A Numerical quadrature for the Schwarz-Chimera Method

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Abstract

Chimera [10] happens to be a version of Schwarz' method and of Lions' space decomposition method (SDM). It was analyzed by Brezzi et al [1] but an estimate was missing for for numerical quadrature. We give it here with new numerical tests.

1 Introduction

Consider a Hilbert space V, a continuous bilinear form $a(u, \hat{u})$ symmetric with a coercivity constant $\alpha > 0$ and f regular for well posedness of

$$a(u, \hat{u}) = (f, \hat{u}) \ \ \forall \hat{u} \in V, \tag{1}$$

We assume that $V = V_1 + V_2 \qquad V_1 \cap V_2$ of non zero measure (i.e. overlapping) where each V_i is a closed subspace of V. We will need also two continuous symmetric bilinear forms $b_i(u, \hat{u}), \ i = 1, 2$ coervice enough so that

$$\sum_1^2 b_i(\hat{u}_i) + a(\hat{u}_i) \geq a(\sum_1^2 \hat{u}_i) \quad \forall \hat{u}_i \in V_i \tag{2}$$

A typical example is the Dirichlet problem for $-\Delta u = f$ in $\Omega = \Omega_1 \cup \Omega_2$ and such that $\Omega_1 \cap \Omega_2 \neq \emptyset$; denote by $S_i = \partial\Omega_i \cap \Omega_j, \ j \neq i$. Then set

$$V_i = \{v \in L^2(\Omega) : \quad v|_{\Omega_i} \in V(\Omega_i), \ \ v|_{\Omega - \Omega_i} = 0\} \tag{3}$$

Algorithm 1 (Schwarz)
Begin loop with a Chosen $v_i^0 \in V_i$, and $n = 0$.
Find v_i^{n+1} such that $v_i^{n+1} - v_j^n \in V_i, \ \ i, j = 1, 2, j \neq i$ by solving

$$a(v_i^{n+1}, \hat{v}_i) = (f, \hat{v}_i) \quad \forall \hat{v}_i \in V_i \tag{4}$$

End loop
The convergence has been analyzed by P.L. Lions[6] in a general setting. In search for precision, we present the following alternative; it uses $b_i(u,v) = b(u,v) = (\beta u, v),\ i=1,2$ for some positive scalar β and two arbitrary functions $u_i^0 \in V_i$.

Algorithm 2 (SDM)
Begin loop with $n=0$:
Find $u_i^{n+1} \in V_i$ by solving

$$\begin{aligned} &b(u_1^{n+1}-u_1^n,\hat{u}_1)+a(u_1^{n+1}+u_2^n,\hat{u}_1)=(f,\hat{u}_1) \quad \forall \hat{u}_1\in V_1\\ &b(u_2^{n+1}-u_2^n,\hat{u}_2)+a(u_1^n+u_2^{n+1},\hat{u}_2)=(f,\hat{u}_2) \quad \forall \hat{u}_2\in V_2 \end{aligned} \tag{5}$$

End loop
When $\beta=0$ Algorithm 2 is identical to Algorithm 1 with $u_i^{n+1}=v_i^{n+1}-v_j^n,\ i,j=1,2,\ j\neq i$. If the decomposition is done with m members with $m\geq 2$ then u^{n+1} is found by solving

$$b(u_i^{n+1}-u_i^n,\hat{u}_i)+a(u_i^{n+1}-u_i^n+\sum_{j=1}^m u_j^n,\hat{u}_i)=(f,\hat{u}_i) \quad \forall \hat{u}_i\in V_i \tag{6}$$

Theorem 1. *(J.L. Lions[4]) We assume (1-2). Then Algorithm (6) is convergent in the following sense: as $n\to\infty$, $u_i^n\to u_i^*$ with $u_1^*+u_2^*=u$ solution of (1) and the decomposition is uniquely defined by*

$$\begin{aligned} &(\beta+A)u_1=\frac{1}{2}(\beta+A)(u+u_1^0-u_2^0) \quad \textit{in} \quad \Omega_1\cap\Omega_2, \quad u_1|_{S_1}=0,\ \ u_1|_{S_2}=u\\ &(\beta+A)u_2=\frac{1}{2}(\beta+A)(u+u_2^0-u_1^0) \quad \textit{in} \quad \Omega_1\cap\Omega_2, \quad u_2|_{S_2}=0,\ \ u_2|_{S_1}=u\\ &Au_i=f \quad \textit{in} \quad \Omega_i\backslash\Omega_1\cap\Omega_2, \quad u_i|_{\partial\Omega_i}=0 \end{aligned} \tag{7}$$

2 Discretization

Let $\mathcal{T}_{1h}$ (resp $\mathcal{T}_{2h}$) be a triangulation of Ω_1 (resp Ω_2), quasi-uniform [2], in the sense that, if h_M and h_m are the maximum and minimum edges in $\mathcal{T}_{1h}$, and H_M and H_m are the maximum and minimum edges in $\mathcal{T}_{2h}$, then there exists two constants C_{1T} and C_{2T} such that $h_M \leq C_{1T}h_m \qquad H_M \leq C_{2T}H_m$. Without loss of generality we can also assume, that $h_M \leq H_M$. For clarity we assume that the Ω_i are polygonal and that $a(,)$ is the Laplace operator with Dirichlet conditions. Let V_{1h} and V_{2h} be two Lagrange conforming continuous finite element approximation spaces of order p of $V_1=H_0^1(\Omega_1)$ and $V_2=H_0^1(\Omega_2)$. Then the discrete version of Algorithm 2 is to find for i=1,2, $u_{ih}^{n+1}\in V_{ih}$ such that $\forall v_{ih}\in V_{ih}$

$$\int_{\Omega_i}(\beta(u_{ih}^{n+1}-u_{ih}^n)v_{ih}+\nabla(u_{1h}^{n+1}+u_{2h}^n)\nabla v_{ih})=\int_{\Omega_i} f v_{ih}, \tag{8}$$

Theorem 2. *(see Hecht et al. [4]) Assume that the solution of (1) is in* $H^{p+1}(\Omega)$ *for some* $p \geq 1$. *Assume that in (7)* $u_i|_{\Omega_i} \in H^{p+1}(\Omega_i)$. *If* $u_h = \lim(u^n_{1h} + u^n_{2h})$ *is computed with Lagrange conforming finite elements of order* p*, then*

$$\|u - u_h\|_{1,\Omega} \leq Ch^p(\|u_1\|_{p+1,\Omega_1} + \|u_2\|_{p+1,\Omega_2}). \tag{9}$$

3 Numerical Quadrature

As such, the scheme is too costly to implement because it requires the intersection of triangulations. Recall that the quadrature formula with integration points at the vertices is exact for polynomials of degree less than or equal to one. In particular, for a given triangle $\hat{T}$ one has

$$\int_{\hat{T}} g\,dxdy \;=\; \frac{|\hat{T}|}{3} \sum_{i=1,2,3} g(q_i) \quad \forall g \in P_1(\hat{T}). \tag{10}$$

Hence we introduce the following quadrature rule.

$$\begin{aligned}(\nabla u, \nabla v)_h := \textstyle\sum_{T\in\mathcal{T}_{1h}} \frac{|T|}{3} \sum_{i=1,2,3} \frac{\nabla(u_{|T})\cdot\nabla v}{I_{\Omega_1}+I_{\Omega_2}}\Big|_{q_i(T)} \\ + \textstyle\sum_{K\in\mathcal{T}_{2h}} \frac{|K|}{3} \sum_{j=1,2,3} \frac{\nabla(v_{|K})\cdot\nabla u}{I_{\Omega_1}+I_{\Omega_2}}\Big|_{q_j(K)}.\end{aligned} \tag{11}$$

where $I_{\Omega_j}(x) = 1$ if $x \in \Omega_j$ and zero otherwise ($j = 1, 2$). The notation $\nabla(u_{|T})$ is used to indicate that we first restrict the function u to T, and then we compute its gradient (which is actually constant in T). A similar interpretation holds for $\nabla(v_{|K})$. With such definitions we propose to solve the discrete problems:

- Find $u^{n+1}_{ih} \in V_{ih}$ such that $\forall v_{ih} \in V_{ih}$

$$\begin{aligned}b(u^{n+1}_{1h} - u^n_{1h}, \hat{u}_{1h}) + a_h(u^{n+1}_{1h} + u^n_{2h}, \hat{u}_{1h}) = (f, \hat{u}_{1h}) \quad \forall \hat{u}_{1h} \in V_{1h} \\ b(u^{n+1}_{2h} - u^n_{2h}, \hat{u}_{2h}) + a_h(u^n_{1h} + u^{n+1}_{2h}, \hat{u}_{2h}) = (f, \hat{u}_{2h}) \quad \forall \hat{u}_{2h} \in V_{2h}\end{aligned} \tag{12}$$

Clearly these define u^{n+1}_{ih} uniquely. At convergence the problem solved is

- Find $u_{ih} \in V_{ih}$ such that $\forall \hat{u}_{ih} \in V_{ih}$

$$a_h(u_{1h} + u_{2h}, \hat{u}_{1h} + \hat{u}_{2h}) = (f, \hat{u}_{1h} + \hat{u}_{2h}). \tag{13}$$

Under a mild assumption on the triangulations this discrete problem has a unique solution at least when linear elements are used ($p = 1$):

each vertex of $\mathcal{T}_{1h}$ is internal to a triangle K of $\mathcal{T}_{2h}$, and conversely. (14)

This is because of the coercivity of the bilinear form and of the uniqueness of the decomposition $u_h = u_{1h} + u_{2h}$:

Theorem 3. *(Brezzi)*
Assume (14) holds. If two functions $u_{ih} \in V_{ih},\ i = 1, 2$ *coincide on a connected subset* $\mathcal{X}$ *of* $\Omega_1 \cap \Omega_2$*, then both* u_{ih} *are linear (not just piecewise linear) in* $\mathcal{X}$*. Furthermore*

$$a_h(u_{1h} + u_{2h}, u_{1h} + u_{2h}) \geq c\|u_{1h} + u_{2h}\|^2 \textit{ for all } u_{ih} \in V_{ih}$$

One more property is needed, the continuity of a_h, and then we can apply Strang's lemma and obtain the following estimate:

Proposition 1. *(see [4]) Assume that the triangulations of Ω_1 and Ω_2 are compatible in the sense that they give a coercive bilinear form. Assume that a_h is uniformly continuous for all h. Then the error between the approximate problem (13) and the continuous problem is*

$$\|u - u_h\| < Ch(\|u_1\|_{2,\Omega_1} + \|u_2\|_{2,\Omega_2})$$

4 Continuity of the Approximate Bilinear Form

4.1 The One Dimensional Case

We begin with the mono dimensional case because the proof is easier to follow. The same argument will be extended to 2D.

Proposition 2. *In one dimension the constant of continuity C in*

$$\begin{aligned}&|\nabla u_H + \nabla u_h|_h \le C|\nabla u_H + \nabla u_h| \\ &\textit{satisfies} \quad C^2 \le \frac{1}{2}\max\{\max_{i\in K}\frac{|x_{i+1}-x_i|}{|x_i - X_{j(i)}|}, \max_{i\in L}\frac{|X_{i+1}-X_i|}{|X_i - x_{j(i)}|}\}\end{aligned} \tag{15}$$

where K (resp. L) is the set of i such that $j(i)$ exists with $X_{j(i)} \in [x_i, x_{i+1}]$ (resp $x_{j(i)} \in [X_i, X_{i+1}]$). Consequently C is bounded by the square root of half the largest interval length divided by the smallest distance between two vertices.

Proof For any real valued function f,

$$\max_{u_h,u_H} f(\nabla u_H + \nabla u_h) \le \max_{U_h,U_H} f(U_H + U_h)$$

where u_h, u_H are real valued continuous- piecewise linear functions on their mesh and U_H, U_h are piecewise constant vector valued on their meshes, because every ∇u is a U and the opposite is not true when boundary conditions exist at both ends. Denote $V = U_H + U_h$. As V is piecewise constant, by definition

$$\begin{aligned}4|\ V|_h^2 &= \sum_i |x_{i+1}-x_i|(|V|(x_i^+)^2 + |V|(x_{i+1}^-)^2) + \sum_j |X_{j+1}-X_j|(|V|(X_j^+)^2 + |V|(X_{j+1}^-)^2) \\ 2|\ V|_0^2 &= \sum_{i,j\in K} |X_j - x_i|(|V|(X_j^-)^2 + |V|(x_i^+)^2) + \sum_{i,j\in L} |x_i - X_j|(|V|(X_j^+)^2 + |V|(x_i^-)^2) \\ &+ \sum_{i\in I} |x_{i+1}-x_i|(|V|(x_i^+)^2 + |V|(x_{i+1}^-)^2) + \sum_{j\in J} |X_{j+1}-X_j|(|V|(X_j^+)^2 + |V|(X_{j+1}^-)^2)\end{aligned} \tag{16}$$

where I, J are the set of intervals completely inside an interval of the other mesh, i.e.

$$I = \{i \,:\, \exists j \; s.t. \, [x_i, x_{i+1}] \subset [X_j, X_{j+1}\} \qquad J = \{j \,:\, \exists i \; s.t. \, [X_j, X_{j+1}] \subset [x_i, x_{i+1}]\}$$

Denote by N the set of values of V_k of V right or left of x_i or X_j. As $f(V) = |V|_h^2/|V|_0^2$ we see that it is of the type $f(V) = \sum_{k\in N} \alpha_k |V|_k^2 \,/\, \sum_{k\in N} \beta_k |V|_k^2$ with α_i equal to a fourth of $x_{i+1} - x_i$ or $X_{i+1} - X_i$, and β_i equal half of $x_{i+1} - x_i$ or $X_{i+1} - X_i$ or $x_i - X_{j(i)}$ or $X_i - x_{j(i)}$ a sum of two of those. Of course it is important to notice that all values appear both in the nominator and denominator.
With a change of variable this is also

$$f(W) = \frac{\sum \frac{\alpha_k}{\beta_k} W_k^2}{\sum W_k^2}. \text{ Then } \quad \max f(W) = \max_k \frac{\alpha_k}{\beta_k}$$

Now that this is established we can answer much simply the problem of finding $\max \alpha_k/\beta_k$: it is the largest ratio of coefficients multiplying $V(x_i^{\pm})$ or $V(X_j^{\pm})$ in the expressions for $|V|_h$ and $|V|_0^2$, i.e. in (16).

4.2 The Two Dimensional Case

A similar argument applies in two dimensions. Assume we have two triangulations with triangles $\{T_k\}_1^N$ and $\{t_k\}_1^n$ respectively and vertices Q_i and q_i. Recall that

$$|V|_h^2 = \frac{1}{6}\sum_{k=1}^{N}\sum_{j=1,2,3} |V_{T_k}(Q_{i_j}|^2 |T_k| + \frac{1}{6}\sum_{k=1}^{n}\sum_{j=1,2,3} |V_{t_k}(q_{i_j}|^2 |t_k| \qquad (17)$$

where $i_j,\ j = 1, 2, 3$ are the numbers of the 3 vertices for each triangle. On the other hand the exact value $|V|_0^2$ is

$$|V|_0^2 = \sum_{k,l}\sum_{j=1,2,3} |V_{R_{kl}}(\xi_{kl})|^2 |R_{kl}| \qquad (18)$$

where $R_{kl} = T_k \cap t_l$ and ξ_{kl} is any point in R_{kl}.
For each Q_{i_j} (resp q_{i_j}) in (17) there is a R_{kl} which contains it. For these R let us choose in (18) $\xi_{kl} = Q_{i_j}$ and q_{i_j}. Then for every term in $|V|_h^2$ there is a corresponding term in $|V|_0^2$:

$$\frac{1}{6}|V_{T_k}(Q_{i_j}|^2 |T_k| \text{ in correspondence with } |V_{T_k}(Q_{i_j}|^2 |T_k \cap t_l| \qquad (19)$$

where l is such that $Q_{i_j} \in t_l$; and similarly with q_{i_j}.
However in this construction we will assign as many ξ to a R as the number of vertices it contains. So the safest is to divide by 3 the second term in (19).
Notice that some R do not contain any vertex; if we leave these aside we obtain

$$\frac{|V|_h^2}{|V|_0^2} \le \frac{1}{2}\max_{k,l}\{\frac{\max\{|T_k|, |t_l|\}}{|T_k \cap t_l|} \ :\ T_k \cap t_l \text{ contains at least one vertex } \} \qquad (20)$$

So we have proved the following

Proposition 3. *In two dimensions, the constant of continuity between the approximate norm* $|\nabla u_H + \nabla u_h|_h$ *and the exact one is proportional to the square root of the biggest ratio of area between a triangle* T *and one of its polygons* $T \cap t$ *where* t *is a triangle of the other triangulation containing a vertex of* T.

The proof is similar, except that in the exact norm there are terms which do not exist in the approximate norm; but these are positive and appear in the denominator of the expression which bounds C.

Remark 1. Consider the case where each triangle of the mesh h has no more than one vertex of the mesh H inside. Assume that this vertex is near the center of the triangle (or segment in one-D). Assume that all angles between two intersecting edges are bounded away from 0 and π when $h, H \to 0$ and that H/h and h/H do not tend to 0. Then C is strictly posivite in the limit. However it is difficult in practice to insure that no angle tend to zero when the mesh is refined

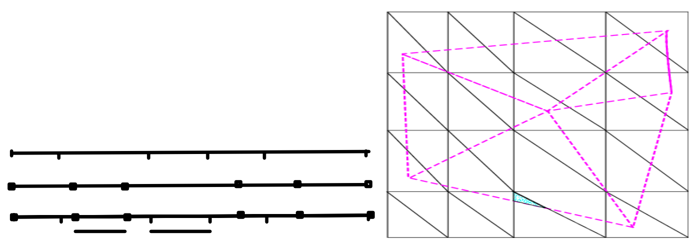

Fig. 1. Top: Two meshes in 1D and the intersected mesh. Two intervals have been singled out as they are strictly inside an interval of the other mesh; the continuity constant is proportional to the ratio of the smallest interval in the intersected mesh to the biggest interval in both meshes neighbor to that smallest one. Bottom: The continuity constant is proportional to the smallest polygon containing a vertex (shown with a texture) divided by the area of the biggest neighbor triangle in both meshes. Notice that some edges pass right through a vertex in this example, so if one mesh is shifted slightly the continuity constant estimate suddenly deteriorates.

5 Numerical Test

In all the numerical tests that follow, errors are evaluated on the intersected mesh, using exact quadrature formula. The problem to solve is $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$. Data are chosen so that $u(x, y) = \sin(x) \times \sin(y)$.

5.1 Exact quadrature

This formula is introduced so as to give an exact computation for integral like $\int_{T_h \cap T_H} \Phi\Psi$. Where Φ and Ψ see FIG.2 below are $P1$-lagrange functions on the

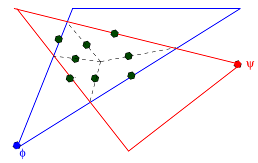

Fig. 2. Quadrature points for exact evaluation of $\int_{T_h \cap T_H} \Phi\Psi$.

triangle T_h and T_H respectively. It is based on the intersection of the two meshes. Ω_1 is a circle of radius 1 centered at $(0,0)$ and Ω_2 is the square $(-0.5, 0.5)^2$. Ω_2 is going to be meshed with uniform triangles so that by diadic refinement, order of convergence should be easily evaluated see Table1.

5.2 First quadrature formula

Table1 displays the results when(11) is used. Notice that by taking $u \in V_h, v \in V_h$, we don't recover the ordinary approximated bilinear form for the Laplace equation on the domain Ω_1. So for a parallel implementation of (12), instead, we must find $u^{n+1} \in V_{0h}$ such that (here $b \equiv 0$), $\forall \hat{v} \in V_{0h}(\Omega_1)$

$$(\nabla u_1^{n+1}, \nabla \hat{v})_h = (f, \hat{v}) - (\nabla u_2^n, \nabla \hat{v})_{hH} - \frac{1}{2}(\nabla u_1^n, \nabla \hat{v})_h + \frac{1}{2}(\nabla u_1^n, \nabla \hat{v})_H.$$

Here $(.,.)_h, (.,.)_H$ don't need quadrature. For the numerical experiments, we have taken $\Omega_2 = (-2,3) \times (-3,2)$ and $\Omega_1 = (-\frac{4}{3}, \frac{5}{3}) \times (-\frac{5}{3}, \frac{4}{3})$.

5.3 Second quadrature formula

On the way, we have also tried, for $u_1, v_1 \in V_h,\ u_2, v_2 \in V_H$

$$\begin{aligned}
(\nabla u_1, \nabla v_2)_{hH,\Omega_1\cap\Omega_2} &:= \sum_{K\in\mathcal{K}_H} \frac{|K|}{3} \sum_{j=1,2,3} \left(\nabla(u_1)\cdot\nabla(v_2|_K)\right)(q_j(K)) \\
(\nabla u_2, \nabla v_1)_{Hh,\Omega_1\cap\Omega_2} &:= \sum_{T\in\tau_h} \frac{|T|}{3} \sum_{j=1,2,3} \left(\nabla(u_2)\cdot\nabla(v_1|_T)\right)(q_j(T)).
\end{aligned} \tag{21}$$

5.4 Schwarz algorithm with quadrature

Finally, to compare with Schwarz' algoritm, let $\pi_{hH} : V_h \mapsto V_H$ and $\pi_{Hh} : V_H \mapsto V_h$ the P^1 interpolation operators. Then the Schwarz method is implemented as

$$\begin{cases} (\nabla(u^{n+1} + \pi_{Hh} v^n), \nabla\hat{u})_h = (f, \hat{u})_h \quad \forall \hat{u} \in V_{0h} \\ (\nabla(v^{n+1} + \pi_{hH} u^n), \nabla\hat{v})_H = (f, \hat{v})_H \ \forall \hat{v} \in V_{0H} \end{cases} \tag{22}$$

		$u-(u_1+u_2)$			
$N1$	$N2$	L^2 error	rate	∇L^2 error	rate
Exact Quadrature					
10	5	$1.54E-02$	–	$2.25E-01$	–
20	10	$3.78E-03$	2.02	$1.11E-01$	1.02
40	20	$8.24E-04$	2.2	$5.03E-02$	1.15
First Quadrature					
3	5	$4.64E-01$	–	$1.00E-00$	–
6	10	$8.18E-02$	2.50	$5.44E-01$	0.89

		$u-(u_1+u_2)$			
$N1$	$N2$	L^2 error	rate	∇L^2 error	rate
Second Quadrature					
10	5	$1.85E-02$	–	$2.32E-01$	–
20	10	$5.66E-03$	1.71	$1.16E-01$	1.00
40	20	$1.03E-03$	2.45	$5.34E-02$	1.12
Schwarz overlapping					
10	5	$1.68E-02$	–	$2.29E-01$	–
20	10	$3.49E-03$	2.26	$1.09E-01$	1.06
40	20	$9.15E-04$	1.93	$5.13E-02$	1.09

Table 1. Numerical L^2 errors, and convergence rate, for P1 polynomials with different quadrature formula. $N_i, i=1,2$ is the number of vertices on the boundary of the domain Ω_i.

Conclusion

The results show that the first quadrature formula has optimal errors numerically but the results are very sensitive to the position of the grid points. Good results are obtained with the second quadrature formula, which is also easy to implement in 3D but no error analysis are available yet.

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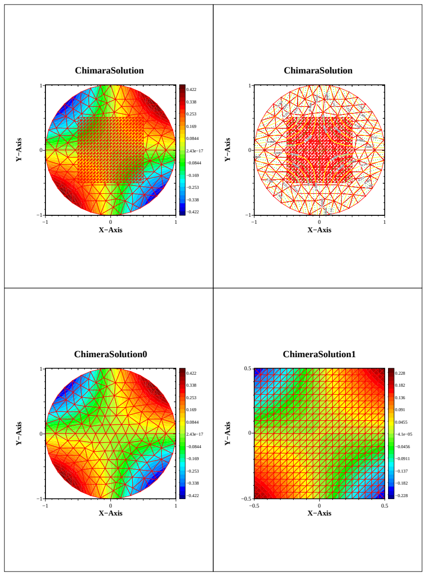

Fig. 3. Chimera solution of test case with exact quadrature formula. Bottom : solution on each subdomain.

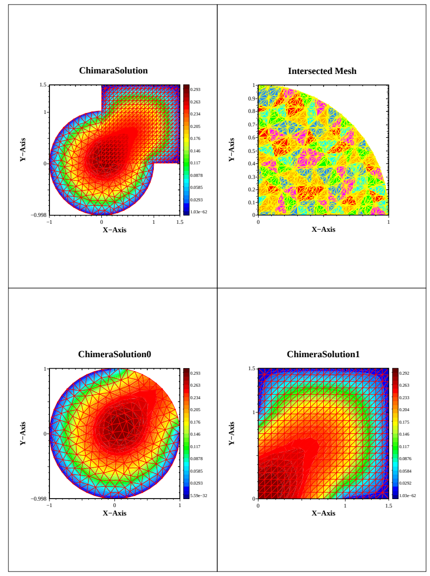

Fig. 4. Chimera solution of ($\Delta u = 1 \; on \; \Omega, \; u = 0 \; on \; \partial\Omega$) with second quadrature formula. Top right: intersected mesh. Bottom : solution on each subdomain.