PROBLEMS FOR BASIC PROBABILITY, FALL 2017

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Instructions. You are allowed to work on solutions in groups, but you are required to write up solutions on your own. Please give complete solutions, all claims need to be justified. Late homework will not be accepted.

Please let me know if you find any misprints or mistakes.

1. The complete first homework assignment, due on September 20

1. Determine which of the following sets are countable:
   (a) all intervals on $\mathbb{R}$ with rational endpoints;
   (b) all circles in the plane with rational radii and centers on the diagonal $x = y$;
   (c) all sequences of 0's and 1's.
2. (a) How many ways are there to split 12 people into 3 teams A, B, and C, where team A has 3 people, team B has 4 people, team C has 5 people?
   (b) How many ways are there to split 12 people into 3 teams where one team has 3 people, one team has 4 people, and one team has 5 people?
3. (a) There are n particles labeled by numbers $1, \dots, n$. Each of those particles is placed into one of M bins enumerated by numbers $1, \dots, M$. Assuming that all placements of particles are equally likely, prove that the probability $P(n, M, k)$ that bin no.1 contains exactly k particles is given by

$$P(n, M, k) = \binom{n}{k} \frac{(M-1)^{n-k}}{M^n}, \quad k \geq 0.$$

   (b) Let $P(n, M, k)$ be defined in the previous problem. Prove that as $n \to \infty$ and $M \to \infty$ so that $n/M \to \lambda > 0$,

$$P(n, M, k) \to e^{-\lambda} \frac{\lambda^k}{k!}.$$

4. Suppose there are N balls in a box. M of them are blue, the remaining ones are red. We pick n balls at random from the box. Prove that the probability to find m blue balls among the pick equals

$$\frac{\binom{M}{m}\binom{N-M}{n-m}}{\binom{N}{n}}, \quad \text{if } 0 \leq m \leq M,\ 0 \leq n - m \leq N - M.$$

5. The standard 52 card deck (4 suits with 13 cards in each) is shuffled and two cards are chosen at once at random. Find the probability that these cards are of the same suit.
6. In class, I gave a proof of the following version of the Poisson theorem: If a number sequence (p_n) satisfies $np_n = \lambda$ for some λ and all n, then

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} \to e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Give a detailed proof of the same claim under the following weaker assumption: (p_n) is a sequence of numbers satisfying $p_n \in [0,1]$ and $np_n \to \lambda$.
7. Let $\lambda > 0$ and $p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$ (the Poisson distribution with rate λ). Find the maximal value of $p(k)$ and the value of k where the maximum is attained. Hint: consider $p(k+1)/p(k)$.
8. Using the relations

$$\ln n! = \sum_{k=2}^{n} \ln k$$

and

$$\ln(n-1)! < \int_1^n \ln t \, dt < \ln n!,$$

where

$$\int_1^n \ln t \, dt = n \ln n - n + 1,$$

derive

$$e\left(\frac{n}{e}\right)^n < n! < en\left(\frac{n}{e}\right)^n.$$

Remark. In fact, this can be upgraded to the following Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O\left(\frac{1}{n^4}\right)\right).$$

9. Consider the binomial distribution with parameters $(2n, 1/2)$. Let us denote the associated probabilities by $P_{2n}(k)$, $k = 0, 1, 2 \ldots, 2n$.
   (a) Find

$$\lim_{n\to\infty} (P_{2n}(n)\sqrt{n}).$$

   (b) Let (h_n) be a sequence of integers satisfying $h_n/\sqrt{n} \to z$ for some z. Compute

$$\lim_{n\to\infty} \frac{P_{2n}(n+h_n)}{P_{2n}(n)}.$$

2. Due on October 4, in class

1. Exercise 2.2 from the textbook [JP04]
2. Exercise 2.6 from the textbook [JP04]
3. Exercise 2.14 from the textbook [JP04]
4. Exercise 2.15 from the textbook [JP04]
5. Exercise 2.17 from the textbook [JP04]
6. Exercise 3.6 from the textbook [JP04]
7. Exercise 3.10 from the textbook [JP04]
8. Exercise 3.11 from the textbook [JP04]

3. Due on Oct 11, in class

1. Exercise 3.8 from the textbook [JP04]
2. Exercise 3.9 from the textbook [JP04]
3. Suppose $n \in \mathbb{N}$, $p \in (0,1)$, and $A_1, A_2, \dots, A_n$ are mutually independent events such that $\mathsf{P}(A_k) = p$ for all $k = 1, \dots, n$. Let $m \in \mathbb{N} \cup \{0\}$. What is the probability that exactly m of these n events happen?
4. Exercise 7.1 from the textbook [JP04]
5. Exercise 7.16 from the textbook [JP04]
6. Exercise 7.17 from the textbook [JP04]
7. Exercise 7.18 from the textbook [JP04]

4. Due on Oct 25, in class

1. Exercise 5.4 from the textbook [JP04]
2. Find the expectation and variance of a Bernoulli r.v. with parameter p.
3. Find the expectation and variance of a Binomial r.v. with parameters (n, p), using the "unpleasant computation" referred to on p.30 of the textbook [JP04].
4. Exercise 5.12 from the textbook [JP04]
5. Exercise 5.13 from the textbook [JP04]
6. Exercise 5.16 from the textbook [JP04]. Be careful. The authours are not firm on which of the two versions of the geometric distribution (starting with 0 or 1), they are using. Choose the right one.
7. Exercise 5.19 from the textbook [JP04]. Be careful. Had the sum been finite, this would be an easy exercise, but the sum is infinite. Do properties of expectation extend to infinite linear combinations?
8. Exercise 5.20 from the textbook [JP04]
9. Exercise 5.21 from the textbook [JP04]
10. Suppose $X_1, X_2, \dots$ are r.v.'s on $(\Omega, \mathcal{F}, \mathsf{P})$. Suppose that for all $\omega \in \Omega$, the sum of the series
$$S(\omega) = X_1(\omega) + X_2(\omega) + X_3(\omega) + \dots$$
is well-defined (i.e., the series converges). Prove that S is a r.v. using results from Chapter 8 of [JP04].

5. Due on Nov 8, in class

1. Exercise 9.1 from the textbook [JP04]
2. Exercise 9.2 from the textbook [JP04] Hint: prove that the distribution function of X can only be equal to 0 or 1.
3. Exercise 9.5 from the textbook [JP04]
4. Exercise 9.10 from the textbook [JP04]
5. Exercise 9.12 from the textbook [JP04]
6. Exercise 9.13 from the textbook [JP04]
7. Exercise 9.16 from the textbook [JP04]
8. Exercise 9.19 from the textbook [JP04] Hint: use the fundamental theorem of calculus to represent the function on the right-hand side via the integral of its derivative from x to $+\infty$.
9. Exercise 9.20 from the textbook [JP04]

6. Due on Nov 15, in class

1. Let X and Y be two independent r.v.'s. Let X have exponential distribution with rate parameter λ and Y have exponential distribution with rate parameter μ. Find the distribution function F_Z of r.v. $Z = \min\{X, Y\}$. Hint: it is easier to work with $G_Z(x) = 1 - F_Z(x)$.
2. Exercise 10.5 from the textbook [JP04].
3. Independence is an important requirement in the second half of the Borel–Cantelli lemma. Without it, the conclusion may be violated. Consider the unit interval $[0, 1]$ with Borel σ-algebra and Lebesgue measure (i.e. the uniform distribution) on it. Give an example of a sequence of events (A_n) satisfying $\sum \mathsf{P}(A_n) = \infty$ and $\mathsf{P}(A_n \text{ i.o.}) = 0$.
4. Exercise 11.3 from the textbook [JP04].
5. Exercise 11.6 from the textbook [JP04].
6. Exercise 11.13 from the textbook [JP04].
7. Exercise 11.16 from the textbook [JP04].
8. Exercise 12.2 from the textbook [JP04].

7. Due on Nov 29, in class

1. Exercise 12.8 from the textbook [JP04].
2. Suppose X, Y are independent standard Gaussian r.v.'s. Find the density of the random variable $R = \sqrt{X^2 + Y^2}$ using the following approach:
   (a) Rewrite X, Y via polar coordinates (R, Θ), where the angle $\Theta \in [0, 2\pi]$.
   (b) Find $p_{R,\Theta}(r, \theta)$, the joint density of (R, Θ), using the Jacobian ormula.
   (c) To find the density of R, integrate over the other variable: $p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r, \theta) d\theta$.
3. Exercise 12.15 from the textbook [JP04]
4. Exercise 10.6 from the textbook [JP04]
5. Exercise 10.9 from the textbook [JP04]

6. Exercise 14.6 from the textbook [JP04]
7. Exercise 14.7 from the textbook [JP04]
8. Exercise 14.8 from the textbook [JP04]
9. Exercise 14.18 from the textbook [JP04]
10. Exercise 15.1 from the textbook [JP04]
11. Exercise 15.2 from the textbook [JP04]
12. Exercise 15.3 from the textbook [JP04]

8. Due on Dec 6, in class

1. Suppose that random variables $X_1 \sim N(a_1, \sigma_1^2)$ and $X_2 \sim N(a_2, \sigma_2^2)$ are independent. Prove that their sum $Z = X_1 + X_2$ is $N(a_1 + a_2, \sigma_1^2 + \sigma_2^2)$ using (i) density convolutions; (ii) characteristic functions.
2. Suppose the random variables X_1 and X_2 are independent and have Poisson distribution with means $\lambda_1 > 0$ and $\lambda_2 > 0$. Prove that $Z = X_1 + X_2$ is also a Poisson r.v. (i) computing $\mathsf{P}\{Z = k\}$, $k \geq 0$ directly, and (ii) using characteristic functions.
3. Let X_λ be a r.v. with Poisson distribution with mean $\lambda > 0$. Show that the characteristic function of $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges pointwise to the characteristic function of $N(0,1)$ as $\lambda \to +\infty$.
4. Use characteristic functions to prove the following version of the Poisson limit theorem: Let $X_1^{(n)}, \dots, X_n^{(n)}$ be i.i.d. Bernoulli r.v.'s with parameter p_n (i.e. $\mathsf{P}\{X_1^{(n)} = 1\} = p_n$ and $\mathsf{P}\{X_1^{(n)} = 0\} = 1 - p_n$) for each $n \in \mathbb{N}$. Assuming $np_n \to \lambda$ as $n \to \infty$, prove that $X_1^{(n)} + \dots, +X_n^{(n)}$ converges in distribution to a Poisson r.v. with parameter λ.
5. Suppose each r.v. X_n is constant, i.e., $\mathsf{P}\{X_n = c_n\} = 1$ for some constant $c_n \in \mathbb{R}$. Prove that $X_n \xrightarrow{d} X$ if and only if for some $c \in \mathbb{R}$, $\mathsf{P}\{X = c\} = 1$ and $c_n \to c$, $n \to \infty$.
6. Exercise 17.2 from the textbook [JP04]
7. Exercise 17.14 from the textbook [JP04]
8. Exercise 18.9 from the textbook [JP04]
9. Exercise 18.15 from the textbook [JP04]
10. Exercise 19.1 from the textbook [JP04]
11. Exercise 19.2 from the textbook [JP04]
12. Exercise 19.3 from the textbook [JP04]

9. Due on Dec 13, in class

1. Let X_n be an i.i.d. sequence of exponential r.v.'s with parameter 1. Using the Borel–Cantelli lemma, prove that for any $b > 1$,

$$\frac{X_n}{(\ln n)^b} \to 0 \text{ a.s.}$$

2. Let X_n be an i.i.d. sequence of standard Gaussian r.v.'s. Using the Borel–Cantelli lemma and problem 9.19 from [JP04], prove that for any $a > 1/2$,
$$\frac{X_n}{(\ln n)^a} \to 0 \text{ a.s.}$$
3. Using the unit interval with Lebesgue measure as the probability space, construct r.v.'s $(X_n)_{n\in\mathbb{N}}$ such that $X_n \xrightarrow{\mathsf{P}} X$, but $X_n \overset{\text{a.s.}}{\not\to} X$.
4. Exercise 20.4 from the textbook [JP04]
5. Exercise 20.6 from the textbook [JP04]
6. Exercise 20.7 from the textbook [JP04]
7. Exercise 20.8 from the textbook [JP04]
8. Exercise 21.2 from the textbook [JP04]
9. Exercise 21.10 from the textbook [JP04]
10. [This is not for grading, just letting you know:] In the discussion of conditional expectations and martingales at the last lecture, I will asssume that you know Chapter 22 of [JP04].]

10. Some practice exercises (not to be graded!)

Exercises 23.1-23.6, 23.8 [Hint for the latter: prove $E(X-Y)^2 = 0$.]

References

[JP04] Jean Jacod and Philip Protter. *Probability essentials*. Universitext. Springer-Verlag, Berlin, second edition, 2003 or 2004.