New and improved Johnson-Lindenstrauss embeddings via the Restricted Isometry Property

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The Johnson-Lindenstrauss (JL) Lemma states that any set of $p$ points in high dimensional Euclidean space can be embedded into $O(\delta^{-2} \log(p))$ dimensions, without distorting the distance between any two points by more than a factor between $1 - \delta$ and $1 + \delta$. We establish a new connection between the JL Lemma and the Restricted Isometry Property (RIP), a well-known concept in the theory of sparse recovery often used for showing the success of $\ell_1$-minimization.

Consider an $m \times N$ matrix satisfying the $(k, \delta_k)$-RIP with randomized column signs and an arbitrary set $E$ of $O(e^k)$ points in $\mathbb{R}^N$. We show that with high probability, such a matrix with randomized column signs maps $E$ into $\mathbb{R}^m$ without distorting the distance between any two points by more than a factor of $1 \pm 4\delta_k$. Consequently, matrices satisfying the Restricted Isometry of optimal order provide optimal Johnson-Lindenstrauss embeddings up to a logarithmic factor in $N$. Moreover, our results yield the best known bounds on the necessary embedding dimension $m$ for a wide class of structured random matrices. In particular, for partial Fourier and partial Hadamard matrices, our method optimizes the dependence of $m$ on the distortion $\delta$: We improve the recent bound $m = O(\delta^{-4} \log(p) \log^4(N))$ of Ailon and Liberty (2010) to $m = O(\delta^{-2} \log(p) \log^4(N))$, which is optimal up to the logarithmic factors in $N$. Our results also have a direct application in the area of compressed sensing for redundant dictionaries.

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