1. We will estimate the parameters of a few GARCH-type models for the S&P500 data from Jan 2, 1987 to Oct 16, 2009.

   (a) Estimate the parameters of a GARCH(1,1) model. Compute the residuals of the model, \( \hat{\epsilon}_t = (r_t - \hat{\mu})/\hat{\sigma}_t \). Compare the autocorrelation of squared returns, \( \text{corr}(r_t^2, r_{t-1}^2) \), with the autocorrelation of squared residuals from the model, \( \text{corr}(\hat{\epsilon}_t^2, \hat{\epsilon}_{t-1}^2) \). Also compare the kurtosis of returns, \( \text{kurt}(r_t) \), with the kurtosis of the residuals from the model, \( \text{kurt}(\hat{\epsilon}_t) \). Did the GARCH model standardize the residuals, i.e., do the squared residuals have zero autocorrelation and do the residuals have zero excess kurtosis? Perform the portmanteau test of Box and Pierce.

   (b) Estimate a GARCH(1,1) model with conditional returns having a Generalized Error Distribution.

   (c) Estimate a TGARCH(1,1) model with conditional returns having a normal distribution.

   (d) Compare the maximum value of the likelihood function for the models in parts (a), (b), and (c) to rank the three models. The maximum value of the likelihood function for parts (b) and (c) can be compared directly since they have the same number of parameters. Use the Schwartz Bayesian Criteria to compare likelihood functions when the numbers of parameters is different.

2. Suppose \( X_1, X_2, \ldots, X_N \) is an i.i.d. random sample from the density function given by

\[
f(x) = \begin{cases} 
\theta x^{-(\theta+1)}, & \text{if } x \geq 1; \theta > 1 \\
0, & \text{otherwise}
\end{cases}
\]

   (a) Derive the method of moments estimate of \( \theta \). Also derive the maximum likelihood estimate of \( \theta \).

   (b) Generate \( N \) random samples of \( X_i \) with \( \theta = 3 \) by using the inverse transform technique, i.e. \( X_i = F^{-1}(R_i) \) where \( R_i \) are uniform \((0,1)\) random numbers and \( F(x) \) is the cumulative distribution function of \( f(x) \), i.e., \( F(x) = \int_1^x f(t) \, dt \). Estimate \( \theta \) using the method of moments estimate and the maximum likelihood estimate that you derived above to verify that the estimates of \( \theta \) are close to the true value of 3.

3. Suppose \( X_1, X_2, \ldots, X_N \) is an i.i.d. random sample from the student’s t distribution, where the density function is given by

\[
f(x) = \frac{\Gamma[\nu + 1/2]}{\Gamma(\nu/2)\sqrt{\pi\nu}} [1 + x^2/\nu]^{-(\nu+1)/2}
\]
The second and fourth moments (odd moments are zero because the distribution is symmetric around zero) are

\[ E[X_i^2] = \frac{\nu}{\nu - 2} \]
\[ E[X_i^4] = \frac{3\nu^2}{(\nu - 2)(\nu - 4)} \]

Generate \( N \) random samples from the student’s \( t \) distribution with 5 degrees of freedom. Recall that if \( Z \) is a standard normal random variable and \( \chi^2 \) is a chi-squared random variable with \( \nu \) degrees of freedom, and \( Z \) and \( \chi^2 \) are independent, then \( X = Z/\sqrt{\chi^2/\nu} \) has a student’s \( t \) distribution with \( \nu \) degrees of freedom, so to generate each random number \( X_i \), generate 6 standard normal random numbers, \( Z_{1,i}, \ldots, Z_{6,i} \), and \( X_i = Z_{1,i}/\sqrt{\sum_{j=2}^{6} Z_{j,i}^2}/5 \). Use GMM with the two moment conditions to estimate the parameter \( \nu \) and verify that the estimated parameter is close to the true parameter of 5. There is no need to use the Newey-West estimate because the \( X_i \) are i.i.d., but go through at least one iteration of estimating the weighting matrix.