PROBLEMS FOR BASIC PROBABILITY, FALL 2016

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Instructions. You are allowed to work on solutions in groups, but you are required to write up solutions on your own. Please give complete solutions, all claims need to be justified. Late homework will not be accepted.

Please let me know if you find any misprints or mistakes.

1. Due by September 14, 7pm (5 problems)

1. Determine which of the following sets are countable:
   (a) all intervals on $\mathbb{R}$ with rational endpoints;
   (b) all circles in the plane with rational radii and centers on the diagonal $x = y$;
   (c) all sequences of 0's and 1's.
2. (a) How many ways are there to split 12 people into 3 teams A, B, and C, where team A has 3 people, team B has 4 people, team C has 5 people?
   (b) How many ways are there to split 12 people into 3 teams where one team has 3 people, one team has 4 people, and one team has 5 people?
3. (a) There are n particles labeled by numbers $1, \ldots, n$. Each of those particles is placed into one of M bins enumerated by numbers $1, \ldots, M$. Assuming that all placements of particles are equally likely, prove that the probability $P(n, M, k)$ that bin no.1 contains exactly k particles is given by

$$P(n, M, k) = \binom{n}{k} \frac{(M-1)^{n-k}}{M^n}, \quad k \geq 0.$$

   (b) Let $P(n, M, k)$ be defined in the previous problem. Prove that as $n \to \infty$ and $M \to \infty$ so that $n/M \to \lambda > 0$,

$$P(n, M, k) \to e^{-\lambda} \frac{\lambda^k}{k!}.$$

4. Suppose there are N balls in a box. M of them are blue, the remaining ones are red. We pick n balls at random from the box. Prove that the probability to find m blue balls among the pick equals

$$\frac{\binom{M}{m}\binom{N-M}{n-m}}{\binom{N}{n}}, \quad \text{if } 0 \leq m \leq M,\ 0 \leq n - m \leq N - M.$$

5. The standard 52 card deck (4 suits with 13 cards in each) is shuffled and two cards are chosen at once at random. Find the probability that these cards are of the same suit.

2. Due by September 28, 7pm, 10 problems.

(I have corrected a misprint in problem 1)

1. In class, I gave a proof of the following version of the Poisson theorem: If a number sequence (p_n) satisfies $np_n = \lambda$ for some λ and all n, then

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} \to e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots.$$

Give a detailed proof of the same claim under the following weaker assumption: (p_n) is a sequence of numbers satisfying $p_n \in [0,1]$ and $np_n \to \lambda$.

2. Let $\lambda > 0$ and $p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 0,1,2,\ldots$ (the Poisson distribution with rate λ). Find the maximal value of $p(k)$ and the value of k where the maximum is attained.
3. Using the relations

$$\ln n! = \sum_{k=2}^{n} \ln k$$

and

$$\ln(n-1)! < \int_1^n \ln t \, dt < \ln n!,$$

where

$$\int_1^n \ln t \, dt = n \ln n - n + 1,$$

derive

$$e\left(\frac{n}{e}\right)^n < n! < en\left(\frac{n}{e}\right)^n.$$

Remark. In fact, this can be upgraded to the following Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O\left(\frac{1}{n^4}\right)\right).$$

4. Consider the binomial distribution with parameters $(2n, 1/2)$. Let us denote the associated probabilities by $P_{2n}(k),\ k = 0,1,2\ldots,2n$.
   (a) Find

$$\lim_{n\to\infty} (P_{2n}(n)\sqrt{n}).$$

   (b) Let (h_n) be a sequence of integers satisfying $h_n/\sqrt{n} \to z$ for some z. Compute

$$\lim_{n\to\infty} \frac{P_{2n}(n+h_n)}{P_{2n}(n)}.$$

5. Consider the binomial distribution with parameters (n, p), where $n = 100$ and $p = 0.1,\ 0.3,\ 0.5$. Using computer tools of your choice, compare the following probabilities with their normal (Gaussian) and Poissonian approximations:
$$\mathsf{P}\{10 < \omega \le 15\}, \quad \mathsf{P}\{30 < \omega \le 35\}, \quad \mathsf{P}\{50 < \omega \le 55\}.$$
6. Exercise 2.2 from the textbook [JP04]
7. Exercise 2.6 from the textbook [JP04]
8. Exercise 2.14 from the textbook [JP04]
9. Exercise 2.15 from the textbook [JP04]
10. Exercise 2.17 from the textbook [JP04]

3. Due on Oct 12, 7pm. 10 Problems

1. Exercise 7.1 from the textbook [JP04]
2. Exercise 7.11 from the textbook [JP04]
3. Exercise 7.16 from the textbook [JP04]
4. Exercise 7.17 from the textbook [JP04]
5. Exercise 7.18 from the textbook [JP04]
6. Exercise 5.4 from the textbook [JP04]
7. Exercise 5.16 from the textbook [JP04]
8. Exercise 5.19 from the textbook [JP04]
9. Exercise 5.22 from the textbook [JP04]
10. Suppose U is a random variable that has uniform distribution on $[0, 1]$. Let $a > 0$. Find the density of the distribution of the random variable $-(\ln U)/a$.

4. Due on Oct 19, 7pm.

1. Exercise 3.9 from the textbook [JP04]
2. Exercise 3.11 from the textbook [JP04]
3. Exercise 3.17 from the textbook [JP04]
4. Exercise 10.12 from the textbook [JP04]
5. Independence is an important requirement in the second half of the Borel–Cantelli lemma. Without it, the conclusion may be violated. Consider the unit interval $[0, 1]$ with Borel σ-algebra and Lebesgue measure (i.e. the uniform distribution) on it. Give an example of a sequence of events (A_n) satisfying $\sum \mathsf{P}(A_n) = \infty$ and $\mathsf{P}(A_n \text{ i.o.}) = 0$.

5. Due on Oct 26, 7pm, (6 problems)

1. Exercise 9.2 from the textbook [JP04]
2. Exercise 9.12 from the textbook [JP04]
3. Exercise 9.16 from the textbook [JP04]
4. Exercise 9.20 from the textbook [JP04]
5. Exercise 11.6 from the textbook [JP04]
6. Exercise 11.13 from the textbook [JP04]

6. Due on Nov 16, 7pm, (10 problems)

1. Prove that if a random variable X satisfies $\mathsf{E}|X|^p < \infty$ for some $p > 0$, then $\mathsf{E}|X|^q < \infty$ for all $q \in (0, p)$. In other words, $L^p(\Omega, \mathcal{F}, \mathsf{P}) \subset L^q(\Omega, \mathcal{F}, \mathsf{P})$ if $0 < q < p$.
2. Let X and Y be two independent r.v.'s. Let X have exponential distribution with rate parameter λ and Y have exponential distribution with rate parameter μ. Find the distribution function of r.v. $Z = \min\{X, Y\}$
3. Exercise 10.5 from the textbook [JP04]
4. Exercise 10.18 from the textbook [JP04]
5. Exercise 12.8 from the textbook [JP04]
6. Exercise 12.15 from the textbook [JP04]
7. Exercise 14.6 from the textbook [JP04]
8. Exercise 14.7 from the textbook [JP04]
9. Exercise 15.1 from the textbook [JP04]
10. Exercise 15.3 from the textbook [JP04]

7. Due on Nov 30, 7pm

1. Suppose that random variables $X_1 \sim N(a_1, \sigma_1^2)$ and $X_2 \sim N(a_2, \sigma_2^2)$ are independent. Prove that their sum $Z = X_1 + X_2$ is $N(a_1 + a_2, \sigma_1^2 + \sigma_2^2)$ using (i) density convolutions; (ii) characteristic functions.
2. Suppose the random variables X_1 and X_2 are independent and have Poisson distribution with means $\lambda_1 > 0$ and $\lambda_2 > 0$. Prove that $Z = X_1 + X_2$ is also a Poisson r.v. (i) computing $\mathsf{P}\{Z = k\}$, $k \geq 0$ directly, and (ii) using characteristic functions.
3. Exercise 14.10 from the textbook [JP04]
4. Exercise 14.18 from the textbook [JP04]
5. Let X_λ be a r.v. with Poisson distribution with mean $\lambda > 0$. Show that the charactresistic function of $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges pointwise to the characteristic function of $N(0, 1)$ as $\lambda \to +\infty$.

8. Due on December 7, 7pm (5 problems)

1. Let X_n be an i.i.d. sequence of exponential r.v.'s with parameter 1. Using the Borel–Cantelli lemma, prove that for any $b > 1$,
$$\frac{X_n}{(\ln n)^b} \to 0 \text{ a.s.}$$
2. Exercise 17.2 from the textbook [JP04]
3. Exercise 17.14 from the textbook [JP04]
4. Exercise 18.15 from the textbook [JP04]
5. Suppose each r.v. X_n is constant, i.e., $\mathsf{P}\{X_n = c_n\} = 1$ for some constant $c_n \in \mathbb{R}$. Prove that $X_n \xrightarrow{d} X$ if and only if for some $c \in \mathbb{R}$, $\mathsf{P}\{X = c\} = 1$ and $c_n \to c$, $n \to \infty$.

9. Due on December 14, 7pm (5 problems)

1. Use characteristic functions to prove the following version of the Poisson limit theorem: Let $X_1^{(n)}, \dots, X_n^{(n)}$ be i.i.d. Bernoulli r.v.'s with parameter p_n (i.e. $\mathsf{P}\{X_1^{(n)} = 1\} = p_n$ and $\mathsf{P}\{X_1^{(n)} = 0\} = 1 - p_n$) for each $n \in \mathbb{N}$. Assuming $np_n \to \lambda$ as $n \to \infty$, prove that $X_1^{(n)} + \dots, +X_n^{(n)}$ converges in distribution to a Poisson r.v. with parameter λ.
2. Exercise 20.4 from the textbook [JP04]
3. Exercise 21.2 from the textbook [JP04]
4. Exercise 23.3 from the textbook [JP04]
5. Exercise 23.5 from the textbook [JP04]

References

[JP04] Jean Jacod and Philip Protter. *Probability essentials*. Universitext. Springer-Verlag, Berlin, second edition, 2003 or 2004.