1. Suppose the random variable X has a standard Gaussian distribution, i.e., its density is

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

(For the final exam you will need to know Gaussian distributions by heart). Compute the density of the random variable $Y = -\log |X|$.

For all $y \in \mathbb{R}$

$$p_Y(y) = \sum_{x:\, -\log|x| = y} \frac{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}{|(-\log|x|)'|} = 2 \cdot \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{e^{-2y}}{2}}}{1/e^{-y}} = \sqrt{\frac{2}{\pi}}\, e^{-y - \frac{e^{-2y}}{2}}$$

$x = \pm e^{-y}$

the two terms are equal to each other

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2. For a sequence of events $(A_n)_{n=1}^{\infty}$ we recall the definitions

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$
$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Is it possible to have $\mathsf{P}(\limsup A_n \cap \limsup A_n^c) \neq 0$ for some sequence $(A_n)_{n=1}^{\infty}$ on some probability space? Is it possible to have $\mathsf{P}(\liminf A_n \cap \liminf A_n^c) \neq 0$ for some sequence $(A_n)_{n=1}^{\infty}$ on some probability space?

For each question, if your answer is *yes*, give a supporting example. If your answer is *no*, prove your answer.

Question 1: Yes. Choose, for example, any $(\Omega, \mathcal{F}, P)$ and

$$A_n = \begin{cases} \Omega, & n \text{ even} \\ \emptyset, & n \text{ odd} \end{cases} \qquad A_n^c = \begin{cases} \emptyset, & n \text{ even} \\ \Omega, & n \text{ odd} \end{cases}$$

Then $\limsup A_n = \Omega$ and $\limsup A_n^c = \Omega$, so $P(\limsup A_n \cap \limsup A_n^c) = P(\Omega) = 1$.

Question 2: No. We must prove that no ω can belong to both $\liminf A_n$ and $\liminf A_n^c$. If $\omega \in \liminf A_n$, then there is n such that $\omega \in A_k$ for all $k \geq n$, so $\omega \in A_k^c$ can hold only for finitely many values of k. Therefore, $\omega \notin \liminf A_n^c$.

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3. Suppose your chance to win \$200 000 in a lottery is 0.000001. Let X denote the amount you win. Find $\mathsf{E}X$ and $\mathsf{Var}X$.

$P\{X = 2\cdot 10^5\} = 10^{-6}$
$P\{X = 0\} = 1 - 10^{-6}$

$EX = 2\cdot 10^5 \cdot 10^{-6} + 0\cdot(1-10^{-6}) = 2\cdot 10^{-1} = 0.2$

$EX^2 = 4\cdot 10^{10}\cdot 10^{-6} + 0\cdot(1-10^{-6}) = 4\cdot 10^4 = 40000$

$\mathrm{Var}\,X = EX^2 - (EX)^2 = 40000 - 0.04 = 39999.96$

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4. Suppose there are three boxes. Box no.1 contains 1 blue ball and 1 red ball. Box no.2 contains 1 blue ball and 3 red balls. Box no.3 contains only 1 blue ball. A box is chosen at random, and then a ball is chosen at random from that box. Given that the result of this procedure is a blue ball, what is the conditional probability that it was chosen from Box no.1?

$A = \{\text{blue ball chosen}\}$

$B_k = \{\text{Box 1 chosen}\}$, $k = 1, 2, 3$

$P(B_k) = \frac{1}{3}$, $k = 1, 2, 3$.

$P(A | B_1) = \frac{1}{2}$

$P(A | B_2) = \frac{1}{4}$

$P(A | B_3) = 1$

The Bayes formula

$$P(B_1 | A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4} + 1} = \frac{1/2}{7/4} = \underline{\underline{\frac{2}{7}}}.$$

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5. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\mathsf{E}X = 1$ and $X(\omega) \geq 0$ for all ω.

   (a) Prove that the function $Q : \mathcal{F} \to \mathbb{R}$ defined by $Q(A) = \mathsf{E}[X\mathbb{1}_A]$ for $A \in \mathcal{F}$ is a probability measure on $(\Omega, \mathcal{F})$.

   (b) Prove that if $\mathsf{P}(A) = 0$, then $Q(A) = 0$.

   (c) Give an example showing that in general $Q(A) = 0$ does not imply $\mathsf{P}(A) = 0$.

(a) Need to check (1) $Q(\Omega)=1$ (2) σ-additivity.

(1): $Q(\Omega) = EX\mathbb{1}_\Omega = EX = 1$

(2) Let $A_1, A_2, \ldots$ be disjoint.

$$Q\left(\bigcup_{k=1}^\infty A_k\right) = E\left[X \mathbb{1}_{\bigcup_{k=1}^\infty A_k}\right] = E\left[X \sum_{k=1}^\infty \mathbb{1}_{A_k}\right] = E\left[\sum_{k=1}^\infty X\mathbb{1}_{A_k}\right] = E\left[\lim_{n\to\infty}\sum_{k=1}^n X\mathbb{1}_{A_k}\right]$$

$$= \begin{bmatrix}\text{by Monotone} \\ \text{Convergence Thm}\end{bmatrix} = \lim_{n\to\infty} E\left[\sum_{k=1}^n X\mathbb{1}_{A_k}\right] = \lim_{n\to\infty}\sum_{k=1}^n EX\mathbb{1}_{A_k} = \lim_{n\to\infty}\sum_{k=1}^n Q(A_k)$$

$$= \sum_{k=1}^\infty Q(A_k)$$

(b) If $P(A)=0$, then $E\underbrace{X\mathbb{1}_A}_{=0 \text{ a.s.}} = 0$ for all r.v.'s. [it is true for simple r.v.'s, hence for all]

(c) Let $\Omega = \{0,1\}$, $\mathcal{F} = 2^{\{0,1\}}$, $P\{0\} = P\{1\} = \frac{1}{2}$, $X(0)=0$, $X(1)=2$.

Then $Q\{0\} = 0$, but $P\{0\} \neq 0$.

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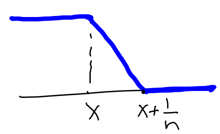
6. Let P and Q be two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ Prove that if $\int_{\mathbb{R}} f(x)\mathsf{P}(dx) = \int_{\mathbb{R}} f(x)\mathsf{Q}(dx)$ for all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$, then $\mathsf{P} = \mathsf{Q}$. Hint: it suffices to check that the distribution functions of these two measures coincide.

Let $F(x) = \mathbb{P}\big((-\infty, x]\big)$ and $G(x) = \mathbb{Q}\big((-\infty, x]\big)$ for $x \in \mathbb{R}$

Since Borel measures are uniquely defined by their distribution functions, it suffices to show that $F(x) = G(x)$ for all $x \in \mathbb{R}$.

Let us introduce functions $f_n, n \in \mathbb{N}$, with the following graph:



Then f_n are continuous, bounded by 1. By assumption, $\int_{\mathbb{R}} f_n(y) \mathbb{P}(dy) = \int_{\mathbb{R}} f_n(y) \mathbb{Q}(dy)$.

Also, for all $y \in \mathbb{R}$, $f_n(y) \searrow \mathbb{1}_{(-\infty, x]}(y)$.

Therefore, by one of the convergence theorems (monotone convergence Thm / dominated conv. Thm / monotone conv. Thm)

$$F(x) = \mathbb{P}\big((-\infty, x]\big) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty, x]}(y) \mathbb{P}(dy) = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(y) \mathbb{P}(dy) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(y) \mathbb{P}(dy)$$

$$\|$$

$$G(x) = \mathbb{Q}\big((-\infty, x]\big) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty, x]}(y) \mathbb{Q}(dy) = \int_{\mathbb{R}} \lim f_n(y) \mathbb{Q}(dy) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(y) \mathbb{Q}(dy)$$

So $F(x) = G(x)$.

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