1 Problems

1.1 Determinants, Trace and Rank

January 1995 Problem 2

Suppose $A$ is an $n \times n$ matrix and that

\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_n \\
\end{bmatrix}, \quad x = \begin{bmatrix}
y_1 \\
\vdots \\
y_n \\
\end{bmatrix}, \quad y^* = (y_1 \ldots y_n)
\]

Suppose that all the entries of $A$, $x$ and $y$ are real.

Part 1
Show that there exist number $a$ and $b$ so that $\det(A + xy^*) = a + bs$.

Part 2
Show that if $\det(A) \neq 0$ then $a = \det(A)$ and $b = \det(A) y^* A^{-1} x$.

Part 3
Is it true that $a = 0$ if $\det(A) = 0$?

January 2012 Problem 2

Let $H = (h_{ij})$ be an $n \times n$ matrix such that (i) All coefficients $h_{ij} \in \{+1,-1\}$ and (ii) The row vectors $h_i = (h_{i1}, \ldots, h_{in})$ are mutually orthogonal

Part 1
Find a simple expression for $HH^T$

Part 2
Find $\det(H)$.

Part 3
Let $u = (u_1, \ldots, u_n)$ with all $u_j = \pm 1$. Prove that at least one coordinate of $Hu^T$ has absolute value at least $\sqrt{n}$. (Hint: Find the Euclidean norm of $Hu^T$).

January 2014 Problem 1

Consider $n \times n$ real matrices $A, B$ and $C$ with $ABC = 0$.

- What can be the maximal possible rank of $CBA$?
- What is a maximal possible rank of $CBA$ is we assume that $C, A$ are diagonal?
- What is a maximal possible rank of $CBA$ is we assume that $A, B, C$ are symmetric?
1.2 Projection and Orthogonal Matrix

January 2014 Problem 1

A complex $n \times n$ matrix $U$ is unitary if it satisfies $U^H U = I$. A real unitary matrix is called orthogonal.

1. If $U$ is $n \times n$ unitary, show that
$$\max_{j,k} |U_{j,k}| \geq \frac{1}{\sqrt{n}}$$
with equality if and only if all of the elements of $U$ are equal in magnitude.

2. Give an example of a $3 \times 3$ unitary matrix $U$ for which all of the elements of $U$ are equal in magnitude.

3. Prove the existence for every positive integer $n$ of an $n \times n$ unitary matrix $U$ for which all of the elements of $U$ are equal in magnitude.

4. Now consider orthogonal matrices. For which $n \in \{2, 3, 4, 5, 6\}$ do there exist orthogonal matrices with all of the (real) elements equal in magnitude? For those $n \in \{2, 3, 4, 5, 6\}$ for which such matrices exist, construct one; and for the remaining $n \in \{2, 3, 4, 5, 6\}$, prove that no such orthogonal matrix exists.

5. Find an infinite set $S$ of positive integers such that for every $n \in S$ there exists an $n \times n$ orthogonal matrix all of whose elements are equal in magnitude.

September 2012 Problem 2

Find $3 \times 3$ matrices $A, B$ and $C$ that correspond to the following three linear operations. In all steps, explain your answers.

Part 1
Let $A$ represent the orthogonal projection onto the plane $x - y + z = 0$.

Part 2
Let $B$ represent the reflection across the plane $x - y + z = 0$.

Part 3
Let $C$ represent rotation by $\pi$ about the line $x = -y = z$. (Fortunately, it doesn’t matter the sign of rotation, as either direction is the same.)

Part 4
Evaluate $A^4, B^2$ and $C^3$.

Part 5
Evaluate $AB - BA, AC - CA$, and $BC - CB$.

September 2015 Problem 3

In this problem, $\mathcal{M}_n(\mathbb{C})$ is the set of square matrices with size $n \times n$ with coefficients in $\mathbb{C}$, and for any matrix $A$, $A^*$ is its conjugate transpose, defined such that $A_{ij}^* = \overline{A_{ji}}$, $U_n(\mathbb{C})$ is the set of unitary matrices in $\mathcal{M}_n(\mathbb{C})$, i.e. matrices $U$ whose adjoints $U^*$ are also their inverses: $U^* U = U U^* = I_n$

Let $k$ be an integer in $[1, n]$. We consider a matrix $P \in \mathcal{M}_n(\mathbb{C})$ that satisfies the conditions $(P_k)$ given by:

$$(P_k) \quad P^2 = P = P^*, \quad rk(P) = k$$

1. Show that the entries of $P$ satisfy
$$\forall i \in \mathbb{N} \text{ s.t. } 1 \leq i \leq n, 0 \leq P_{ii} \leq 1,$$
\[
\sum_{i=1}^{n} P_{ii} = k
\]

2. Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) be \( n \) real numbers and \( D \) the diagonal matrix such that \( D_{ii} = \lambda_i \) for all integers \( i \) such that \( 1 \leq i \leq n \). Show that for any matrix \( P \) satisfying \((P_k)\)

\[
\text{Trace}(PD) \leq \sum_{i=1}^{k} \lambda_i
\]

Find a matrix \( Q \) that satisfies \((P_k)\) and such that \( \text{Trace}(PD) = \sum_{i=1}^{k} \lambda_i \)

3. Show that if \( P_1 \) and \( P_2 \) are two matrices satisfying \((P_k)\), there exists \( U \in U_n(\mathbb{C}) \) such that \( P_2 = UP_1U^* \).

Show then that

\[
\sum_{i=1}^{k} \lambda_i = \max_{U \in U_n(\mathbb{C})} \text{Trace}(UPU^*D)
\]

where \( P \) is a matrix satisfying \((P_k)\).

1.3 Inner Product and Norms

January 2016 Problem 4

Introduction: you might recall the double-angle formulas

\[
2 \sin^2 \theta = (1 - \cos 2\theta) \quad \text{and} \quad 2 \cos^2 \theta = (1 + \cos 2\theta).
\]

These formulae, along with the change-of-variables \( x = \sin \theta \), can be used to show:

\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx = \pi, \quad \text{and} \quad \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx = \pi/2
\]

\[
\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}} \, dx = 3\pi/8, \quad \text{and} \quad \int_{-1}^{1} \frac{x^6}{\sqrt{1-x^2}} \, dx = 5\pi/16
\]

Now consider the vector-space \( P_n \) of polynomials with real coefficients of degree \( n \) or less defined on the interval \( x \in [-1, 1] \). Endow this space with the following inner product:

\[
\langle f, g \rangle_T = \int_{-1}^{1} f(x)g(x) \frac{dx}{\sqrt{1-x^2}}
\]

(a) Find a basis \( \{t_0(x), t_1(x), t_2(x)\} \) for \( P_2 \) that is orthonormal with respect to \( \langle \ldots \rangle_T \) where \( t_0 \) is constant, \( t_1(x) \) is linear, and \( t_2(x) \) is quadratic.

(b) Express the integration operator \( \int_{-1}^{1} f(x) \, dx \) as a linear operator on \( P_2 \) in terms of the basis \( t_0, t_1, t_2 \).

(c) Now consider the polynomial

\[
a(x) = \sqrt{\frac{2}{\pi}} \left[ 4x^3 + 2x^2 - 3x + x - 1 - \frac{1}{\sqrt{2}} \right] \in P_3
\]

Find the polynomial \( b(x) \in P_2 \) that is 'closest' to \( a(x) \) in the \( T \)-norm. That is, find the \( b(x) \) that minimizes:

\[
\langle a - b, a - b \rangle_T = \int_{-1}^{1} (a(x) - b(x))^2 \frac{dx}{\sqrt{1-x^2}}.
\]

(d) Now, with \( n \) arbitrary, find a map from \( P_n \) to \( P_2 \) that sends a polynomial \( a(x) \in P_n \) to the polynomial \( b(x) \in P_2 \) which is closest to \( a(x) \) in the \( T \)-norm.
1.4 Jordan Form

January 2014 Problem 3

Find $\lim_{N \to \infty} \frac{||A^N(x)||}{||B^N(x)||}$ as a function of $x \in \mathbb{R}^2$ if the matrices $A, B$ are:

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 \\ 2 & 2 \end{bmatrix}$$

1.5 Related Problems for practice

S97Q5, J98Q2, J99Q3, J00Q4, S00Q2, J01Q5, S03Q1, S03Q2, J03Q5, S04Q5, J05Q1, J05Q2-Q3, J06Q3, J07Q2, J08Q3, J08Q5, J09Q3, S10Q1, S11Q3, S12Q2, S12Q4