho_1^{(N)}$ converges to the uniform measure on the unit circle,

$$\frac{1}{\pi} \mathbf{1}_{|z| < 1} dA(z). \quad (1.3)$$

This limiting law also holds for real Gaussian entries [12], for which a more detailed analysis was performed in [7, 15, 26].

For non-Gaussian entries, Girko [17] argued that the macroscopic limiting spectrum is still given by (1.3). His main insight is commonly known as the *Hermitization technique*, which converts the convergence of complex empirical measures into the convergence of logarithmic transforms of a family of Hermitian matrices. If we denote the original non-Hermitian matrix by X and the eigenvalues of X by μ_j , then for any C^2 function F we have the identity

$$\frac{1}{N} \sum_{j=1}^N F(\mu_j) = \frac{1}{4\pi N} \int \Delta F(z) \operatorname{Tr} \log(X^* - z^*)(X - z) dA(z). \quad (1.4)$$

Due to the logarithmic singularity at 0, it is clear that the small eigenvalues of the Hermitian matrix $(X^* - z^*)(X - z)$ play a special role. A key question is to estimate the small eigenvalues of $(X^* - z^*)(X - z)$, or in other words, the small singular values of $(X - z)$. This problem was not treated in [17], but the gap was remedied in a series of papers. First Bai [3] was able to treat the logarithmic singularity assuming bounded density and bounded high moments for the entries of the matrix (see also [4]). Lower bounds on the smallest singular values were given in Rudelson, Vershynin [24, 25], and subsequently Tao, Vu [28], Pan, Zhou [21] and Götze, Tikhomirov [18] weakened the moments and smoothness assumptions for the circular law, till the optimal L^2 assumption, under which the circular law was proved in [29].

In the previous article [8], we proved a local version of the circular law, up to the optimal scale $N^{-1/2+\varepsilon}$, in the bulk of the spectrum. More precisely, we considered an $N \times N$ matrix X with independent real¹ centered entries with variance N^{-1} . Let μ_j , $j \in \llbracket 1, N \rrbracket$ denote the eigenvalues of X . To state the local circular law, we first define the notion of *stochastic domination*.

¹For the sake of notational simplicity we do not consider complex entries in this paper, but the statements and proofs are similar.

Definition 1.1. Let $W = (W_N)_{N \geq 1}$ be family a random variables and $\Psi = (\Psi_N)_{N \geq 1}$ be deterministic parameters. We say that W is stochastically dominated by Ψ if for any $\sigma > 0$ and $D > 0$ we have

$$\mathbb{P}\left[|W_N| > N^\sigma \Psi_N\right] \leq N^{-D}$$

for sufficiently large N . We denote this stochastic domination property by

$$W \prec \Psi, \quad \text{or} \quad W = O_{\prec}(\Psi).$$

In this paper, as in [8], we assume that the probability distributions of the matrix elements satisfy the following uniform subexponential decay property:

$$\sup_{(i,j) \in [1,N]^2} \mathbb{P}\left(|\sqrt{N}X_{i,j}| > \lambda\right) \leq \vartheta^{-1} e^{-\lambda^\vartheta} \quad (1.5)$$

for some constant $\vartheta > 0$ independent of N . This condition can of course be weakened to an hypothesis of boundedness on sufficiently high moments, but the error estimates in the following Theorem would be weakened as well.

Let $f : \mathbb{C} \rightarrow \mathbb{R}$ be a fixed smooth compactly supported function, and $f_{z_0}(\mu) = N^{2a} f(N^a(\mu - z_0))$, where z_0 depends on N and $||z_0| - 1| > \tau$ for some $\tau > 0$ independent of N , and a is a fixed scaling parameter in $(0, 1/2]$. Theorem 2.2 of [8] asserts that the following estimate holds:

$$\left(N^{-1} \sum_j f_{z_0}(\mu_j) - \frac{1}{\pi} \int f_{z_0}(z) dA(z)\right) \prec N^{-1+2a}. \quad (1.6)$$

This implies that the circular law holds after zooming, in the bulk, up to scale $N^{-1/2+\varepsilon}$. In particular, there are neither clusters of eigenvalues nor holes in the spectrum at such scales.

We aim at understanding the circular law close to the edge of the spectrum, i.e., $|z_0| - 1 = o(1)$. The following is our main result.

Theorem 1.2. Let X be an $N \times N$ matrix with independent centered entries of variances $1/N$ and vanishing third moments. Suppose that the distributions of the matrix elements satisfy the subexponential decay property (1.5). Let f_{z_0} be defined as previously and D denote the unit disk. Then for any $a \in (0, 1/2]$ and any $z_0 \in \mathbb{C}$, we have

$$\left(N^{-1} \sum_j f_{z_0}(\mu_j) - \frac{1}{\pi} \int_D f_{z_0}(z) dA(z)\right) \prec N^{-1+2a}. \quad (1.7)$$

Notice that the main assertion of (1.7) is for $|z_0| - 1 = o(1)$ since the other cases were proved in [8], stated in (1.6).

Without the third moment vanishing assumption, we have the following weaker estimate. This estimate does not imply the local law up to the finest scale $N^{-1/2+\varepsilon}$, but it asserts that the circular law near the spectral edge holds at least up to scale $N^{-1/4+\varepsilon}$.

Theorem 1.3. Suppose that X is an $N \times N$ matrix with independent centered entries, variance $1/N$, satisfying the subexponential decay property (1.5). Let f_{z_0} be defined as previously, with $|z_0| \leq C$, and D denote the unit disk. Then for any $a \in (0, 1/4]$ and any $z_0 \in \mathbb{C}$, we have

$$\left(N^{-1} \sum_j f_{z_0}(\mu_j) - \frac{1}{\pi} \int_D f_{z_0}(z) dA(z)\right) \prec N^{-1/2+2a}. \quad (1.8)$$

Shortly after the preprint [8] appeared, a version of local circular law was proved by Tao and Vu [30] under the assumption that the first three moments matching a Gaussian distribution both in the bulk and near the edge. Both the assumptions and conclusions of Theorem 20 [30], when restricted to near the edge, are thus very similar to Theorem 1.2. On the other hand, in the bulk case the assumption of vanishing third moments were not needed in [8]. Our proof in this paper follows the approach of the companion article [8], except that for small eigenvalues we will use the Green function comparison theorem [13]. On the other hand, the method in [30] relies on Jensen's formula for determinants.

A main tool in the proof of the local circular law in [8] was a detailed analysis of the self-consistent equations of the Green functions

$$G_{ij}(w) = [(X^* - z^*)(X - z) - w]_{ij}^{-1}.$$

We were able to control $G_{ij}(E + i\eta)$ for the energy parameter E in any compact set and for sufficiently small η so as to use the formula (1.4) for functions F at scales $N^{-1/2+\varepsilon}$. We proved that, in particular, the Stieltjes transform $m = N^{-1} \sum G_{ii}$ converges to $m_c(w, z)$, a fixed point of the self-consistent equation

$$g(m) = \frac{1 + w m(1 + m)^2}{|z|^2 - 1} \quad (1.9)$$

when $||z| - 1| \geq \tau > 0$. In this region of z , the fixed point equation is stable. However, for $|z| - 1 = o(1)$, (1.9) becomes unstable when $|w|$ is small. In this paper, we will show that m is still close to the fix point of this equation under the additional condition that the third moments of the matrix entries vanish. With this condition, we can compare m of our model with the corresponding quantity for the Gaussian ensemble. In order to carry out this comparison, we prove a local circular law for the Ginibre ensemble near the edge $|z| - 1 = o(1)$ in Section 4. We now outline the proof of the main result, Theorem 1.2. Step 1: we use the Hermitization technique to convert the Theorem 1.2 into the problem on the distribution of the eigenvalues λ_i 's of $(X - z)^*(X - z)$. Step 2: for the eigenvalues of order 1, i.e. $\lambda_i \geq \varepsilon$ for some $\varepsilon > 0$, we will control $\sum \log(\lambda_i)$ via an analysis of the self-consistent equations of the Green functions

$$G_{ij}(w) = [(X^* - z^*)(X - z) - w]_{ij}^{-1}.$$

Step 3: for the eigenvalues of order $o(1)$, we will show that $\sum \log(\lambda_i)$ of our model has the same asymptotic distribution as the Ginibre ensemble, via the Green function comparison method first used in [13]. The local circular law for the Ginibre ensemble is proved in Appendix 4. Step 4: combining the previous steps, we conclude the proof of the local circular law on the edge.

2 PROOF OF THEOREM 1.2 AND THEOREM 1.3

2.1 Hermitization. In the following, we will use the notation

$$Y_z = X - zI$$

where I is the identity operator. Let $\lambda_j(z)$ be the j -th eigenvalue (in the increasing ordering) of $Y_z^* Y_z$. We will generally omit the z -dependence in these notations. Thanks to the Hermitization technique of Girko [17], the first step in proving the local circular law is to understand the local statistics of eigenvalues of $Y_z^* Y_z$. More precisely, (1.4) yields

$$N^{-1} \sum_j f_{z_0}(\mu_j) = \frac{1}{4\pi} N^{-1+2a} \int (\Delta f)(\xi) \sum_j \log \lambda_j(z) dA(\xi), \quad (2.1)$$

where

$$z = z_0 + N^{-a}\xi. \quad (2.2)$$

Roughly speaking, for any fixed $\varepsilon > 0$, the sum $N^{-1} \sum_{\lambda_i \geq \varepsilon} \log \lambda_i(z)$ will approach $\int_{x \geq \varepsilon} (\log x) \rho_c(x) dx$, where ρ_c is the limiting spectral density of $Y_z^* Y_z$. To estimate the error term in this convergence, we need to know some properties of ρ_c (Subsection 2.2) and, most importantly, the strong local Green function estimates of Subsection 2.3. The $\lambda_i \leq \varepsilon$ part will be studied with the Green function comparison method in Section 3.

2.2 Properties of m_c and ρ_c . Define the Green function of $Y_z^* Y_z$ and its trace by

$$G(w) := G(w, z) = (Y_z^* Y_z - w)^{-1}, \quad m(w) := m(w, z) = \frac{1}{N} \operatorname{Tr} G(w, z) = \frac{1}{N} \sum_{j=1}^N \frac{1}{\lambda_j(z) - w}, \quad w = E + i\eta.$$

We will also need the following version of the Green function later on:

$$\mathcal{G}(w) := \mathcal{G}(w, z) = (Y_z Y_z^* - w)^{-1}.$$

As we will see, for some regions of (w, z) , with high probability $m(w, z)$ converges to $m_c(w, z)$ pointwise, as $N \rightarrow \infty$ where $m_c(w, z)$ is the unique solution of

$$m_c^{-1} = -w(1 + m_c) + |z|^2(1 + m_c)^{-1} \quad (2.3)$$

with positive imaginary part (see Section 3 in [18] for the existence and uniqueness of such a solution). The limit $m_c(w, z)$ is the Stieltjes transform of a density $\rho_c(x, z)$ and we have

$$m_c(w, z) = \int_{\mathbb{R}} \frac{\rho_c(x, z)}{x - w} dx \quad (2.4)$$

whenever $\eta > 0$. The function $\rho_c(x, z)$ is the limiting eigenvalue density of the matrix $Y_z^* Y_z$ (cf. Lemmas 4.2 and 4.3 in [3]). This measure is compactly supported and satisfies the following bounds. Let

$$\lambda_{\pm} := \lambda_{\pm}(z) := \frac{(\alpha \pm 3)^3}{8(\alpha \pm 1)}, \quad \alpha := \sqrt{1 + 8|z|^2}. \quad (2.5)$$

Note that λ_- has the same sign as $|z| - 1$. It is well-known that the density $\rho_c(x, z)$ can be obtained from its Stieltjes transform $m_c(x + i\eta, z)$ via

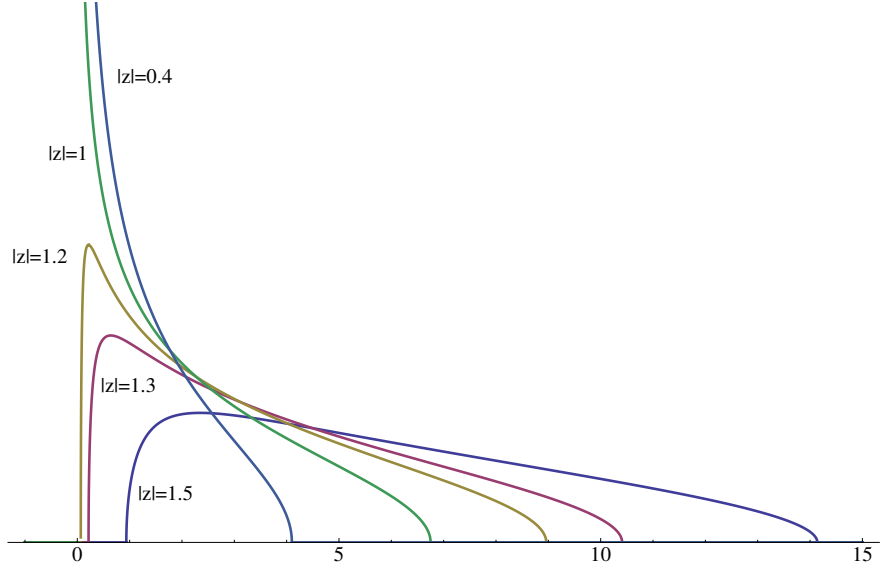
$$\rho_c(x, z) = \frac{1}{\pi} \operatorname{Im} \lim_{\eta \rightarrow 0^+} m_c(x + i\eta, z) = \frac{1}{\pi} \mathbf{1}_{x \in [\max\{0, \lambda_-\}, \lambda_+]} \operatorname{Im} \lim_{\eta \rightarrow 0^+} m_c(x + i\eta, z).$$

The propositions and lemmas in this subsection summarize the properties of ρ_c and m_c that we will need in this paper. They contain slight extensions (to the case $|z| = 1 + o(1)$) of the analogous statements [8]. We omit the proof, strictly similar to the calculation mentioned in [8].

In the following, we use the notation $A \sim B$ when $cB \leq A \leq c^{-1}B$, where $c > 0$ is independent of N .

Proposition 2.1. *The limiting density ρ_c is compactly supported and the following properties hold.*

- (i) *The support of $\rho_c(x, z)$ is $[\max\{0, \lambda_-\}, \lambda_+]$.*
- (ii) *As $x \rightarrow \lambda_+$ from below, the behavior of $\rho_c(x, z)$ is given by $\rho_c(x, z) \sim \sqrt{\lambda_+ - x}$.*
- (iii) *For any fixed $\varepsilon > 0$, if $\max\{0, \lambda_-\} + \varepsilon \leq x \leq \lambda_+ - \varepsilon$, then $\rho_c(x, z) \sim 1$.*
- (iv) *Near $\max\{0, \lambda_-\}$, if z is allowed to change with N such that $|z| = 1 + \varepsilon_N$ (with $\varepsilon_N = o(1)$), then $\lambda_- = o(1)$ and the behavior of $\rho_c(x, z)$ depends on ε_N . We will not need this detailed property and will only need the following upper bound: for any $\delta > 0$ there is $C > 0$ such that for any $|z| \leq \delta^{-1}$, $x \in [0, \delta]$, we have $\rho_c(x, z) \leq C/\sqrt{x}$.*



The limiting eigenvalues density $\rho_c(x, z)$ for $z = 1.5, 1.3, 1.2, 1, 0.4$.

From this proposition, we obtain the following bound regarding the Stieltjes transform of ρ_c (see (2.4)).

Proposition 2.2. *Uniformly in z and w in any compact set, the Stieltjes transform of ρ_c is bounded by*

$$m_c(w, z) = O(|w|^{-1/2}). \quad (2.6)$$

Moreover, for any fixed $\varepsilon > 0$, uniformly in $\varepsilon < |w| < \varepsilon^{-1}$, $|z| < \varepsilon^{-1}$, we have $m_c \sim 1$.

We now collect some technical properties of m_c used in this paper. Define $\kappa := \kappa(w, z)$ as the distance from E to $\{\lambda_+, \lambda_-\}$:

$$\kappa = \min\{|E - \lambda_-|, |E - \lambda_+|\}. \quad (2.7)$$

For $|z| \leq 1$, we have $\lambda_- < 0$, so in this case we define $\kappa := |E - \lambda_+|$.

The following two lemmas, concerning the case $c \leq |w| \leq c^{-1}$, are analogous to Lemma 4.1-4.3 of [8], and the proofs are essentially the same. Notice that the properties of $\rho_c(x)$ used in [8] when $c \leq |w| \leq c^{-1}$ can be summarized as follows: (1) $\rho_c(x) \sim \sqrt{\lambda_+ - x}$ if $0 < c \leq x \leq \lambda_+$. (2) $\int_0^{o(1)} \rho_c(x) = o(1)$. From Proposition 2.1, these two estimates hold uniformly for all $|z| \leq C$, including $|z| - 1 = o(1)$. We therefore can use the proofs in [8] to obtain the following two Lemmas.

Lemma 2.3. *There exists $\tau_0 > 0$ such that for any $\tau \leq \tau_0$, if $||z| - 1| \leq \tau$ and $\tau \leq |w| \leq \tau^{-1}$ then the following properties concerning m_c hold (all constants in the following estimates depending on τ). Notice that by (2.5), we have in this case $\lambda_- = \frac{(\alpha-3)^3}{8(\alpha-1)} = O(|z| - 1|^3) = O(\tau^3)$.*

Case 1: $E \geq \lambda_+$ and $|w - \lambda_+| \geq \tau$. We have

$$|\operatorname{Re} m_c| \sim 1, \quad 0 \geq \operatorname{Re} m_c \geq -\frac{1}{2}, \quad \operatorname{Im} m_c \sim \eta. \quad (2.8)$$

Case 2: $|w - \lambda_+| \leq \tau$. We have

$$m_c(w, z) = -\frac{2}{3 + \alpha} + \sqrt{\frac{8(1 + \alpha)^3}{\alpha(3 + \alpha)^5}} (w - \lambda_+)^{1/2} + \mathcal{O}(\lambda_+ - w), \quad (2.9)$$

and

$$\operatorname{Im} m_c \sim \begin{cases} \frac{\eta}{\sqrt{\kappa}} & \text{if } \kappa \geq \eta \text{ and } E \geq \lambda_+, \\ \sqrt{\eta} & \text{if } \kappa \leq \eta \text{ or } E \leq \lambda_+. \end{cases} \quad (2.10)$$

Case 3: $|w - \lambda_+| \geq \tau$ and $E \leq \lambda_+$ (notice that E is allowed to be smaller than λ_-). We have

$$|m_c| \sim 1, \quad \operatorname{Im} m_c \sim 1. \quad (2.11)$$

Lemma 2.4. *There exists $\tau_0 > 0$ such that for any $\tau \leq \tau_0$ if the conditions $||z| - 1| \leq \tau$ and $\tau \leq |w| \leq \tau^{-1}$ hold, then we have the following three bounds concerning m_c (all constants in the following estimates depend on τ):*

$$|m_c + 1| \sim |m_c| \sim |w|^{-1/2}, \quad (2.12)$$

$$\left| \operatorname{Im} \frac{1}{w(1 + m_c)} \right| \leq C \operatorname{Im} m_c, \quad (2.13)$$

$$\left| (-1 + |z^2|) \left(m_c - \frac{-2}{3 + \alpha} \right) \left(m_c - \frac{-2}{3 - \alpha} \right) \right| \geq C \frac{\sqrt{\kappa + \eta}}{|w|}. \quad (2.14)$$

2.3 The Strong Green function estimates. We now state precisely the estimate regarding the convergence of m to m_c . Since the matrix $Y_z^* Y_z$ is symmetric, we will follow the approach of [14]. We will use extensively the following definition of high probability events.

Definition 2.5 (High probability events). *Define*

$$\varphi := (\log N)^{\log \log N}. \quad (2.15)$$

Let $\zeta > 0$. We say that an N -dependent event Ω holds with ζ -high probability if there is some constant C such that

$$\mathbb{P}(\Omega^c) \leq N^C \exp(-\varphi^\zeta)$$

for large enough N .

For $\alpha \geq 0$, define the z -dependent set

$$\mathbb{S}(\alpha) := \{w \in \mathbb{C} : \max(\lambda_-/5, 0) \leq E \leq 5\lambda_+, \varphi^\alpha N^{-1} |m_c|^{-1} \leq \eta \leq 10\},$$

where φ is defined in (2.15). Here we have suppressed the explicit z -dependence.

Theorem 2.6 (Strong local Green function estimates). *Let $\varepsilon > 0$ be given. Suppose that z is bounded in \mathbb{C} , uniformly in N . Then for any $\zeta > 0$, there exists $C_\zeta > 0$ such that the following event holds with ζ -high probability:*

$$\bigcap_{w \in \underline{\mathcal{S}}(C_\zeta), |w| > \varepsilon} \left\{ |m(w) - m_c(w)| \leq \varphi^{C_\zeta} \frac{1}{N\eta} \right\}. \quad (2.16)$$

Moreover, the individual matrix elements of the Green function satisfy, with ζ -high probability,

$$\bigcap_{w \in \underline{\mathcal{S}}(C_\zeta), |w| > \varepsilon} \left\{ \max_{ij} |G_{ij} - m_c \delta_{ij}| \leq \varphi^{C_\zeta} \left(\sqrt{\frac{\text{Im } m_c}{N\eta}} + \frac{1}{N\eta} \right) \right\}. \quad (2.17)$$

Proof. The proof mimics the one of Theorem 3.4 in [8], in the case of general z but restricted to $|w| \geq \varepsilon$. Indeed, in the special case $|w| \geq \varepsilon$, Theorem 3.4 in [8] (which is stated for $||z| - 1| > 0$) actually also applies for $|z| = 1$: its proof only needs some properties of m_c in this case, which are Lemmas 4.1 and 4.2 in [8], and have as analogues (even for $|z| = 1$) Lemmas 2.3 and 2.4 from the previous subsection. \square

2.4 Conclusion. Following the Hermitization explained in the previous section, in order to prove Theorem 1.2, we need to properly bound (2.1). Now $\log \lambda_j(z)$ needs to be evaluated even for z close to the edge, and for this we first define a proper approximation of the logarithm, with a cutoff at scale $N^{-2+\varepsilon}$.

Definition 2.7. *Let $h(x)$ be a smooth increasing function supported on $[1, +\infty]$ with $h(x) = 1$ for $x \geq 2$ and $h(x) = 0$ for $x \leq 1$. For any $\varepsilon > 0$, define ϕ on \mathbb{R}_+ by*

$$\phi(x) = \phi_\varepsilon(x) = h(N^{2-2\varepsilon}x) (\log x) \left(1 - h\left(\frac{x}{2\lambda_+}\right) \right). \quad (2.18)$$

We now prove that ϕ is a proper approximation for \log in the sense that

$$\text{Tr}(\log(Y^*Y) - \phi(Y^*Y)) \prec N^{C\varepsilon} \quad (2.19)$$

uniformly in z in any compact set, where C is an absolute constant independent of N .

For this purpose, we first bound the number of small eigenvalues of Y^*Y . This is the aim of the following lemma, proved in Subsection 3.2.

Lemma 2.8. *Suppose that $|w| + |z| \leq M$ for some constant M independent of N . Under the same assumptions as Theorem 1.3, for any $\zeta > 0$, there exists C_ζ such that if*

$$\eta \geq \varphi^{C_\zeta} N^{-1} |w|^{1/2} \quad (2.20)$$

then we have

$$|m(w, z)| \leq (\log N) |w|^{-1/2} \quad (2.21)$$

with ζ -high probability (notice that if we take $\eta \sim |w|$ then the restriction is $|w| \geq N^{-2}\varphi^{2C_\zeta}$).

Using

$$|\{j : \lambda_j \in [E - \eta, E + \eta]\}| \leq C\eta \text{Im } m(E + i\eta) \quad (2.22)$$

and Lemma 2.8 with $E = 0$ and $\eta = N^{-2+2\varepsilon}$, we obtain

$$\frac{1}{N} |\{j : |\lambda_j| \leq N^{-2+2\varepsilon}\}| \prec N^\varepsilon. \quad (2.23)$$

Moreover, we have the following lower bound on the smallest eigenvalue.

Lemma 2.9. *Under the same assumptions as Theorem 1.3,*

$$|\log \lambda_1(z)| \prec 1$$

holds uniformly for z in any fixed compact set.

Proof. This lemma follows from [25] or Theorem 2.1 of [28], which gives the required estimate uniformly in z . Note that the typical size of λ_1 is N^{-2} [25], and we need a much weaker bound of type $\mathbb{P}(\lambda_1(z) \leq e^{-N^{-\varepsilon}}) \leq N^{-C}$ for any $\varepsilon, C > 0$. This estimate is very simple to prove if, for example, the entries of X have a density bounded by N^C , which was discussed in Lemma 5.2 of [8]. \square

The preceding lemma together with (2.23) implies

$$\mathrm{Tr} \left(\log(Y^*Y) - \tilde{\phi}(Y^*Y) \right) \prec N^{C\varepsilon}, \quad (2.24)$$

where $\tilde{\phi}(x) = h(N^{2-2\varepsilon}x)(\log x)$. The proof of (2.19) is then complete thanks to the following easy lemma bounding the contribution of exceptionally large eigenvalues.

Lemma 2.10. *Under the same assumptions as Theorem 1.3, for any $\varepsilon > 0$,*

$$\mathrm{Tr} \left(\phi(Y^*Y) - \tilde{\phi}(Y^*Y) \right) \prec 1 \quad (2.25)$$

holds uniformly for z in any fixed compact set.

Proof. Notice first that, for any $c_0 > 0$, there is a constant $C > 0$ such that, uniformly in $|z| < c_0$, $\lambda_N \prec N^C$ (this is for example an elementary consequence of the inequality $\lambda_N \leq \mathrm{Tr}(Y^*Y)$). The proof of the lemma will therefore be complete if

$$|\{i : \lambda_i \geq 2\lambda_+\}| \prec 1. \quad (2.26)$$

Let $(\lambda_k^{(\delta)})_{k \in [1, N]}$ be the increasing eigenvalues of $Y^{(\delta)*}Y^{(\delta)}$, where $Y^{(\delta)} = X - (1 - \delta)z$. By the Weyl inequality for the singular values, for any $\delta > 0$

$$\lambda_k^{1/2} \leq \lambda_k^{(\delta)1/2} + \delta. \quad (2.27)$$

Moreover, as $|(1 - \delta)z| < 1$, we are in the context of Lemma 5.1 in [8], which yields, for any $\zeta > 0$, the existence of $C_\zeta > 0$ such that

$$\lambda_{N-\varphi^{C_\zeta}}^{(\delta)} \leq \lambda_+^{(\delta)} \quad (2.28)$$

with ζ -high probability. Equations (2.27) and (2.28) give

$$\lambda_{N-\varphi^{C_\zeta}} \leq \lambda_+^{(\delta)} + 2\delta\lambda_+^{(\delta)1/2} + \delta^2.$$

Together with $\lambda_+^{(\delta)} \rightarrow \lambda_+$ as $\delta \rightarrow 0$ (uniformly in $|z| < c_0$), this concludes the proof of (2.26) by choosing δ small enough. \square

Remark 2.11. *With some more technical efforts, the techniques used in this paper allow to prove the stronger result $\lambda_N - \lambda_+ \prec N^{-2/3}$. This rigidity on the edge could be proved similarly to the rigidity of the largest eigenvalues of X^*X , for non-square matrices [22].*

It is known, by Lemma 4.4 of [3], that

$$\int_0^\infty (\log x) \Delta_z \rho_c(x, z) dx = 4\chi_D(z). \quad (2.29)$$

By definition of ϕ and ρ_c

$$\left| \int \phi(x) \rho_c(x) dx - \int \log(x) \rho_c(x) dx \right| \leq N^{-1+C\varepsilon} \quad (2.30)$$

Then from equations (2.1) and (2.19), Theorem 1.2 follows if we can prove estimate

$$\left| \int \Delta f(\xi) \left(\text{Tr} \phi(Y^*Y) - N \int \phi(x) \rho_c(x) dx \right) d\xi d\bar{\xi} \right| \prec N^{C\varepsilon}. \quad (2.31)$$

We first evaluate $\text{Tr} \phi(Y^*Y)$ in the following lemma.

Lemma 2.12. *Let χ be a smooth cutoff function supported in $[-1, 1]$ with bounded derivatives and $\chi(y) = 1$ for $|y| \leq 1/2$. For the same ε presented in the definition of ϕ_ε , we define the domain in \mathbb{C}*

$$I = \left\{ N^{-1+\varepsilon} \sqrt{E} \leq \eta, |w| \leq \varepsilon \right\}. \quad (2.32)$$

There is some $C > 0$ such that for any $\varepsilon > 0$, (recall: $\phi := \phi_\varepsilon$ from Def. 2.7)

$$\left| \text{Tr} \phi(Y^*Y) - N \int \phi(x) \rho_c(x) dx - \frac{N}{\pi} \int_I \chi(\eta) \phi'(E) \text{Re}(m(w) - m_c(w)) dE d\eta \right| \prec N^{C\varepsilon} \quad (2.33)$$

uniformly in z in any compact set.

Proof. Suppose that (2.33) holds with the integration range I replaced by \tilde{I} where

$$\tilde{I} = \left\{ N^{-1+\varepsilon} \sqrt{E} \leq \eta \right\}.$$

Define the notation $\Delta m = m - m_c$. One can easily check that if $w \in \tilde{I}$ and $\phi'(E) \neq 0$, then $\eta \geq N^{-1+\varepsilon} |w|^{1/2}$ (if $\eta \leq E$ this last equation exactly means $\omega \in \tilde{I}$ and if $\eta \geq E$ it is equivalent to $\eta \geq N^{-2+2\varepsilon}$, which is true because $\eta \geq N^{-1+\varepsilon} \sqrt{E}$ and $E \geq N^{-2+2\varepsilon}$ when $\phi'(E) \neq 0$).

Using Proposition 2.2 and Theorem 2.6, we therefore have $\text{Re}(\Delta m(w)) = O(\varphi^{C\varepsilon}/(N\eta))$ on \tilde{I}/I , so

$$\int_{\tilde{I}/I} \chi(\eta) \phi'(E) \text{Re}(\Delta m(w)) dE d\eta \prec \int_{\varepsilon \leq |w| \leq C, N^{-1+\varepsilon} \sqrt{E} \leq \eta \leq C} \frac{dE d\eta}{NE\eta} \prec N^{-1}.$$

Hence (2.33) holds with integration range given by I .

We now prove (2.33) with I replaced by \tilde{I} . The Helffer-Sjöstrand functional calculus (see e.g. [11]), gives for any smooth function $q: \mathbb{R} \rightarrow \mathbb{R}$,

$$q(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{iyq''(x)\chi(y) + i(q(x) + iyq'(x))\chi'(y)}{\lambda - x - iy} dx dy.$$

As a consequence,

$$\text{Tr} \phi(Y^*Y) = \frac{N}{\pi} \int_{\eta>0} \left(i\eta\phi''(E)\chi(\eta) + i\phi(E)\chi'(\eta) - \eta\phi'(E)\chi'(\eta) \right) m(E + i\eta) dE d\eta, \quad (2.34)$$

$$N \int \phi(x) \rho_c(x) dx = \frac{N}{\pi} \int_{\eta>0} \left(i\eta\phi''(E)\chi(\eta) + i\phi(E)\chi'(\eta) - \eta\phi'(E)\chi'(\eta) \right) m_c(E + i\eta) dE d\eta. \quad (2.35)$$

By definition, $\chi'(\eta) \neq 0$ only if $|\eta| \in [1/2, 1]$. Using (2.16), we have

$$\frac{N}{\pi} \int_{\eta>0} \left(i\phi(E)\chi'(\eta) - \eta\phi'(E)\chi'(\eta) \right) \Delta m(E + i\eta) dE d\eta \prec 1.$$

hence the difference between (2.34) and (2.35) is

$$\text{Tr } \phi(Y^*Y) - N \int \phi(x)\rho_c(x)dx = \frac{N}{\pi} \int_{\eta>0} i\eta\phi''(E)\chi(\eta)\Delta m(E + i\eta)dE d\eta + O_{\prec}(1).$$

Since the left side of the equation, χ and ϕ are all real, this can be rewritten as

$$\text{Tr } \phi(Y^*Y) - N \int \phi(x)\rho_c(x)dx = -\frac{N}{\pi} \int_{\eta>0} \eta\phi''(E)\chi(\eta) \text{Im } \Delta m(E + i\eta)dE d\eta + O_{\prec}(1).$$

Furthermore, for $\eta \leq N^{-1+\varepsilon}\sqrt{E}$ and $E \geq N^{-2+2\varepsilon}$, we have $\eta \text{Im } m(E + i\eta) \prec N^{-1}$, because $\eta \mapsto \eta \text{Im } m(E + i\eta)$ is increasing, and at $\eta_0 = N^{-1+\varepsilon}\sqrt{E}$ the result is true thanks to (2.21) ($\eta \leq N^{-1+\varepsilon}\sqrt{E}$ and $E \geq N^{-2+2\varepsilon}$ imply that $\eta > N^{-1+\varepsilon}|w|^{1/2}$ as we saw previously). Together with the easy bound $|\phi''(E)| \leq C(1 + |\log(E)|)E^{-2}$, the above estimate yields

$$N \int_{\tilde{I}^c} \eta\phi''(E)\chi(\eta) \text{Im } m(w)dE d\eta \prec \int_{N^{-2} \leq E \leq O(1)} CN^{-1+C\varepsilon}E^{-3/2}dE \prec N^{C\varepsilon}. \quad (2.36)$$

The same inequality holds when replacing m with m_c , hence

$$N \int_{\tilde{I}^c} \eta\phi''(E)\chi(\eta) \text{Im}(\Delta m(w))dE d\eta \prec N^{C\varepsilon}.$$

To estimate $N \int_{\tilde{I}} \eta\phi''(E)\chi(\eta) \text{Im}(\Delta m(w))dE d\eta$, we integrate this term by parts first in E , then in η (and use the Cauchy-Riemann equation $\frac{\partial}{\partial E} \text{Im}(\Delta m) = -\frac{\partial}{\partial \eta} \text{Re}(\Delta m)$) so that

$$\begin{aligned} N \int_{\tilde{I}} \eta\phi''(E)\chi(\eta) \text{Im}(\Delta m(w))dE d\eta &= N \int \eta\chi(\eta)\phi'(N^{2-2\varepsilon}\eta^2) \text{Im}(\Delta m(N^{2-2\varepsilon}\eta^2 + i\eta))d\eta \\ &\quad + N \int N^{-1+\varepsilon}\sqrt{E}\phi'(E)\chi(N^{-1+\varepsilon}\sqrt{E}) \text{Re}(\Delta m(E + iN^{-1+\varepsilon}\sqrt{E}))dE \\ &\quad - N \int_{\tilde{I}} \chi(\eta)\phi'(E) \text{Re}(\Delta m(w))dE d\eta \\ &\quad - N \int_{\tilde{I}} \eta\chi'(\eta)\phi'(E) \text{Re}(\Delta m(w))dE d\eta. \end{aligned} \quad (2.37)$$

The last term is nonzero only if $|\eta| \in [1/2, 1]$. By (2.16), this term is of order $O_{\prec}(1)$.

Concerning the first two terms, they can be bounded in the following way: as we already saw, if $w \in \tilde{I}$ and $\phi(E) \neq 0$, then $\eta \geq N^{-1+\varepsilon}|w|^{1/2}$; one can therefore use (2.21): $|m| + |m_c| \leq (\log N)|w|^{-1/2}$. This together with $|\phi'(E)| \leq C(1 + |\log(E)|)E^{-1}$ proves that the first two terms are $O_{\prec}(N^{C\varepsilon})$, completing the proof: only the third term remains. \square

To complete the proof of Theorem 1.2, we estimate the third term in (2.37) under the additional assumption that the third moments of the matrix entries vanish. The following lemma provides such an estimate whose proof will be postponed to Section 3.3. To state this lemma, we introduce the following notation,

$$Z_{X_1, X_2}^{(g)} = N \int \Delta g(\xi) \int_{\tilde{I}} \chi(\eta)\phi'(E) \text{Re}(m_1(w) - m_2(w))dE d\eta dA(\xi),$$

for any given smooth function g and initial random matrices X_1, X_2 , with associated matrices $Y_z^{(1)*} Y_z^{(1)}$, $Y_z^{(2)*} Y_z^{(2)}$ having respective Stieltjes transforms m_1 and m_2 (the parameter z and ξ are related by (2.2)). Moreover, we will write

$$Z_{X,c}^{(g)} = N \int \Delta g(\xi) \int_I \chi(\eta) \phi'(E) \operatorname{Re}(m(w) - m_c(w)) dE d\eta dA(\xi).$$

Notice that $Z_{X_1, X_2}^{(g)}$ and $Z_{X,c}^{(g)}$ depend on ε through the definition of ϕ in Def. 2.7.

Lemma 2.13. *Under the assumptions of Theorem 1.2, there exists a constant $C > 0$ such that for any small enough $\varepsilon > 0$, we have*

$$Z_{X,c}^{(f)} \prec N^{C\varepsilon} c_f,$$

where c_f is a constant depending only on the function f .

Combining this lemma with (2.33), we obtain (2.31) and complete the proof of Theorem 1.2.

To prove Theorem 1.3, instead of Lemma 2.13, we only need to prove the following lemma which does not assume the vanishing third moment condition. This lemma will be proved at the end of Section 3. This concludes the proof of Theorem 1.3.

Lemma 2.14. *Under the assumptions of Theorem 1.3, for some fixed $C > 0$, for any small enough $\varepsilon > 0$, we have*

$$Z_{X,c}^{(f)} \prec N^{1/2+C\varepsilon} c_f,$$

where c_f is a constant depending only on the function f .

3 PROOF OF LEMMAS 2.8 AND 2.13

3.1 Preliminary lemmas. This subsection summarizes some elementary results from [8], based on large deviation estimates. Note that all the inequalities in this subsection hold uniformly for bounded z , no matter its distance to the unit circle. We first introduce some notations.

Definition 3.1. *Let $\mathbb{T}, \mathbb{U} \subset \llbracket 1, N \rrbracket$. Then we define $Y^{(\mathbb{T}, \mathbb{U})}$ as the $(N - |\mathbb{U}|) \times (N - |\mathbb{T}|)$ matrix obtained by removing all columns of Y indexed by $i \in \mathbb{T}$ and all rows of Y indexed by $i \in \mathbb{U}$. Notice that we keep the labels of indices of Y when defining $Y^{(\mathbb{T}, \mathbb{U})}$.*

Let \mathbf{y}_i be the i -th column of Y and $\mathbf{y}_i^{(\mathbb{S})}$ be the vector obtained by removing $\mathbf{y}_i(j)$ for all $j \in \mathbb{S}$. Similarly we define y_i be the i -th row of Y . Define

$$\begin{aligned} G^{(\mathbb{T}, \mathbb{U})} &= \left[(Y^{(\mathbb{T}, \mathbb{U})})^* Y^{(\mathbb{T}, \mathbb{U})} - w \right]^{-1}, \quad m_G^{(\mathbb{T}, \mathbb{U})} = \frac{1}{N} \operatorname{Tr} G^{(\mathbb{T}, \mathbb{U})}, \\ \mathcal{G}^{(\mathbb{T}, \mathbb{U})} &= \left[Y^{(\mathbb{T}, \mathbb{U})} (Y^{(\mathbb{T}, \mathbb{U})})^* - w \right]^{-1}, \quad m_{\mathcal{G}}^{(\mathbb{T}, \mathbb{U})} = \frac{1}{N} \operatorname{Tr} \mathcal{G}^{(\mathbb{T}, \mathbb{U})}. \end{aligned}$$

By definition, $m^{(\emptyset, \emptyset)} = m$. Since the eigenvalues of $Y^* Y$ and $Y Y^*$ are the same except the zero eigenvalue, it is easy to check that

$$m_G^{(\mathbb{T}, \mathbb{U})}(w) = m_{\mathcal{G}}^{(\mathbb{T}, \mathbb{U})} + \frac{|\mathbb{U}| - |\mathbb{T}|}{Nw} \quad (3.1)$$

For $|\mathbb{U}| = |\mathbb{T}|$, we define

$$m^{(\mathbb{T}, \mathbb{U})} := m_G^{(\mathbb{T}, \mathbb{U})} = m_{\mathcal{G}}^{(\mathbb{T}, \mathbb{U})} \quad (3.2)$$

Lemma 3.2 (Relation between G , $G^{(\mathbb{T},\emptyset)}$ and $G^{(\emptyset,\mathbb{T})}$). For $i, j \neq k$ ($i = j$ is allowed) we have

$$G_{ij}^{(k,\emptyset)} = G_{ij} - \frac{G_{ik}G_{kj}}{G_{kk}}, \quad \mathcal{G}_{ij}^{(\emptyset,k)} = \mathcal{G}_{ij} - \frac{\mathcal{G}_{ik}\mathcal{G}_{kj}}{\mathcal{G}_{kk}}, \quad (3.3)$$

$$G^{(\emptyset,i)} = G + \frac{(G\mathbf{y}_i^*)(\mathbf{y}_i G)}{1 - \mathbf{y}_i G \mathbf{y}_i^*}, \quad G = G^{(\emptyset,i)} - \frac{(G^{(\emptyset,i)}\mathbf{y}_i^*)(\mathbf{y}_i G^{(\emptyset,i)})}{1 + \mathbf{y}_i G^{(\emptyset,i)} \mathbf{y}_i^*}, \quad (3.4)$$

and

$$\mathcal{G}^{(i,\emptyset)} = \mathcal{G} + \frac{(\mathcal{G}\mathbf{y}_i)(\mathbf{y}_i^* \mathcal{G})}{1 - \mathbf{y}_i^* \mathcal{G} \mathbf{y}_i}, \quad \mathcal{G} = \mathcal{G}^{(i,\emptyset)} - \frac{(\mathcal{G}^{(i,\emptyset)}\mathbf{y}_i)(\mathbf{y}_i^* \mathcal{G}^{(i,\emptyset)})}{1 + \mathbf{y}_i^* \mathcal{G}^{(i,\emptyset)} \mathbf{y}_i}.$$

Furthermore, the following crude bound on the difference between m and $m_G^{(\mathbb{U},\mathbb{T})}$ holds: for $\mathbb{U}, \mathbb{T} \subset \llbracket 1, N \rrbracket$ we have

$$|m - m_G^{(\mathbb{U},\mathbb{T})}| + |m - m_{\mathcal{G}}^{(\mathbb{U},\mathbb{T})}| \leq \frac{|\mathbb{U}| + |\mathbb{T}|}{N\eta}. \quad (3.5)$$

Definition 3.3. In the following, \mathbb{E}_X means the integration with respect to the random variable X . For any $\mathbb{T} \subset \llbracket 1, N \rrbracket$, we introduce the notations

$$Z_i^{(\mathbb{T})} := (1 - \mathbb{E}_{\mathbf{y}_i}) \mathbf{y}_i^{(\mathbb{T})} G^{(\mathbb{T},i)} \mathbf{y}_i^{(\mathbb{T})*}$$

and

$$\mathcal{Z}_i^{(\mathbb{T})} := (1 - \mathbb{E}_{\mathbf{y}_i}) \mathbf{y}_i^{(\mathbb{T})*} \mathcal{G}^{(i,\mathbb{T})} \mathbf{y}_i^{(\mathbb{T})}.$$

Recall by our convention that \mathbf{y}_i is a $N \times 1$ column vector and \mathbf{y}_i is a $1 \times N$ row vector. For simplicity we will write

$$Z_i = Z_i^{(\emptyset)}, \quad \mathcal{Z}_i = \mathcal{Z}_i^{(\emptyset)}.$$

Lemma 3.4 (Identities for G , \mathcal{G} , Z and \mathcal{Z}). For any $\mathbb{T} \subset \llbracket 1, N \rrbracket$, we have

$$G_{ii}^{(\emptyset,\mathbb{T})} = -w^{-1} \left[1 + m_{\mathcal{G}}^{(i,\mathbb{T})} + |z|^2 \mathcal{G}_{ii}^{(i,\mathbb{T})} + \mathcal{Z}_i^{(\mathbb{T})} \right]^{-1}, \quad (3.6)$$

$$G_{ij}^{(\emptyset,\mathbb{T})} = -w G_{ii}^{(\emptyset,\mathbb{T})} G_{jj}^{(i,\mathbb{T})} \left(\mathbf{y}_i^{(\mathbb{T})*} \mathcal{G}^{(ij,\mathbb{T})} \mathbf{y}_j^{(\mathbb{T})} \right), \quad i \neq j, \quad (3.7)$$

where, by definition, $\mathcal{G}_{ii}^{(i,\mathbb{T})} = 0$ if $i \in \mathbb{T}$. Similar results hold for \mathcal{G} :

$$\left[\mathcal{G}_{ii}^{(\mathbb{T},\emptyset)} \right]^{-1} = -w \left[1 + m_G^{(\mathbb{T},i)} + |z|^2 G_{ii}^{(\mathbb{T},i)} + Z_i^{(\mathbb{T})} \right] \quad (3.8)$$

$$\mathcal{G}_{ij}^{(\mathbb{T},\emptyset)} = -w \mathcal{G}_{ii}^{(\mathbb{T},\emptyset)} \mathcal{G}_{jj}^{(\mathbb{T},i)} \left(\mathbf{y}_i^{(\mathbb{T})} G^{(\mathbb{T},ij)} \mathbf{y}_j^{(\mathbb{T})*} \right), \quad i \neq j. \quad (3.9)$$

Lemma 3.5 (Large deviation estimate). For any $\zeta > 0$, there exists $Q_\zeta > 0$ such that for $\mathbb{T} \subset \llbracket 1, N \rrbracket$, $|\mathbb{T}| \leq N/2$ the following estimates hold with ζ -high probability:

$$|Z_i^{(\mathbb{T})}| = \left| (1 - \mathbb{E}_{\mathbf{y}_i}) \left(\mathbf{y}_i^{(\mathbb{T})} G^{(\mathbb{T},i)} \mathbf{y}_i^{(\mathbb{T})*} \right) \right| \leq \varphi^{Q_\zeta/2} \sqrt{\frac{\text{Im } m_G^{(\mathbb{T},i)} + |z|^2 \text{Im } G_{ii}^{(\mathbb{T},i)}}{N\eta}}, \quad (3.10)$$

$$|\mathcal{Z}_i^{(\mathbb{T})}| = \left| (1 - \mathbb{E}_{\mathbf{y}_i}) \left(\mathbf{y}_i^{(\mathbb{T})*} \mathcal{G}^{(i,\mathbb{T})} \mathbf{y}_i^{(\mathbb{T})} \right) \right| \leq \varphi^{Q_\zeta/2} \sqrt{\frac{\text{Im } m_{\mathcal{G}}^{(i,\mathbb{T})} + |z|^2 \text{Im } \mathcal{G}_{ii}^{(i,\mathbb{T})}}{N\eta}}.$$

Furthermore, for $i \neq j$, we have

$$\left| (1 - \mathbb{E}_{\mathbf{y}_i \mathbf{y}_j}) \left(\mathbf{y}_i^{(\mathbb{T})} G^{(\mathbb{T}, ij)} \mathbf{y}_j^{(\mathbb{T})*} \right) \right| \leq \varphi^{Q_\zeta/2} \sqrt{\frac{\operatorname{Im} m_G^{(\mathbb{T}, ij)} + |z|^2 \operatorname{Im} G_{ii}^{(\mathbb{T}, ij)} + |z|^2 \operatorname{Im} G_{jj}^{(\mathbb{T}, ij)}}{N\eta}}, \quad (3.11)$$

$$\left| (1 - \mathbb{E}_{\mathbf{y}_i \mathbf{y}_j}) \left(\mathbf{y}_i^{(\mathbb{T})*} \mathcal{G}^{(ij, \mathbb{T})} \mathbf{y}_j^{(\mathbb{T})} \right) \right| \leq \varphi^{Q_\zeta/2} \sqrt{\frac{\operatorname{Im} m_{\mathcal{G}}^{(ij, \mathbb{T})} + |z|^2 \operatorname{Im} \mathcal{G}_{ii}^{(ij, \mathbb{T})} + |z|^2 \operatorname{Im} \mathcal{G}_{jj}^{(ij, \mathbb{T})}}{N\eta}}, \quad (3.12)$$

where

$$\mathbb{E}_{\mathbf{y}_i \mathbf{y}_j} \left(\mathbf{y}_i^{(\mathbb{T})} G^{(\mathbb{T}, ij)} \mathbf{y}_j^{(\mathbb{T})*} \right) = |z|^2 G_{ij}^{(\mathbb{T}, ij)}, \quad \mathbb{E}_{\mathbf{y}_i \mathbf{y}_j} \left(\mathbf{y}_i^{(\mathbb{T})*} \mathcal{G}^{(ij, \mathbb{T})} \mathbf{y}_j^{(\mathbb{T})} \right) = |z|^2 \mathcal{G}_{ij}^{(ij, \mathbb{T})}. \quad (3.13)$$

3.2 Proof of Lemma 2.8. By (3.10) and (3.5), with ζ -high probability, we have for some large constant C

$$\begin{aligned} |Z_i^{(i)}| &\leq \varphi^{C_\zeta} \sqrt{\frac{\operatorname{Im} m^{(i, i)}}{N\eta}} \leq \varphi^{C_\zeta} \frac{1}{N\eta} + \varphi^{C_\zeta} \sqrt{\frac{|\operatorname{Im} m|}{N\eta}} \\ &\leq \varphi^{3C_\zeta} \frac{1}{N\eta} + \frac{|\operatorname{Im} m|}{(\log N)^C}, \end{aligned} \quad (3.14)$$

where we have used the Cauchy-Schwarz inequality in the last step. Moreover, choosing $\mathbb{T} = \{i\}$ in (3.8), we have

$$\left[\mathcal{G}_{ii}^{(i, \emptyset)} \right]^{-1} = -w \left(1 + m_G^{(i, i)} + Z_i^{(i)} \right). \quad (3.15)$$

If

$$\varphi^{3C} \frac{1}{N\eta} \ll \frac{|m|}{(\log N)^C} \quad \text{and} \quad |m| \geq 3, \quad (3.16)$$

then by (3.15) and (3.5) we have

$$\left| \mathcal{G}_{ii}^{(i, \emptyset)} \right|^{-1} \geq |w| \left| 1 + m_G^{(i, i)} + Z_i^{(i)} \right| \geq \frac{2}{3} |w| \left| m - \frac{|m|}{(\log N)^C} \right| \geq \frac{1}{2} |w| |m| \quad (3.17)$$

Similarly, with ζ -high probability, we can bound Z_i by

$$|Z_i| \leq \varphi^{C_\zeta} \sqrt{\frac{\operatorname{Im} m_{\mathcal{G}}^{(i, \emptyset)} + |z|^2 \operatorname{Im} \mathcal{G}_{ii}^{(i, \emptyset)}}{N\eta}} \leq \varphi^{C_\zeta} \frac{1}{N\eta} + \frac{\operatorname{Im} m + |z|^2 \operatorname{Im} \mathcal{G}_{ii}^{(i, \emptyset)}}{(\log N)^C} \quad (3.18)$$

If (2.21) is violated, i.e., $|m| \geq (\log N) |w|^{-1/2}$, then (3.16) is implied by the assumption (2.20). Hence (3.17) holds. By (3.6),

$$|w G_{ii}|^{-1} = |1 + m_G^{(i, \emptyset)} + |z|^2 \mathcal{G}_{ii}^{(i, \emptyset)} + Z_i| \geq |m|/2 - C - |w|^{-1/2}$$

and we conclude that $|G_{ii}| \leq |w|^{-1/2}$ with ζ -high probability, for any $1 \leq i \leq N$. This contradicts the assumption $|m| \geq (\log N) |w|^{-1/2}$ and we have thus proved (2.21).

3.3 Proof of Lemma 2.13. We begin with the following estimates on the elements of the Green function, weaker than those in Theorem 2.6 but valid for any w and z .

Lemma 3.6. *Suppose $|w| + |z| \leq C$. For any $\zeta > 0$, there exists C_ζ such that if the assumption (2.20) holds then the following estimates hold:*

$$\max_i |G_{ii}| \leq 2(\log N)|w|^{-1/2}, \quad (3.19)$$

$$\max_i |w| |G_{ii}| |\mathcal{G}_{ii}^{(i,\emptyset)}| \leq (\log N)^4, \quad (3.20)$$

$$\max_{ij} |G_{ij}| \leq C(\log N)^2 |w|^{-1/2}, \quad (3.21)$$

with ζ -high probability. Furthermore, for eigenvectors u_α of Y^*Y are delocalized: with ζ -high probability

$$\max_\alpha \|u_\alpha\|_\infty^2 \leq \varphi^{C_\zeta} N^{-1}. \quad (3.22)$$

Proof. From equations (6.37) and (6.38) of [8], the following estimate holds²:

$$\mathcal{G}_{ii}^{(i,\emptyset)} = -\frac{1}{w(1+m)} + \mathcal{E}_1 \quad (3.23)$$

where

$$\mathcal{E}_1 = w^{-1} \frac{1}{(1+m)^2} \left[m_G^{(i,i)} - m + Z_i^{(i)} \right] + \mathcal{O} \left(\frac{|Z_i^{(i)}|^2 + \frac{1}{(N\eta)^2}}{|w||1+m|^3} \right) = \mathcal{O}(\varphi^{Q_\zeta/2} \Psi) \quad (3.24)$$

$$\Psi = \left(\sqrt{\frac{\operatorname{Im} m_c + |m - m_c|}{N\eta}} + \frac{1}{N\eta} \right).$$

By using (2.16)-(2.17), (3.23), (3.24) and (2.12), for $|w| \geq \varepsilon$, one can be easily prove: $|G_{ij}| + |\mathcal{G}_{ii}^{(i,\emptyset)}| \leq |w|^{-1/2}$, which implies (3.19)-(3.21). Moreover, if $|z| \leq 1/2$, (2.16)-(2.17) still hold without the restriction $|w| \leq \varepsilon$ (see Theorem 3.4 in [8]). It implies (3.19)-(3.21) in the case $|z| \leq 1/2$. From now on, we therefore assume that for some small enough $\varepsilon > 0$,

$$|w| \leq \varepsilon, \quad |z| \geq 1/2.$$

From (3.6), (3.5) and (3.8), we have

$$|G_{ii}^{-1} - G_{jj}^{-1}| \leq \frac{C|w|}{N\eta} + W_{ij} + w|Z_i| + w|Z_j|,$$

where

$$W_{ij} = |z|^2 \left((1 + m_G^{(i,i)} + Z_i^{(i)})^{-1} - (1 + m_G^{(j,j)} + Z_j^{(j)})^{-1} \right). \quad (3.25)$$

If $\varepsilon^{1/4}|w|^{-1/2} \leq |m| \leq (\log N)|w|^{-1/2}$ then by (3.14) we have $|Z_i^{(i)}| \leq (\log N)^{-C}|m|$ due to the restriction of η in (2.20). Thus we can estimate W_{ij} by

$$|W_{ij}| \leq |z|^2 \left[|Z_i^{(i)}| + |Z_j^{(j)}| + (N\eta)^{-1} \right] (1 + m^{(i,i)} + Z_i^{(i)})^{-1} (1 + m^{(j,j)} + Z_j^{(j)})^{-1} \quad (3.26)$$

$$\leq C|z|^2 |1+m|^{-2} \left[|Z_i^{(i)}| + |Z_j^{(j)}| + (N\eta)^{-1} \right] \leq (\log N)^{-C} |w|^{1/2} \quad (3.27)$$

²Although these bounds in [8] are stated under the assumption $\|z| - 1| \geq \tau$, the same argument holds also for z close to the unit circle, under the extra assumption $|w| > \varepsilon$.

Under the assumption $|w|$ is small enough and $\varepsilon^{1/4}|w|^{-1/2} \leq |m| \leq (\log N)|w|^{-1/2}$, the condition (3.16) is satisfied and, from (3.18), we have

$$|\mathcal{Z}_i| \leq \left[\frac{1}{N\eta} + \frac{1}{(\log N)^C} \right] |w|^{-1/2} \quad (3.28)$$

Thus we have in this case the estimate

$$|G_{ii} - m| \leq N^{-1} \sum_j |G_{ii} - G_{jj}| \leq |G_{ii}G_{jj}| |G_{ii}^{-1} - G_{jj}^{-1}| \leq |G_{ii}G_{jj}| \left[\frac{1}{N\eta} + \frac{1}{(\log N)^C} \right] |w|^{1/2} \quad (3.29)$$

Define the parameter

$$\gamma(z, w) = \max_i |G_{ii}| |w|^{1/2}$$

Then (3.29) and the assumption $\varepsilon^{1/4}|w|^{-1/2} \leq |m| \leq (\log N)|w|^{-1/2}$ imply that

$$\gamma(z, w) \leq C \log N + \frac{\gamma(z, w)^2}{(\log N)^C}$$

By continuity (in η) method, we have proved $\gamma(z, w) \leq C \log N$ (i.e., (3.19)) assuming that $\varepsilon^{1/4}|w|^{-1/2} \leq |m| \leq (\log N)|w|^{-1/2}$ holds. Since $|m| \leq (\log N)|w|^{-1/2}$ holds with ζ -high probability by (2.21), to prove (3.19), we only have to consider the last case $|m| \leq \varepsilon^{1/4}|w|^{-1/2}$.

By (3.15) and (3.14), we have

$$|\mathcal{G}_{ii}^{(i, \emptyset)}| \geq |w|^{-1} |1 + m^{(i, i)} + \mathcal{Z}_i^{(i)}|^{-1} \geq \varepsilon^{-1/4} |w|^{-1/2} / 2$$

Thus

$$|m_{\mathcal{G}}^{(i, \emptyset)} + |z|^2 \mathcal{G}_{ii}^{(i, \emptyset)}| \geq \varepsilon^{-1/4} |w|^{-1/2} / 2 - \varepsilon^{1/4} |w|^{-1/2} - \frac{1}{N\eta} \geq \varepsilon^{-1/4} |w|^{-1/2} / 4$$

By (3.18) and the notation $m_{\mathcal{G}}^{(i, \emptyset)} + |z|^2 \mathcal{G}_{ii}^{(i, \emptyset)} = A_i + iB_i$, $A_i, B_i \in \mathbb{R}$, we have, for η satisfies (2.20), that

$$|\mathcal{Z}_i| \leq (\log N)^{-1} |B_i| + \frac{(\log N)^C}{N\eta} \quad (3.30)$$

From (3.6), we have

$$\frac{1}{|G_{ii}|} \geq |w| [|1 + A_i + iB_i| - |\mathcal{Z}_i|] \geq |w| \left[\frac{|1 + A_i + iB_i|}{2} - (\log N)^{-C} \right] \geq \varepsilon^{-1/4} |w|^{1/2} / 8 \quad (3.31)$$

We have thus proved (3.19).

We now prove (3.20). From (3.6), we have

$$(wG_{ii})^{-1} + 1 + m_{\mathcal{G}}^{(i, \emptyset)} = -[|z|^2 \mathcal{G}_{ii}^{(i, \emptyset)} + \mathcal{Z}_i] \quad (3.32)$$

By the triangle inequality, we have

$$|wz^2 G_{ii} \mathcal{G}_{ii}^{(i, \emptyset)}| \leq |wG_{ii}| [|z|^2 \mathcal{G}_{ii}^{(i, \emptyset)} + \mathcal{Z}_i|] + |wG_{ii} \mathcal{Z}_i| \quad (3.33)$$

Together with (3.32) and using (3.18) to bound \mathcal{Z}_i , with (2.20) we have

$$\begin{aligned} |wz^2 G_{ii} \mathcal{G}_{ii}^{(i,0)}| &\leq |wG_{ii}[1 + m_{\mathcal{G}}^{(i,0)}]| + 1 + \varphi^{C_\zeta} |wG_{ii}| \sqrt{\frac{\operatorname{Im} m_{\mathcal{G}}^{(i,0)} + |z|^2 \operatorname{Im} \mathcal{G}_{ii}^{(i,0)}}{N\eta}} \\ &\leq |wG_{ii}[1 + m_{\mathcal{G}}^{(i,0)}]| + 1 + \frac{|m| |wG_{ii}|}{(\log N)^C} + \varphi^{-1} \sqrt{|wz^2 G_{ii} \mathcal{G}_{ii}^{(i,0)}|} \end{aligned} \quad (3.34)$$

Then with (2.21) and (3.19) on m and G_{ii} , we obtain (3.20).

For any eigenfunction with eigenvalues $|\lambda_\alpha - E| \leq \eta$, we have

$$\frac{\eta |u_\alpha(i)|^2}{(\lambda_\alpha - E)^2 + \eta^2} \leq |\operatorname{Im} G_{ii}| \leq |G_{ii}|$$

and thus

$$|u_\alpha(i)|^2 \leq 2\eta |G_{ii}| \leq \varphi^{C_\zeta} N^{-1},$$

where we used (3.19) for $\eta = \varphi^{C_\zeta} N^{-1} |w|^{1/2}$. Since this bound holds for all energy E , we have proved (3.22) (notice that the proof of (3.22) for $w \geq \varepsilon$ follows by the same argument).

We now prove (3.21). By the Cauchy-Schwarz inequality, we have

$$|G_{ij}| \leq \sum_\alpha \frac{|\lambda_\alpha - E| |u_\alpha(i)|^2}{(\lambda_\alpha - E)^2 + \eta^2} + \sum_\alpha \frac{|\lambda_\alpha - E| |u_\alpha(j)|^2}{(\lambda_\alpha - E)^2 + \eta^2} + \operatorname{Im} G_{ii} + \operatorname{Im} G_{jj} \quad (3.35)$$

The imaginary parts, $\operatorname{Im} G_{ii} + \operatorname{Im} G_{jj}$, can be bounded by (3.19). The first term on the right hand side is bounded by

$$\sum_\alpha \frac{|\lambda_\alpha - E| |u_\alpha(i)|^2}{(\lambda_\alpha - E)^2 + \eta^2} \leq 2 \sum_{\alpha: \lambda_\alpha \geq E} \frac{(\lambda_\alpha - E) |u_\alpha(i)|^2}{(\lambda_\alpha - E)^2 + \eta^2} + |\operatorname{Re} G_{ii}| \quad (3.36)$$

The real part, $\operatorname{Re} G_{ii}$ is bounded again by (3.19). Let $E_k = E + (2^k - 1)\eta$ and $\eta_k = 2^{k-1}\eta$. From (3.19), we have

$$\sum_{\alpha: \lambda_\alpha \geq E} \frac{(\lambda_\alpha - E) |u_\alpha(i)|^2}{(\lambda_\alpha - E)^2 + \eta^2} \leq \sum_{k=0}^{C \log N} \sum_{\alpha: E_k \leq \lambda_\alpha \leq E_{k+1}} \frac{C\eta_k |u_\alpha(i)|^2}{(\lambda_\alpha - E_k)^2 + \eta_k^2} \leq C \log N |w|^{-1/2} \quad (3.37)$$

This concludes the proof of (3.21) and Lemma 3.6. \square

Lemma 3.7. *Suppose that $|w| + |z| = O(1)$. For any $\zeta > 0$, there exists C_ζ such that if (2.20) holds then we have*

$$|[YG(z)]_{kl}| + |[GY^*(z)]_{kl}| \leq C(\log N)^2 \quad (3.38)$$

and

$$|[YGY^*(z)]_{kl} - \delta_{kl}| \leq C(\log N)^2 |w|^{1/2} \quad (3.39)$$

and

$$\max_{ij} |[G^2]_{ij}| \leq \max_{ij} ||G^2_{ij}| \leq \frac{C(\log N)^2}{\eta} |w|^{-1/2} \quad (3.40)$$

and

$$\max_{ij} |[Y G^2]_{ij}| \leq \frac{C(\log N)^2}{\eta} \quad (3.41)$$

and

$$\max_{ij} |[YG^2Y^*]_{ij}| \leq \frac{C(\log N)^2}{\eta} |w|^{1/2} \quad (3.42)$$

with ζ -high probability.

Proof. We first prove (3.38) and (3.39). From the general identity

$$A(A^*A - w)^{-1}A^* = 1 + w(AA^* - w)^{-1},$$

we have

$$(YGY^*)_{ij} = \delta_{ij} + w\mathcal{G}_{ij}$$

This proves (3.39) with (3.19). To prove (3.38), we notice first the identity

$$YG = \mathcal{G}Y, \quad GY^* = Y^*\mathcal{G}.$$

Recall the rank one perturbation formula

$$(A + \mathbf{v}^*\mathbf{v})^{-1} = A^{-1} - \frac{(A^{-1}\mathbf{v}^*)(\mathbf{v}A^{-1})}{1 + \mathbf{v}A^{-1}\mathbf{v}^*}$$

and Lemma 3.2. We have

$$\mathcal{G}\mathbf{y}_i = \frac{\mathcal{G}^{(i,\emptyset)}\mathbf{y}_i}{1 + \langle \mathbf{y}_i, \mathcal{G}^{(i,\emptyset)}\mathbf{y}_i \rangle}.$$

Together with the equation

$$[G_{ii}]^{-1} = -w - w\mathbf{y}_i^\dagger \mathcal{G}^{(i,\emptyset)}\mathbf{y}_i$$

(see Equation (6.18) in [8] for a derivation) and the definition of \mathbf{y}_i as the i -th column of Y , we have

$$(\mathcal{G}\mathbf{y}_i)_j = -wG_{ii}(\mathcal{G}^{(i,\emptyset)}\mathbf{y}_i)_j = -wzG_{ii}\mathcal{G}_{ji}^{(i,\emptyset)} - wG_{ii} \sum_k \mathcal{G}_{jk}^{(i,\emptyset)} X_{ki} \quad (3.43)$$

From the large deviation lemma, we can bound

$$\left| \sum_k \mathcal{G}_{jk}^{(i,\emptyset)} X_{ki} \right| \leq \varphi^C \sqrt{N^{-1} \sum_k |\mathcal{G}_{jk}^{(i,\emptyset)}|^2} \leq \varphi^C \sqrt{\frac{\text{Im } \mathcal{G}_{ii}^{(i,\emptyset)}}{N\eta}} \leq \text{Im } \mathcal{G}_{ii}^{(i,\emptyset)} + \frac{\varphi^C}{N\eta} \quad (3.44)$$

with high probability. Hence

$$|(\mathcal{G}Y)_{ji}| = |(\mathcal{G}\mathbf{y}_i)_j| \leq \mathbf{1}_{i \neq j} |wzG_{ii}\mathcal{G}_{ji}^{(i,\emptyset)}| + C(\log N)^4 \quad (3.45)$$

where we have used (3.19), (3.20) and (2.20). Now we estimate $\mathcal{G}_{ji}^{(i,\emptyset)}$, with (3.6), (3.11), (3.13), we have

$$\left| wG_{ii}\mathcal{G}_{ji}^{(i,\emptyset)} \right| \leq \varphi^C \left| w^2 G_{ii}\mathcal{G}_{ii}^{(i,\emptyset)}\mathcal{G}_{jj}^{(i,i)} \right| \sqrt{\frac{\text{Im } m_G^{(i,ij)} + |z|^2 \text{Im } G_{jj}^{(i,ij)}}{N\eta}} \quad (3.46)$$

Then using $wG_{ii}\mathcal{G}_{ii}^{(i,\emptyset)} \leq (\log N)^4$ and $w\mathcal{G}_{jj}^{(i,i)}G_{jj}^{(i,ij)} \leq (\log N)^4$ (see (3.20)), we obtain it is less than $(\log N)^2$. This proves (3.38).

By the Cauchy-Schwarz inequality, we have

$$|G|_{ij}^2 \leq \sqrt{\sum_k |G_{ik}|^2} \sqrt{\sum_k |G_{jk}|^2} \leq \sqrt{\frac{\operatorname{Im} G_{ii}}{\eta}} \sqrt{\frac{\operatorname{Im} G_{jj}}{\eta}},$$

which implies (3.40). Using $|w| \geq \operatorname{Im} w = \eta$, (3.21) and (3.40), we have

$$|(YG^2Y)_{ij}| = |\mathcal{G}_{ij} + w\mathcal{G}_{ij}^2| \leq C|w|^{-1/2} + C|w|^{-1/2}\eta^{-1} \leq C|w|^{-1/2}\eta^{-1}$$

This proves (3.42). Notice that we claimed that all estimates proved for G are valid for \mathcal{G} as well. This is because that we can write YY^* in \mathcal{G} as A^*A where $A = Y^*$. Hence all estimates hold for G will hold for \mathcal{G} .

By the the Cauchy-Schwarz inequality, we have

$$|(YG^2)_{ij}| \leq \left| \sum_k (YG)_{ik} G_{kj} \right| \leq |(YGG^*Y^*)_{ii}| |(G^*G)_{ii}|.$$

Together (3.40) and (3.42), we have proved (3.41). \square

Remark 3.8. *The previous lemmas 3.6 and 3.7 also hold if one entry of X is supposed to vanish (this will be useful for us in the following of this subsection). Indeed, we just used the following facts: the independence of the entries, the subexponential decay, $\mathbb{E}X_{ab} = 0$, and $\mathbb{E}|X_{ab}|^2 = 1/N$. We notice that only this last condition is changed, and it was are only (3.6) and (3.8). Furthermore, for (3.6), this affects the case $i = a$ as*

$$G_{aa}^{(0,T)} = -w^{-1} \left[1 + m_{\mathcal{G}}^{(a,T)} - \frac{1}{N} \mathcal{G}_{bb}^{(a,T)} + |z|^2 \mathcal{G}_{aa}^{(a,T)} + \mathcal{Z}_a^{(T)} \right]^{-1}$$

The difference is therefore just $\frac{1}{N} \mathcal{G}_{bb}^{(a,T)}$ which is of order $1/N$ of $m_{\mathcal{G}}^{(a,T)}$, so the proof holds for such matrices with a vanishing entry.

The following lemma is not useful for the proof of Theorem 2.13, but it helps to understand how the Green's function comparison method works at here. Furthermore, some by-products are very useful for the whole proof of Theorem 2.13.

Lemma 3.9. *Suppose that $|w| + |z| < c$ for some fixed $c > 0$, and that η satisfies (2.20). Assume that we have two ensembles X_1, X_2 , both of them satisfying (1.5), and with matrix elements moments matching up to order 3. Then we have*

$$|\mathbb{E}_{X^{(1)}}(m(w, z)) - \mathbb{E}_{X^{(2)}}(m(w, z))| \leq \frac{\varphi^C}{N\eta}. \quad (3.47)$$

Proof. For $k \in \llbracket 0, N^2 \rrbracket$, define the following matrix X_k interpolating between $X^{(1)}$ and $X^{(2)}$:

$$X_k(i, j) = \begin{cases} X^{(1)}(i, j) & \text{if } k < N(i-1) + j \\ X^{(2)}(i, j) & \text{if } k \geq N(i-1) + j \end{cases}.$$

Note that $X^{(1)} = X_0$ and $X^{(2)} = X_{N^2}$. A sufficient condition for (3.47) is that for any $k \geq 1$

$$|\mathbb{E}_{X_k} m(w, z) - \mathbb{E}_{X_{k-1}} m(w, z)| \leq \frac{\varphi^C}{N^3\eta} \quad (3.48)$$

We are going to compare the Stieltjes transforms corresponding to X_k and X_{k-1} with a third one, corresponding to the matrix Q hereafter with deterministic k -th entry: noting $k = aN + b$ ($a \in \llbracket 0, N-1 \rrbracket$, $b \in \llbracket 1, N \rrbracket$) we define the following $N \times N$ matrices (hereafter, $Y_\ell = X_\ell - zI$):

$$v = v_{ab} \mathbf{e}_{ab} = X^{(1)}(a, b) \mathbf{e}_{ab}, \quad (3.49)$$

$$u = u_{ab} \mathbf{e}_{ab} = X^{(2)}(a, b) \mathbf{e}_{ab}, \quad (3.50)$$

$$Q = Y_{k-1} - v = Y_k - u, \quad (3.51)$$

$$R = (Q^*Q - wI)^{-1} \quad (3.52)$$

$$\mathcal{R} = (QQ^* - wI)^{-1}$$

$$S = (Y_{k-1}^* Y_{k-1} - wI)^{-1} \quad (3.53)$$

$$T = (Y_k^* Y_k - wI)^{-1} \quad (3.54)$$

Then (3.48) holds if we can prove that

$$\mathrm{Tr} R - \mathbb{E}_{v_{ab}} \mathrm{Tr} S = F + (\log N)^C \mathcal{O}(N^{-2}\eta^{-1}), \quad (3.55)$$

$$\mathrm{Tr} R - \mathbb{E}_{u_{ab}} \mathrm{Tr} T = F + (\log N)^C \mathcal{O}(N^{-2}\eta^{-1}),$$

holds with ζ -high probability where $\mathbb{E}_{v_{ab}}$ (resp. $\mathbb{E}_{u_{ab}}$) means an integration only with respect to the random variable v_{ab} (resp. u_{ab}), and F is a random variable, identical for both equations. We prove the first equation, the proof of the other one being obviously the same. For this, we want to compare S and R . Denoting

$$U = (1 + RQ^*v)^{-1}R(1 + v^*QR)^{-1},$$

we first compare $\mathrm{Tr} U$ and $\mathrm{Tr} R$, and then $\mathrm{Tr} S$ and $\mathrm{Tr} U$ (S and U are related by the simple formula (3.58) hereafter, with the notation (3.57)). For this first comparison, we introducing the notations

$$p = (RQ^*)_{ba}v_{ab}, \quad q = (QR)_{ab}v_{ba}^*,$$

appearing in the following identity obtained by expansion³:

$$\begin{aligned} \mathrm{Tr} U &= \mathrm{Tr}(1 + v^*QR)^{-1}(1 + RQ^*v)^{-1}R = \sum_{k,l \geq 0} (-1)^{k+l} \mathrm{Tr}((v^*QR)^k(RQ^*v)^l R) \\ &= \mathrm{Tr} R + \sum_{k \geq 1} (-1)^k \mathrm{Tr}((v^*QR)^k R) + \sum_{l \geq 1} (-1)^l \mathrm{Tr}((RQ^*v)^l R) + \sum_{k,l \geq 1} (-1)^{k+l} \mathrm{Tr}((v^*QR)^k(RQ^*v)^l R) \\ \mathrm{Tr} U &= \mathrm{Tr} R - \frac{v_{ba}^*(QR^2)_{ab}}{1+q} - \frac{(R^2Q^*)_{ba}v_{ab}}{1+p} + \frac{(QR^2Q^*)_{aa}|v_{ab}|^2 R_{bb}}{(1+p)(1+q)}. \end{aligned} \quad (3.56)$$

For this last equality, we extensively used that if $v_{ij} \neq 0$ then $(i, j) = (a, b)$.

To compare now $\mathrm{Tr} S$ and $\mathrm{Tr} U$, we introduce the notation

$$\Omega = v^*(QRQ^* - 1)v. \quad (3.57)$$

A routine calculation yields $X_{k-1}^* X_{k-1} - w = U^{-1} - \Omega$, hence

$$S = (1 - U\Omega)^{-1}U. \quad (3.58)$$

³All the expansions considered here converge with high probability. Anyways, they aim at proving identities between rational functions, which just need to be checked for small values of the perturbation.

As $QRQ^* - I = w\mathcal{R}$, we have $\Omega_{ij} = r\delta_{ib}\delta_{jb}$ where

$$r = |v_{ab}|^2 w \mathcal{R}_{aa}.$$

Consequently, expanding (3.58) we get

$$S_{ij} = U_{ij} + U_{ib}rU_{bj} + U_{ib}rU_{bb}rU_{bj} + U_{ib}rU_{bb}rU_{bb}rU_{bj} \dots$$

so after summation over $i = j$,

$$\text{Tr } S = \text{Tr } U + [U^2]_{bb} r(1 - U_{bb}r)^{-1} \quad (3.59)$$

An argument similar to the one leading to (3.56) yields that, for any matrix M ,

$$[(1 + RQ^*v)^{-1}M(1 + v^*QR)^{-1}]_{bb} = \frac{M_{bb}}{(1+p)(1+q)},$$

which yields, in our context,

$$U_{bb} = \frac{R_{bb}}{(1+p)(1+q)}, \quad \frac{r}{1 - U_{bb}r} = \frac{r(1+p)(1+q)}{(1+p)(1+q) - rR_{bb}}, \quad [U^2]_{bb} = \frac{(R(1 + v^*QR)^{-1}(1 + RQ^*v)^{-1}R)_{bb}}{(1+p)(1+q)}.$$

The numerator of this last expression is also, by the same reasoning leading to (3.56),

$$(R^2)_{bb} - \frac{R_{bb}v_{ba}^*(QR^2)_{ab}}{1+q} - \frac{(R^2Q^*)_{ba}v_{ab}R_{bb}}{1+p} + \frac{(QR^2Q^*)_{aa}|v_{ab}|^2R_{bb}^2}{(1+p)(1+q)}$$

Substituting the above expressions in (3.59) and combining it with (3.56), we get

$$\begin{aligned} \text{Tr } S - \text{Tr } R &= \frac{r(R^2)_{bb}}{(1+p)(1+q) - rR_{bb}} \\ &+ \frac{(1+p)(1+q)}{(1+p)(1+q) - rR_{bb}} \left(-\frac{v_{ba}^*(QR^2)_{ab}}{1+q} - \frac{(R^2Q^*)_{ba}v_{ab}}{1+p} + \frac{(QR^2Q^*)_{aa}|v_{ab}|^2R_{bb}}{(1+p)(1+q)} \right) \end{aligned} \quad (3.60)$$

In the above formula, v_{ab} only appears through p, q (in a linear way) and r (in a quadratic way). All other terms can be bounded thanks to the estimates for S from Lemmas 3.6 and 3.7, which also holds for R , as stated in Remark 3.8. More precisely, the following bounds hold with ζ -high probability, for any choice of a and b (including possibly $a = b$), and under the assumption (2.20):

$$\begin{aligned} |(RQ^*)_{ba}|, |(QR)_{ab}| &\leq (\log N)^C && \text{by (3.38),} \\ |R_{bb}|, |\mathcal{R}_{aa}| &\leq (\log N)^C |w|^{-1/2} && \text{by (3.19),} \\ |(R^2)_{bb}| &\leq (\log N)^C |w|^{-1/2} \eta^{-1} && \text{by (3.40),} \\ |(QR^2)_{ab}| &\leq (\log N)^C \eta^{-1} && \text{by (3.41),} \\ |(QR^2Q^*)_{aa}| &\leq (\log N)^C |w|^{1/2} \eta^{-1} && \text{by (3.42).} \end{aligned} \quad (3.61)$$

Therefore, if we make an expansion of (3.60) with respect to v_{ab} , we get for example for the first term of the sum in (3.60) with ζ -high probability (remember that v_{ab} satisfies the subexponential decay property)

$$\frac{r(R^2)_{bb}}{(1+p)(1+q) - rR_{bb}} = f + O(w(R^2)_{bb}\mathcal{R}_{aa}|v_{ab}|^4 \max((RQ^*)_{ba}^2, (QR)_{ab}^2, wR_{bb}\mathcal{R}_{aa})) = f + O\left(\frac{\varphi^{C\zeta}}{N^2\eta}\right)$$

where f is a polynomial of degree 3 in v_{ab} . A calculation shows that the same estimate, of type $g + O\left(\frac{\varphi^{C\zeta}}{N^2\eta}\right)$ with g of degree 3 in v_{ab} , holds when expanding the second term of the sum (3.60). This finishes the proof of (3.55). \square

Proof of Lemma 2.13. First since Theorem 1.2 holds in the Gaussian case (cf. Theorem 4.4), then with (2.23), the estimate on the smallest eigenvalues (Lemma 2.9) and the largest eigenvalue (Lemma 2.10), the estimate (2.31) holds in the case of centered and reduced complex Gaussian entries. Furthermore, using (2.33), we deduce that Lemma 2.13 holds in the Gaussian case. We will therefore use the same method as in the proof of Lemma 3.9, replacing the matrix elements one by one to extrapolate from the Ginibre ensemble to the general setting. We assume that $X^{(1)} = X_0$ is the Gaussian case and $X^{(2)} = X_{N^2}$ is the ensemble for which we want to prove Lemma 2.13. We know that $Z_{X^{(1)},c}^{(f)} \prec N^{C\varepsilon}$, and we even know that, for any fixed $p > 0$, $\mathbb{E}(|Z_{X^{(1)},c}^{(f)}|^p) \leq N^{C\varepsilon p}$ (Theorem 4.4 is proved by bounding the moments). We will prove that, for any fixed $p \in 2\mathbb{N}$ and N sufficiently large,

$$\left| \mathbb{E} \left((Z_{X_{N^2},c}^{(f)})^p \right) \right| \leq C_p N^{\tilde{C}\varepsilon p}, \quad (3.62)$$

for some $\tilde{C} > C$ independent of N and p . This is sufficient for our purpose, by the Markov inequality.

From X_k and X_{k-1} , we defined the matrices R, S, T , in (3.52), (3.53), (3.54), From equation (3.60), one can write

$$\text{Tr } S - \text{Tr } R = P_{w,z,Q}(v_{ab}) + B_{w,z,S} v_{ab}^4,$$

where $P_{w,z,Q}$ is a degree 3 polynomial whose coefficients depend only on Q . Moreover, a careful analysis of (3.60), using the estimates (3.61), proves that the coefficients of P are $O_{\prec}(\eta^{-1})$, and that B is also $O_{\prec}(\eta^{-1})$. Note that $v_{ab}^4 = O_{\prec}(N^{-2})$, hence we obtain (hereafter, $\tilde{Q} = X_{k-1} - v = X_k - u$)

$$\begin{aligned} Z_{X_{k-1},c}^{(f)} - Z_{\tilde{Q},c}^{(f)} &= \int \Delta f(\xi) \int_I \chi(\eta) \phi'(E) O_{\prec}(\eta^{-1} N^{-2}) dE d\eta d\xi d\bar{\xi} \\ &\quad + \int \Delta f(\xi) \int_I \chi(\eta) \phi'(E) \text{Re}(P_{w,z,Q}(v_{ab}) - P_{w,z,Q}(0)) dE d\eta d\xi d\bar{\xi} \end{aligned}$$

Noting that $|\phi'(E)| \leq (1 + \log E) E^{-1} \mathbf{1}_{4\lambda_+ > E > N^{\varepsilon-2}}$, the first term in the above sum is

$$O_{\prec} \left(N^{-2} \int |\Delta f(\xi)| \int_{I \cap \{4\lambda_+ > E > N^{\varepsilon-2}\}} \eta^{-1} E^{-1} dE d\eta d\xi d\bar{\xi} \right) = O_{\prec} (N^{-2} |\Delta f|_{L^1}) = O_{\prec} (N^{-2}),$$

where we omit the dependence in f in the previous and next estimates, as f does not depend on N . Concerning the second term, it is of type $\text{Re } \mathcal{P}_{f,Q}(v_{ab})$, where \mathcal{P} has degree 3, vanishes at 0, with coefficients of order $O_{\prec}(1)$ being independent of v_{ab} and u_{ab} . We therefore have

$$\begin{aligned} Z_{X_{k-1},c}^{(f)} &= Z_{\tilde{Q},c}^{(f)} + \Delta_{k-1}, \quad \Delta_{k-1} = \mathcal{P}_{f,Q}(v_{ab}) + O_{\prec}(N^{-2}), \\ Z_{X_k,c}^{(f)} &= Z_{\tilde{Q},c}^{(f)} + \Delta_k, \quad \Delta_k = \mathcal{P}_{f,Q}(u_{ab}) + O_{\prec}(N^{-2}) \end{aligned} \quad (3.63)$$

We can decompose

$$(Z_{X_{k-1},c}^{(f)})^p - (Z_{X_k,c}^{(f)})^p = \sum_{j=0}^{p-1} \binom{p}{j} (Z_{\tilde{Q},c}^{(f)})^j (\Delta_{k-1}^{p-j} - \Delta_k^{p-j}).$$

Since \mathcal{P} has no constant term and the first three moments of v_{ab} and u_{ab} coincide, we get

$$\left| \mathbb{E} \left((Z_{X_{k-1},c}^{(f)})^p \right) - \mathbb{E} \left((Z_{X_k,c}^{(f)})^p \right) \right| \leq \sum_{j=0}^{p-1} \mathbb{E} \left| (Z_{\tilde{Q},c}^{(f)})^j \right| O_{\prec}(N^{-2}) \leq N^{-2} \left(O_{\prec}(1) + \mathbb{E} \left((Z_{\tilde{Q},c}^{(f)})^p \right) \right), \quad (3.64)$$

where we used that p is even, $\mathbb{E} \left| (Z_{\tilde{Q},c}^{(f)})^j \right| \leq \mathbb{E} \left| (Z_{\tilde{Q},c}^{(f)})^p \right|^{j/p}$ and

$$\mathbb{E} \left| (Z_{\tilde{Q},c}^{(f)})^p \right|^{j/p} N^\delta \leq \mathbb{E} \left| (Z_{\tilde{Q},c}^{(f)})^p \right| + N^{\delta \frac{p}{p-j}}, \quad (3.65)$$

by Young's inequality (remember that $j \in \llbracket 0, p-1 \rrbracket$) Moreover, from (3.63),

$$\begin{aligned} \mathbb{E} \left((Z_{\tilde{Q},c}^{(f)})^p \right) &\leq \mathbb{E} \left((Z_{X_{k-1},c}^{(f)})^p \right) + \sum_{j=1}^p \binom{j}{p} \mathbb{E} \left(\left| Z_{X_{k-1},c}^{(f)} \right|^{p-j} |\Delta_{k-1}|^j \right) \\ &\leq \mathbb{E} \left((Z_{X_{k-1},c}^{(f)})^p \right) + \sum_{j=1}^p \binom{j}{p} \mathbb{E} \left(\left| Z_{X_{k-1},c}^{(f)} \right|^p \right)^{\frac{p-j}{p}} \mathbb{E} (|\Delta_{k-1}|^p)^{\frac{j}{p}}, \\ \mathbb{E} \left((Z_{\tilde{Q},c}^{(f)})^p \right) &\leq \left(O_{\prec}(1) + \mathbb{E} \left((Z_{X_{k-1},c}^{(f)})^p \right) \right), \end{aligned}$$

where we used Hölder's inequality and the trivial bound $\Delta_{k-1} = O_{\prec}(1)$ in the last equation. Using the last bound and (3.64), we obtain for any $\delta > 0$

$$\left| \mathbb{E} \left((Z_{X_{k-1},c}^{(f)})^p \right) - \mathbb{E} \left((Z_{X_{k,c}}^{(f)})^p \right) \right| \leq N^{-2} \left(N^\delta + \mathbb{E} \left((Z_{X_{k-1},c}^{(f)})^p \right) \right).$$

It implies

$$\mathbb{E} \left((Z_{X_{k,c}}^{(f)})^p \right) + N^\delta \leq (1 + N^{-2}) \left(\mathbb{E} \left((Z_{X_{k-1},c}^{(f)})^p \right) + N^\delta \right) \quad (3.66)$$

As $\mathbb{E} \left((Z_{X_0,c}^{(f)})^p \right) \leq N^{C\varepsilon p}$ and we obtain (3.62) by iterating (3.66). It completes the proof of Lemma 2.13. \square

Proof of Lemma 2.14. Notice that in the previous proof, the third moment condition was first used in (3.64). So all equations up to and including (3.63) are still valid. Introducing the notation $Y = Z/\sqrt{N}$, we have, instead of (3.64), the following bound

$$\left| \mathbb{E} \left((Y_{X_{k-1},c}^{(f)})^p \right) - \mathbb{E} \left((Y_{X_{k,c}}^{(f)})^p \right) \right| \leq \sum_{j=0}^{p-1} \mathbb{E} \left| (Y_{\tilde{Q},c}^{(f)})^j \right| O_{\prec}(N^{-2}) \leq N^{-2} \left(O_{\prec}(1) + \mathbb{E} \left((Y_{\tilde{Q},c}^{(f)})^p \right) \right). \quad (3.67)$$

Following the rest of the argument in the proof of Lemma 2.13, we have proved that

$$Y_{X,c}^{(f)} \prec N^{C\varepsilon} c_f, \quad (3.68)$$

and this completes the proof of Lemma 2.14. \square

4 THE LOCAL CIRCULAR LAW FOR THE GINIBRE ENSEMBLE

In this section, we derive the local circular law on the edge for the Ginibre ensemble (1.1). This is required in the proof of Lemma 2.13, which proceeds by comparison with the Gaussian case. Along the proof, we will need the following partition of \mathbb{C} , with distinct asymptotics of the correlation function K_N for each domain (in the following we will always take $\varepsilon_N = \frac{\log N}{\sqrt{N}}$):

$$\begin{aligned} \Omega_1 &= \{|z| < 1 - \varepsilon_N\}, \text{ the bulk,} \\ \Omega_2 &= \{|z| > 1 + \varepsilon_N\}, \\ \Omega_3 &= \mathbb{C} - (\Omega_1 \cup \Omega_2), \text{ the edge.} \end{aligned}$$

We first consider the case when the test function is supported in Ω_1 (this step is required even for the circular law on the edge, as the support of the test functions overlaps the bulk), then when it can overlap Ω_3 . The first case is directly adapted from [1], the second one requires some more work.

In the following, we note $e_N(z) = \sum_{\ell=0}^N \frac{z^\ell}{\ell!}$ for the partial sums of the exponential function.

4.1 The bulk case. We prove the following local circular law for the Ginibre ensemble, with some more precision: the local convergence towards the Gaussian free field holds in the bulk, generalizing the global convergence result obtained in [23].

Theorem 4.1. *Let $0 < a < 1/2$, f be a smooth non-negative function with compact support, and z_0 be such that f_{z_0} is supported in Ω_1 . Let μ_1, \dots, μ_N be distributed as the eigenvalues of a Ginibre random matrix (1.1). Then the local circular law (1.6) holds.*

More generally, if f additionally may depend on N , such that $\|f\|_\infty \leq C$, $\|f'\|_\infty \leq N^C$, the local circular law holds when at distance at least ε_N from the unit circle in the following sense. Define $f^{(N)}(z) = f(N^a(z - z_0))$, for z_0 (depending on N) such that $f^{(N)}$ is supported on Ω_1 . Denote $\text{cum}_N(\ell)$ the ℓ -th cumulant of $X_f^{(N)} = \sum f^{(N)}(\mu_j) - N^{1-2a} \frac{1}{\pi} \int_{\mathbb{C}} f(z) dz d\bar{z}$, and denote⁴ $\delta_N = N^{3a - \frac{1}{2}} \text{Vol}(\text{supp}(\nabla f^{(N)})) \|f\|_\infty$. Then as $N \rightarrow \infty$, for $\ell \geq 1$,

$$\text{cum}_N(\ell) = \begin{cases} \frac{1}{4\pi} \int |\nabla f(z)|^2 dA(z) + O(\delta_N) & \text{if } \ell = 2, \\ O(\delta_N) & \text{if } \ell \neq 2. \end{cases} \quad (4.1)$$

In particular, noting $\sigma^2 = \frac{1}{4\pi} \int |\nabla f(z)|^2 dA(z)$, if $\delta_N \ll \sigma^2$, the linear statistics $\frac{1}{\sigma} X_f^{(N)}$ converge in law to a reduced centered Gaussian random variable.

As a first step in the proof, we need the following elementary estimate.

Lemma 4.2. *Let*

$$k_N(z_1, z_2) = \frac{N}{\pi} e^{-\frac{N}{2}(|z_1|^2 + |z_2|^2 - 2z_1\bar{z}_2)}. \quad (4.2)$$

There is some $c > 0$ such that uniformly in $|z_1\bar{z}_2| < 1 - \varepsilon_N$ we have

$$K_N(z_1, z_2) = k_N(z_1, z_2) + O(e^{-c(\log N)^2}),$$

where K_N is the Ginibre kernel (1.2).

Proof. This is elementary from the following calculation, for any $|z_1 z_2| < 1$:

$$\begin{aligned} K_N(z_1, z_2) &= \frac{N}{\pi} e^{-\frac{N}{2}(|z_1|^2 + |z_2|^2)} \left(e^{Nz_1\bar{z}_2} - \sum_{\ell \geq N} \frac{(Nz_1\bar{z}_2)^\ell}{\ell!} \right) \\ &= k_N(z_1, z_2) + O \left(N e^{-\frac{N}{2}(|z_1|^2 + |z_2|^2)} \frac{|Nz_1\bar{z}_2|^N}{N!} \sum_{m \geq 0} |z_1, \bar{z}_2|^m \right) \\ &= k_N(z_1, z_2) + O \left(N e^{-N((|z_1|^2 + |z_2|^2)/2 - 1 - \log |z_1 z_2|)} \frac{1}{1 - |z_1 z_2|} \right) \\ K_N(z_1, z_2) &= k_N(z_1, z_2) + O \left(N e^{-\frac{N}{2}(1 - |z_1 z_2|)^2} \frac{1}{1 - |z_1 z_2|} \right) \end{aligned} \quad (4.3)$$

⁴ $\|f\|_\infty := \max(\|f'\|_\infty^3, \|f'\|_\infty \|f''\|_\infty, \|f^{(3)}\|_\infty)$.

where we used Stirling's formula and the fact that $(|z_1|^2 + |z_2|^2)/2 - 1 - \log |z_1 z_2| > |z_1 z_2| - 1 - \log |z_1 z_2| > \frac{1}{2}(1 - |z_1 z_2|)^2$. \square

The next following estimate about K_N will be used to bound it for any distant z_1 and z_2 .

Lemma 4.3. *There is a constant $c > 0$ such that for any $N \in \mathbb{N}^*$ and any z_1, z_2 in \mathbb{C} ,*

$$|K_N(z_1, z_2)| \leq cN \left(e^{-N \frac{|z_1 - z_2|^2}{2}} + \frac{|z_1 \bar{z}_2|^{N+1}}{1 + \sqrt{N}|1 - z_1 \bar{z}_2|} e^{-N \left(\frac{|z_1|^2 + |z_2|^2}{2} - 1 \right)} \right).$$

Proof. In the case $|z_1 \bar{z}_2 - 1| < 1/\sqrt{N}$, by case (i) in Lemma 4.9, we get

$$K_N(z_1, z_2) = \frac{N}{\pi} e^{-\frac{|z_1|^2 + |z_2|^2}{2}} e^{N z_1 \bar{z}_2} \left(\frac{1}{2} \operatorname{erfc}(\sqrt{N} \mu(z_1 \bar{z}_2)) + O(N^{-1/2}) \right).$$

As erfc is uniformly bounded in \mathbb{C} , we get $|K_N(z_1, z_2)| = O(e^{-N \frac{|z_1 - z_2|^2}{2}})$.

In the case $\frac{1}{\sqrt{N}} \leq |z_1 \bar{z}_2 - 1| < \delta$, for δ fixed and small enough, the estimate was obtained in [2] (cf. the proof of Lemma 8.10 there, based on Lemma 4.9 here). Finally, in the case $|z_1 \bar{z}_2 - 1| \geq \delta$, an elementary calculation from Lemma 4.11 implies that the result holds, no matter that $z_1 \bar{z}_2$ is in $D - U$ or $D^c - U$. \square

Proof of Theorem 4.1. Proving equation (4.1) is sufficient: this convergence of the cumulants is well-known to imply the weak convergence to the Gaussian distribution, and it also yields the local circular law (1.6): if $m_N(\ell)$ denotes the ℓ -th moment of $X_f^{(N)}$, one can write

$$m_N(\ell) = \sum_{i_1 + \dots + i_\ell = \ell} \lambda_{i_1, \dots, i_\ell} \operatorname{cum}_N(i_1) \dots \operatorname{cum}_N(i_\ell)$$

for some universal constants $\lambda_{i_1, \dots, i_\ell}$'s, where the i_j 's are in \mathbb{N}^* . Hence writing $\Phi = (\sigma^2 + \delta_N)^{1/2}$ we get $m_N(\ell) = O(\Phi^\ell)$. Consequently, for any $\varepsilon > 0$,

$$\mathbb{P}(|X_f^{(N)}| > N^\varepsilon \Phi) \leq m_N(\ell) N^{-\ell\varepsilon} \Phi^{-\ell} = O(N^{-\ell\varepsilon}),$$

which concludes the proof of the local circular law by choosing ℓ large enough.

To prove (4.1), first note that from 4.2 evaluated on the diagonal $z_1 = z_2$, $\operatorname{cum}_N(1) = O(e^{-c(\log N)^2})$, so we consider now cumulants of order $\ell \geq 2$. Due to a peculiar integrability structure of determinantal point processes, the cumulants have an easy form for the Gaussian matrix ensembles, as observed first by Costin and Lebowitz [10], and used later by Soshnikov [27]. For our purpose, in the context of the Ginibre ensemble, we will use the following very useful expression (4.5) due to Ameur, Hedenmalm and Makarov [1], which requires first the following notations. The usual differential operators are noted $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$, $\Delta_\ell = \partial_1 \bar{\partial}_1 + \dots + \partial_\ell \bar{\partial}_\ell$, we will also make use of the length- ℓ vectors $z \mathbf{1}_\ell = (z, \dots, z)$, $h = (h_1, \dots, h_\ell)$, and the following function often appearing in the combinatorics of cumulants for determinantal point processes:

$$F_\ell(z_1, \dots, z_\ell) = \sum_{j=1}^{\ell} \frac{(-1)^{j-1}}{j} \sum_{k_1 + \dots + k_j = \ell, k_1, \dots, k_j \geq 1} \frac{\ell!}{k_1! \dots k_j!} \prod_{m=1}^j f^{(N)}(z_m)^{k_m}.$$

Then the ℓ -th cumulant of $X_f^{(N)}$ is then

$$\operatorname{cum}_N(\ell) = \int_{\mathbb{C}^\ell} F_\ell(z_1, \dots, z_\ell) K_N(z_1, z_2) K_N(z_2, z_3) \dots K_N(z_\ell, z_1) dA(z_1, \dots, z_\ell), \quad (4.4)$$

Remarkably, some combinatorics can prove that $F_\ell(z\mathbb{1}_\ell) = 0$: it vanishes on the diagonal. Moreover, the following approximations of F_ℓ close to the diagonal will be useful:

$$F_\ell(z\mathbb{1}_\ell + h) = \sum_{j \geq 1} T_j(z, h), \text{ where } T_j(z, h) = \sum_{|\alpha+\beta|=j} (\partial^\alpha \bar{\partial}^\beta F_\ell)(z\mathbb{1}_\ell) \frac{h^\alpha \bar{h}^\beta}{\alpha! \beta!}.$$

We will also need the third error term after second order approximation, $r(\lambda, h) = F_\ell(z\mathbb{1}_\ell + h) - T_1(z, h) - T_2(z, h)$. The last notation we will need from [1] is $Z_\ell(z) = \sum_{i < j} (\partial_i \bar{\partial}_j F_\ell)(z\mathbb{1}_\ell)$, and the area measure in \mathbb{C}^ℓ will be noted $dA(z, h_1, \dots, h_\ell) = dA(z) dA(h_1) \dots dA(h_\ell)$, where $dA(z) = d^2z$. Then, as proved in [1], for $\ell \geq 2$

$$\text{cum}_N(\ell) = A_N(\ell) + B_N(\ell) + C_N(\ell) + D_N(\ell) + E_N(\ell) \quad (4.5)$$

where

$$\begin{aligned} A_N(\ell) &= \int_{\mathbb{C}^3} \Re \left(\sum_{i \neq j} (\partial_i \bar{\partial}_j F_\ell)(z\mathbb{1}_\ell) h_1 h_2 \right) K_N(z, z + h_1) K_N(z + h_1, z + h_2) K_N(z + h_2, z) dA(z, h_1, h_2) \\ B_N(\ell) &= \Re \int_{\mathbb{C}^2} \sum_i (\partial_i^2 F_\ell)(z\mathbb{1}_\ell) h_1^2 K_N(z, z + h_1) K_N(z + h_1, z) dA(z, h_1) \\ C_N(\ell) &= 2 \int_{\mathbb{C}^3} \Re (Z_\ell(z) h_1 \bar{h}_2) K_N(z, z + h_1) K_N(z + h_1, z + h_2) K_N(z + h_2, z) dA(z, h_1, h_2) \\ D_N(\ell) &= \int_{\mathbb{C}^2} (\Delta_\ell F_\ell)(z\mathbb{1}_\ell) |h_1|^2 K_N(z, z + h_1) K_N(z + h_1, z) dA(z, h_1) \\ E_N(\ell) &= \int_{\mathbb{C}^{\ell+1}} r(z, h) K_N(z, z + h_1) K_N(z + h_1, z + h_2) \dots K_N(z + h_\ell, z) dA(z, h_1, \dots, h_\ell). \end{aligned}$$

Remember that the support of $f^{(N)}$ is at distance ε_N from the unit circle, so each one of the above integrands vanishes if z is out of the disk with radius $1 - \varepsilon_N$. Therefore, by using Lemma 4.3 to restrict the domain, and then Lemma 4.2 to approximate the kernel strictly inside the unit disk, one easily gets that

$$\text{cum}_N(\ell) = \tilde{A}_N(\ell) + \tilde{B}_N(\ell) + \tilde{C}_N(\ell) + \tilde{D}_N(\ell) + \tilde{E}_N(\ell) + O(e^{-c(\log N)^2}),$$

for some $c > 0$, where

$$\begin{aligned} \tilde{A}_N(\ell) &= \int_{\mathbb{C}^3 \cap \{\|h\|_\infty < \varepsilon_N\}} \Re \left(\sum_{i \neq j} (\partial_i \bar{\partial}_j F_\ell)(z\mathbb{1}_\ell) h_1 h_2 \right) k_N(z, z + h_1) k_N(z + h_1, z + h_2) k_N(z + h_2, z) dA(z, h_1, h_2) \\ \tilde{B}_N(\ell) &= \Re \int_{\mathbb{C}^2 \cap \{\|h\|_\infty < \varepsilon_N\}} \sum_i (\partial_i^2 F_\ell)(z\mathbb{1}_\ell) h_1^2 k_N(z, z + h_1) k_N(z + h_1, z) dA(z, h_1) \\ \tilde{C}_N(\ell) &= 2 \int_{\mathbb{C}^3 \cap \{\|h\|_\infty < \varepsilon_N\}} \Re (Z_\ell(z) h_1 \bar{h}_2) k_N(z, z + h_1) k_N(z + h_1, z + h_2) k_N(z + h_2, z) dA(z, h_1, h_2) \\ \tilde{D}_N(\ell) &= \int_{\mathbb{C}^2 \cap \{\|h\|_\infty < \varepsilon_N\}} (\Delta_\ell F_\ell)(z\mathbb{1}_\ell) |h_1|^2 k_N(z, z + h_1) k_N(z + h_1, z) dA(z, h_1) \\ \tilde{E}_N(\ell) &= \int_{\mathbb{C}^{\ell+1} \cap \{\|h\|_\infty < \varepsilon_N\}} r(z, h) k_N(z, z + h_1) k_N(z + h_1, z + h_2) \dots k_N(z + h_\ell, z) dA(z, h_1, \dots, h_\ell). \end{aligned}$$

Following closely the proof in [1], integrating the h_i 's for fixed z , the terms $\tilde{A}_N(\ell), \tilde{B}_N(\ell), \tilde{C}_N(\ell)$ are $O(e^{-c(\log N)^2})$. Moreover, [1] proved that, for $\ell \geq 3$, $\Delta_\ell F_\ell$ vanishes on the diagonal, and that $\Delta_2 F_2(z_1, z_2) |_{z_1=z_2=z} = \frac{1}{2} |\nabla f^{(N)}(z)|^2$. Consequently, $\tilde{D}_N(\ell) = 0$ if $\ell \geq 3$ and

$$\begin{aligned} \tilde{D}_N(2) &= \frac{1}{2} \int_{\mathbb{C}^2} |\nabla f^{(N)}(z)|^2 |h|^2 \frac{N^2}{\pi^2} e^{-N|h|^2} dA(z, h) + O(e^{-c(\log N)^2}) \\ &= \frac{1}{4\pi} \int_{\mathbb{C}} |\nabla f^{(N)}(z)|^2 dA(z) + O(e^{-c(\log N)^2}) = \frac{1}{4\pi} \|\nabla f\|_2^2 + O(e^{-c(\log N)^2}). \end{aligned}$$

To finish the proof, we need to bound the error term $\tilde{E}_N(\ell)$. We divide the set of possible (z, h) in two parts. Following [1], define $Y_{n,\ell} = \{(z, h) \in \mathbb{C}^{\ell+1} : z \in \text{supp}(\nabla f^{(N)})\}$. On the complement of $Y_{n,k}$, $z \notin \text{supp}(\nabla f^{(N)})$, and then $r(z, h) = 0$. On $Y_{n,\ell}$, $r(z, h)$ is of order $O(\|F_\ell^{(3)}\|_\infty \|h\|_\infty^3) = O(N^{3a} \|f\|_\infty \varepsilon_N^3)$, each k_N is $O(N)$, and the domain of integration has size $O(\text{Vol}(\text{supp}(\nabla f^{(N)})) \varepsilon_N^{2\ell})$, so

$$\tilde{E}_N(\ell) = O\left((\log N)^C N^{3a - \frac{1}{2}} \|f\|_\infty \text{Vol}(\text{supp}(\nabla f^{(N)}))\right),$$

concluding the proof. \square

4.2 The edge case. In the edge case, i.e. $|z_0| \in \Omega_3$, we have the following theorem for the Ginibre ensemble.

Theorem 4.4. *Let μ_1, \dots, μ_N be distributed as the eigenvalues of a Ginibre random matrix (1.1), and $z_0 \in \Omega_3$. Suppose that f is smooth and compactly supported, and let $f_{z_0}(z) = N^{2a} f(N^a(z - z_0))$. Then for any $0 < a < 1/2$, the estimate (1.7) holds.*

We begin with the proper bound on the first cumulant, noting as previously $f^{(N)}(z) = f(N^a(z - z_0))$, and $X_f^{(N)} = \sum f(N^a(\mu_j - z_0)) - \frac{N}{\pi} \int_D f^{(N)}(z) dA(z)$.

Lemma 4.5. *With the previous notations, for any $\varepsilon > 0$, $\mathbb{E}(X_f^{(N)}) = O(N^{-\frac{1}{2} + \varepsilon})$ as $N \rightarrow \infty$.*

Proof. From the definition of the 1-point correlation function,

$$\mathbb{E}(X_f^{(N)}) = N \int_{\mathbb{C}} f^{(N)}(z) \left(\frac{1}{N} K_N(z, \bar{z}) - \frac{\chi_D(z)}{\pi} \right) dz d\bar{z}.$$

From Lemma 4.2, as f is bounded,

$$\begin{aligned} N \int_{\Omega_1} f^{(N)}(z) \left(\frac{1}{N} K_N(z, \bar{z}) - \frac{1}{\pi} \right) dz d\bar{z} \\ = N \int_{\Omega_1} f^{(N)}(z) \left(\frac{1}{N} k_N(z, \bar{z}) - \frac{1}{\pi} \right) dz d\bar{z} + O\left(e^{-c(\log N)^2}\right) = O\left(e^{-c(\log N)^2}\right) \end{aligned}$$

where k_N is defined in (4.2). On Ω_2 , by Lemma 4.3, $K_N(z, \bar{z}) = O(e^{-c(\log N)^2})$. Consequently,

$$\int_{\Omega_2} f^{(N)}(z) K_N(z, \bar{z}) dz d\bar{z} = O\left(e^{-c(\log N)^2}\right).$$

Without loss of generality, we consider the case $|z_0| = 1$ for simplicity of notations. On Ω_3 , we now use Lemma 4.10 which yields that, uniformly on $-\varepsilon_N < \varepsilon < \varepsilon_N$,

$$K_N(1 + \varepsilon, 1 + \varepsilon) = \frac{N}{\pi} \left(\mathbb{1}_{\varepsilon < 0} + \frac{1}{\sqrt{2}} \frac{\mu(t)t}{t-1} \text{erfc}(\sqrt{N-1}\mu(t)) \left(1 + O\left(\frac{1}{\sqrt{N}}\right) \right) \right), \quad (4.6)$$

where $t = \frac{N}{N-1}(1+\varepsilon)^2$. Take any smooth function of type $g(1+\varepsilon) = G(N^a\varepsilon)$. By polar integration, we just need to prove that the following quantity is $O(N^{a-\frac{1}{2}+\varepsilon})$:

$$\begin{aligned} & \int_{-\varepsilon_N}^{\varepsilon_N} (1+\varepsilon)g(1+\varepsilon) \left(K_N(1+\varepsilon, 1+\varepsilon) - \frac{N}{2\pi} \right) d\varepsilon \\ &= \int_{-\varepsilon_N}^{\varepsilon_N} (1+\varepsilon)(g(1+\varepsilon) - g(1)) \left(K_N(1+\varepsilon, 1+\varepsilon) - \frac{N}{2\pi} \right) d\varepsilon + O\left(e^{-c(\log N)^2}\right), \end{aligned} \quad (4.7)$$

where in the last equality we used the simple yet useful facts that $\int_{\mathbb{C}} K_N(z, \bar{z}) dA(z) = 1$ and $|K_N(z, \bar{z}) - \frac{N}{\pi} \mathbb{1}_{|z|<1}| = O(e^{-c(\log N)^2})$ uniformly out of Ω_3 . We now can write $g(1+\varepsilon) - g(1) = N^a G'(0)\varepsilon + O(N^{2a}\varepsilon^2)$, and use (4.6) to approximate $K_N(1+\varepsilon, 1+\varepsilon) - \frac{N}{2\pi}$. Using the fact that $\mu(z) = \frac{|z-1|}{\sqrt{2}} + O((z-1)^2)$ for z close to 1, still denoting $t = \frac{N}{N-1}(1+\varepsilon)^2$, we have $\mu(t) = \sqrt{2}\varepsilon + O(\varepsilon^2 + N^{-1})$, so the integral term in (4.7) can be written

$$\begin{aligned} & \frac{N}{\pi} \int_{-\varepsilon_N}^{\varepsilon_N} (1+O(\varepsilon))(N^a G'(0)\varepsilon + O(N^{2a}\varepsilon^2)) \left(\frac{1}{2} + O(\varepsilon + N^{-1}) \right) \\ & \quad \operatorname{erfc}(\sqrt{2N}|\varepsilon| + O(N^{-1/2} + N^{1/2}\varepsilon^2)) \left(1 + O(N^{-1/2}) \right) d\varepsilon \end{aligned}$$

One can easily check by bounding with absolute values that all terms involving O 's, except the one in the erfc function, contribute to $O(N^{a-\frac{1}{2}+\varepsilon})$, so we just need to prove that

$$N \int_{-\varepsilon_N}^{\varepsilon_N} \varepsilon \operatorname{erfc}(\sqrt{2N}|\varepsilon| + O(N^{-1/2} + N^{1/2}\varepsilon^2)) d\varepsilon = O(N^{-\frac{1}{2}}).$$

As $\operatorname{erfc}' = O(1)$ uniformly, this is the same as proving $N \int_{-\varepsilon_N}^{\varepsilon_N} \varepsilon \operatorname{erfc}(\sqrt{2N}|\varepsilon|) d\varepsilon = O(N^{-1/2+\varepsilon})$, which is obvious as it vanishes by parity. \square

We now prove the convergence of the second moment.

Lemma 4.6. *Under the hypothesis of Theorem 4.4, as $N \rightarrow \infty$,*

$$\operatorname{cum}_N(2) = \frac{1}{4\pi} \int |\nabla f(z)|^2 \mathbb{1}_{z \in A} dA(z) + \frac{1}{2} \|f_T\|_{H^{1/2}}^2 + o(1)$$

where we use the notation $A = \{z \mid \Re(z\bar{z}_0) < 0\}$, $f_T(z) = f(iz_0 z)$ and $\|g\|_{H^{1/2}}^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left(\frac{g(x)-g(y)}{x-y} \right)^2 dx dy$.

Proof. Note that by polarization of formula (4.4),

$$\begin{aligned} \operatorname{cum}_N(2) &= \frac{1}{2} \int_{\mathbb{C}^2} (f^{(N)}(z_1) - f^{(N)}(z_2))^2 |K_N(z_1, z_2)|^2 dA(z_1, z_2) \\ &= \frac{1}{2} \int_{\Omega_1^2 \cup \Omega_3^2} (f^{(N)}(z_1) - f^{(N)}(z_2))^2 |K_N(z_1, z_2)|^2 dA(z_1, z_2) + O\left(e^{-c(\log N)^2}\right), \end{aligned}$$

the contributions of the domains $\mathbb{C} \times \Omega_2$ and $\Omega_1 \times \Omega_3$ being easily bounded by Lemma 4.3. For the Ω_1^2 term, one can easily reproduce the method employed in the bulk, in the previous subsection, approximating K_N by k_N thanks to Lemma 4.2, to obtain that the contribution of the above integral on the domain Ω_1^2 is $\frac{1}{4\pi} \|\chi_A \nabla f\|_2^2 + o(1)$.

The most tricky part consists in evaluating the Ω_3^2 term. To calculate it, we will need the following notations, where $\theta_1 = \arg(z_1)$, $\theta_2 = \arg(z_2)$:

$$\begin{aligned}
\text{(i)} &= \frac{1}{2} \int_{\Omega_3^2} (f^{(N)}(z_1) - f^{(N)}(z_2))^2 |K_N(z_1, z_2)|^2 dA(z_1, z_2), \\
\text{(ii)} &= \frac{1}{2} \int_{\Omega} (f^{(N)}(z_1) - f^{(N)}(z_2))^2 |K_N(z_1, z_2)|^2 dA(z_1, z_2), \text{ where } \Omega = (\{(z_1, z_2) \in \Omega_3^2 : |\theta_1 - \theta_2| > N^{-\frac{1}{2}+\varepsilon}\}), \\
&\text{where } \varepsilon > 0 \text{ is fixed and will be chosen small enough,} \\
\text{(iii)} &= \frac{1}{2} \int_{\Omega} (f^{(N)}(e^{i\theta_1}) - f^{(N)}(e^{i\theta_2}))^2 |K_N(z_1, z_2)|^2 dA(z_1, z_2), \\
\text{(iv)} &= \frac{1}{2} \frac{N}{2\pi^3} \int_{\Omega} \frac{(f^{(N)}(e^{i\theta_1}) - f^{(N)}(e^{i\theta_2}))^2}{|e^{i\theta_1} - e^{i\theta_2}|^2} e^{-N(|z_1|^2 + |z_2|^2)} e^{2N|z_1 z_2|^{2N}} dA(z_1, z_2), \\
\text{(v)} &= \frac{1}{8\pi^2} \int_{(-\pi, \pi)^2} \frac{(f^{(N)}(e^{i\theta_1}) - f^{(N)}(e^{i\theta_2}))^2}{|e^{i\theta_1} - e^{i\theta_2}|^2} d\theta_1 d\theta_2, \\
\text{(vi)} &= \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \frac{(f_{\mathbb{T}}(r_1) - f_{\mathbb{T}}(r_2))^2}{(r_1 - r_2)^2} dr_1 dr_2.
\end{aligned}$$

We will prove successively that the six above terms differ by $o(1)$ a $N \rightarrow \infty$. This will conclude the proof as the last one is exactly $\frac{1}{2} \|f_{\mathbb{T}}\|_{\mathbb{H}^{1/2}}^2$. First, the difference between (i) because (ii) is $o(1)$: it can be bounded by

$$\varepsilon_N N^a \|f\|_{\infty}^2 \sup_{z_1 \in \Omega_3} \int_{|z_2 - z_1| < N^{-\frac{1}{2}+\varepsilon}} |z_1 - z_2|^2 |K_N(z_1, z_2)|^2 dA(z_2),$$

where $N^{-a} \varepsilon_N$ corresponds to the position of z_1 such that $|\theta_1 - \theta_2| < N^{-\frac{1}{2}+\varepsilon}$ and $|f^{(N)}(z_1) - f^{(N)}(z_2)| \neq 0$, and $N^{2a} \|f'\|_{\infty}^2$ comes from the Lipschitz property for $f^{(N)}$. Moreover, lemmas 4.9 and 4.11 easily yield

$$|K_N(z_1, z_2)| = O\left(\frac{N}{1 + \sqrt{N}|z_1 - z_2|}\right), \quad (4.8)$$

uniformly in \mathbb{C}^2 . Hence the difference between (i) and (ii) is bounded by (we choose $\varepsilon < \frac{1}{2}(\frac{1}{2} - a)$)

$$\varepsilon_N N^a \|f\|_{\infty}^2 \int_{0 < r < N^{-\frac{1}{2}+\varepsilon}} r^3 \left(\frac{N}{1 + \sqrt{N}r}\right)^2 dr = O(\varepsilon_N N^a N^{2\varepsilon} \|f\|_{\infty}^2) = o(1).$$

Moreover, noting $A = \Omega \cap (\{f^{(N)}(z_1) \neq 0\} \cup \{f^{(N)}(z_2) \neq 0\})$ and doing anan order 1 approximation around $e^{i\theta_1}$ and $e^{i\theta_2}$, (ii)-(iii) is of order at most

$$\begin{aligned}
\varepsilon_N \|f^{(N)'}\|_{\infty} \|f^{(N)}\|_{\infty} \int_A |K_N(z_1, z_2)|^2 dA(z_1, z_2) &\leq \varepsilon_N N^a \int_A \frac{N^2}{(1 + \sqrt{N}|e^{i\theta_1} - e^{i\theta_2}|)^2} dA(z_1, z_2) \\
&\leq \varepsilon_N^3 N^2 \int_{r > N^{-1/2+\varepsilon}} \frac{dr}{(1 + \sqrt{N}r)^2} \leq \varepsilon_N^3 N^{-\frac{3}{2}-\varepsilon},
\end{aligned}$$

where the derivatives are radial ones, so (ii)-(iii) = $o(1)$.

For the difference between (iii) and (iv), we want to get a good approximation for $K_N(z_1, z_2)$. Note that, as $|\theta_1 - \theta_2| \gg N^{-1/2+\varepsilon} \gg 2\varepsilon_N$, one easily has that on Ω , $|\arg(z_1 \bar{z}_2 - 1)| \sim \frac{\pi}{2}$ uniformly, in particular the formula (ii) in Lemma 4.9 provides a good approximation for $e_{N-1}(N z_1 \bar{z}_2)$: together with the asymptotics $\operatorname{erfc}(z) \underset{|z| \rightarrow \infty}{\sim} \frac{e^{-z^2}}{z\sqrt{\pi}}$, for $|\arg z| < \frac{3\pi}{4}$, and noting that $\mu'(z) = \frac{z_1 \bar{z}_2 - 1}{2\mu(z) z_1 \bar{z}_2}$, we get

$$e_{N-1}(N z_1 \bar{z}_2) \underset{N \rightarrow \infty}{\sim} \frac{1}{\sqrt{2\pi N}(z_1 \bar{z}_2 - 1)} e^N (z_1 \bar{z}_2)^N,$$

so in particular

$$|K_N(z_1, z_2)|^2 \underset{N \rightarrow \infty}{\sim} \frac{N}{\pi^2} \frac{1}{2\pi|z_1\bar{z}_2 - 1|^2 - 2} e^{-N(|z_1|^2 + |z_2|^2)} |z_1 z_2|^{2N} \underset{N \rightarrow \infty}{\sim} \frac{N}{\pi^2} \frac{1}{2\pi|e^{i\theta_1} - e^{i\theta_2}|^2} e^{-N(|z_1|^2 + |z_2|^2 - 2)} |z_1 z_2|^{2N},$$

the last equivalence relying still on the fact that, on Ω , $|\theta_1 - \theta_2| \gg |z_1 - e^{i\theta_1}|, |z_2 - e^{i\theta_2}|$. This proves that (iv)-(iii)=o((iii)).

To obtain (v)-(iv)=o((iv)), just note that $|z_1|^2 - 1 - \log(|z_1|^2) = \frac{1}{2}(|z_1|^2 - 1)^2 + O(|z_1|^2 - 1)^3$, so by a simple saddle point expansion we get

$$\int_{1-\varepsilon_N}^{1+\varepsilon_N} x e^{-Nx^2+N} x^{2N} dx \underset{N \rightarrow \infty}{\sim} \int_{1-\varepsilon_N}^{1+\varepsilon_N} x e^{-\frac{N}{2}(x^2-1)^2} dx \underset{N \rightarrow \infty}{\sim} \frac{1}{2} \int_{-\infty}^{\infty} e^{-\frac{N}{2}u^2} du = \sqrt{\frac{\pi}{2N}}$$

By integrating (iv) along the radial parts of z_1, z_2 , we therefore get (v), up to the domain where $|\theta_1 - \theta_2| < N^{-\frac{1}{2}+\varepsilon}$, which can be shown to have a negligible contribution, as previously.

Finally, (v) converges to (vi), by a simple change of variables and dominated convergence. \square

For all cumulants of order $\ell \geq 3$, we begin with the following very crude bound.

Lemma 4.7. *Under the conditions of Theorem D.4, for any fixed function f , $\varepsilon > 0$, and $\ell \geq 3$, we have (as $N \rightarrow \infty$)*

$$\text{cum}_N(\ell) = O(N^3).$$

Proof. In the decomposition (4.5), it is obvious that the terms $A_N(\ell), B_N(\ell), C_N(\ell), D_N(\ell)$ are $O(N^3)$, just by bounding K_N by N in $\Omega_1 \cup \Omega_3$ and using lemma 4.3 if one point of K_N is in Ω_2 . Concerning the term $E_N(\ell)$, by reproducing the argument in the proof of Theorem 4.1 we just need to prove that it is $O(N^3)$ when all points in the integrand are restricted to Ω_3 , i.e. it would be sufficient to prove that

$$\int_{\Omega_3^{\ell+1}} |K_N(z_0, z_1) K_N(z, z_1) \dots K_N(z_\ell, z_0)| dA(z_0, \dots, z_\ell) = O(N^3).$$

using the estimate (4.8) and integrating along the width of w_3 , we just need to prove that for some $\varepsilon > 0$,

$$N^{\frac{\ell+1}{2}} \int_{(-1,1)^{\ell+1}} \frac{1}{1 + \sqrt{N}|x_0 - x_1|} \dots \frac{1}{1 + \sqrt{N}|x_\ell - x_0|} dx_0 \dots dx_\ell = O(N^{3-\varepsilon}). \quad (4.9)$$

A simple calculation yields, for any $-1 < a < b < 1$,

$$\int_{(-1,1)} \frac{1}{1 + \sqrt{N}|a - u|} \frac{1}{1 + \sqrt{N}|u - b|} du = O\left(\frac{\log N}{\sqrt{N}}\right) \frac{1}{1 + \sqrt{N}|a - b|},$$

so by integrating successively the variables x_2, \dots, x_ℓ in (4.9), we therefore obtain that a sufficient condition is $N \int_{(-1,1)^2} \frac{1}{(1 + \sqrt{N}|x_0 - x_1|)^2} dx_0 dx_1 = O(N^{3-\varepsilon})$, which is obvious. \square

Relying on the Marcinkiewicz theorem, the above initial bounds on the cumulants can be widely improved.

Lemma 4.8. *Under the conditions of Theorem D.4, for any fixed function f , $\varepsilon > 0$, and $\ell \geq 3$, we have (as $N \rightarrow \infty$)*

$$\text{cum}_N(\ell) = O(N^\varepsilon).$$

Proof. Let $Y^{(N)} = N^{-\varepsilon} X_f^{(N)}$. Let $\text{c}\ddot{\text{u}}\text{m}_N(\ell)$ be the ℓ -th cumulant of $Y^{(N)}$: $\text{c}\ddot{\text{u}}\text{m}_N(\ell) = N^{-\ell\varepsilon} \text{cum}_N(\ell)$. Hence, as a consequence of Lemma 4.7, there is a rank ℓ_0 such that for any $\ell > \ell_0$, $\text{c}\ddot{\text{u}}\text{m}_N(\ell) \rightarrow 0$ as $N \rightarrow \infty$. We wish to prove that

$$\sup_{\ell \in \llbracket 3, \ell_0 \rrbracket} \text{c}\ddot{\text{u}}\text{m}_N(\ell) \xrightarrow{N \rightarrow \infty} 0 \quad (4.10)$$

as well. Assume the contrary. We therefore can find $\delta > 0$ independent of N , an index $j_1 \in \llbracket 3, \ell_0 \rrbracket$ and a subsequence (N_k) such that

$$|\text{c}\ddot{\text{u}}\text{m}_{N_k}(j_1)| > \delta, \quad |\text{c}\ddot{\text{u}}\text{m}_{N_k}(j_1)|^{1/j_1} = \sup_{\ell \in \llbracket 3, \ell_0 \rrbracket} \{|\text{c}\ddot{\text{u}}\text{m}_{N_k}(\ell)|^{1/\ell}\}.$$

Consider now the random variable $Z^{(k)} = Y^{(N_k)} / |\text{c}\ddot{\text{u}}\text{m}_{N_k}(j_1)|^{1/j_1}$. Then all the cumulants of $Z^{(k)}$ go to 0 except the j_1 -th one, equal to 1, and eventually some other cumulants with indexes in $\llbracket 3, \ell_0 \rrbracket$, which are all uniformly bounded. Let μ_k be the distribution of $Z^{(k)}$. As the first and second cumulants of $Z^{(k)}$ are uniformly bounded, $\mathbb{E}(Z^{(k)2})$ is uniformly bounded so the sequence $(\mu_k)_{k \geq 0}$ is tight.

We therefore can choose a subsequence of (μ_k) , called (μ_{i_k}) , which converges weakly to a measure ν , and we can moreover assume that all of its cumulants (which are uniformly bounded) with indexes in $\llbracket 3, \ell_0 \rrbracket$ converge. Let Z be a random variable with distribution ν . As μ_{i_k} has any given cumulant uniformly bounded, it has any given moment uniformly bounded, so by the Corollary of Theorem 25.12 in [5], Z has moments of all orders, and those of $Z^{(i_k)}$ converge to those of Z . Hence Z has cumulants of all orders and those of $Z^{(i_k)}$ converge to those of Z . Hence ν has all of its cumulants equal to 0 for $\ell > \ell_0$, and its cumulant of order $j_1 \in \llbracket 3, \ell_0 \rrbracket$ equal to 1. From the Marcinkiewicz theorem (cf. the Corollary to Theorem 7.3.2 in [20]), such a distribution ν does not exist, a contradiction.

Equation (4.10) therefore holds, so we proved that for any arbitrary $\varepsilon > 0$ and any $\ell \geq 1$, $\text{cum}_N(\ell) = O(N^{\ell\varepsilon})$. For fixed ℓ , by choosing ε small enough, we get the result. \square

Finally, we note that the previous bounds on all cumulants of $X_f^{(N)}$, each of order at most N^ε , allow to conclude the proof of Theorem 4.4, by the Markov inequality. The explicit asymptotic form of the second cumulant (Lemma 4.6) is more than what we need, we made the calculus explicit as one expects that $X_f^{(N)}$ converges to a Gaussian centered random variable with asymptotic variance $\frac{1}{4\pi} \int |\nabla f(z)|^2 \mathbb{1}_{z \in A} dA(z) + \frac{1}{2} \|f_T\|_{\mathbb{H}^{1/2}}^2$.

4.3 Estimates on the partial exponential function. For the following lemma we need the function $\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-\omega^2} d\omega$.

Lemma 4.9 (Bleher, Mallison [6]). *For $\delta > 0$ small enough, let $\mu(z) = \sqrt{z - \log z - 1}$, be uniquely defined as analytic in $D(1, \delta)$ and $\mu(1+x) > 0$ for $0 < x < \delta$. Then for any $M > 1$, as $N \rightarrow \infty$, $e^{-Nz} e_N(Nz)$ has the following asymptotics:*

- (i) $\frac{1}{2} \text{erfc}(\sqrt{N}\mu(z)) + O(N^{-1/2})$ if $|z-1| < \frac{M}{\sqrt{N}}$;
- (ii) $\frac{1}{2\sqrt{2}\mu'(z)} \text{erfc}(\sqrt{N}\mu(z)) \left(1 + O\left(\frac{1}{(z-1)N}\right)\right)$ if $\frac{M}{\sqrt{N}} \leq |z-1| \leq \delta$ and $|\arg(z-1)| \leq \frac{2\pi}{3}$;
- (iii) $1 - \frac{1}{2\sqrt{2}\mu'(z)} \text{erfc}(-\sqrt{N}\mu(z)) \left(1 + O\left(\frac{1}{(z-1)N}\right)\right)$ if $\frac{M}{\sqrt{N}} \leq |z-1| \leq \delta$ and $|\arg(z-1) - \pi| \leq \frac{2\pi}{3}$.

In case of real z , the above estimates can be gathered as follows.

Lemma 4.10 (Wimp [9]). *Uniformly for $t \geq 0$,*

$$e^{-Nt} e_N(Nt) = \mathbf{1}_{0 \leq t < 1} + \frac{1}{\sqrt{2}} \frac{\mu(t)t}{t-1} \operatorname{erfc}(\sqrt{N}\mu(t)) \left(1 + O\left(\frac{1}{\sqrt{N}}\right) \right),$$

where $\mu(t) = \sqrt{t - \log t - 1}$ is defined to be positive for all t , contrary to Lemma 4.9.

Note that in the above lemma, the asymptotics are coherent at $t = 1$ because

$$\frac{1}{\sqrt{2}} \frac{\mu(t)t}{t-1} \longrightarrow \begin{cases} -\frac{1}{2} & \text{as } t \rightarrow 1^- \\ \frac{1}{2} & \text{as } t \rightarrow 1^+ \end{cases}.$$

Lemma 4.11 (Kriecherbauer, Kuijlaars, McLaughlin, Miller [19]). *For $0 < a < 1/2$, let $U = \{|z-1| < N^{-a}\}$. Then there exists polynomials h_j of degree $2j$ such that for all $r \in \mathbb{N}^*$ we have*

$$e_{N-1}(Nz) = \begin{cases} e^{Nz} (1 - e^N z^N e^{-Nz} F_N(z)) & \text{for } z \in D - U, \\ e^N z^N F_N(z) & \text{for } z \in D^c - U, \end{cases}$$

where

$$F_N(z) = \frac{1}{\sqrt{2\pi N}(1-z)} \left(1 + \sum_{j=1}^{r-1} \frac{h_j(z)}{N^j (z-1)^{2j}} + O\left(\frac{1}{N^r |1-z|^{2r}}\right) \right)$$

uniformly in $\mathbb{C} - U$.

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