An Iterative Substructuring Method for Mortar Nonconforming Discretization of a Fourth-Order Elliptic Problem in two dimensions

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Summary. In this paper we consider an iterative substructuring method for solving system of equations arising from mortar Morley finite element discretization of a model fourth order elliptic problem in 2D. The parallel preconditioner for the interface problem is introduced using Additive Schwarz Method framework. The method is quasi-optimal i.e. the number of CG iterations for the preconditioned problem grows polylogarithmically as the sizes of the meshes decrease and it is independent of the jumps of the coefficients.

1 Introduction

The discretization methods for partial differential equations are usually built on a mesh in a uniform way, however sometimes it is necessary to develop discretization methods which allow us to apply different type of discretization techniques in subdomains. The mortar method introduced in [BMP94] is a domain decomposition method which enable us to introduce independent meshes or discretization methods in non-overlapping subdomains. A general presentation of mortar method in the two and three dimensions for elliptic boundary value problems of second order can be found e.g. in [BMP94, BBM97, Woh01], see also references therein. Mortar approach for discretizations of fourth order elliptic problems was studied in [Bel97] where locally spectral discretizations were utilized, in [Lac98] for DKT local discretizations, and in [Mar02] for HCT and Morley finite element discretizations. Many parallel algorithms for solving a discrete problem were also developed, see e.g. [AMW99, Mar99, Woh01] and the references therein.

In this paper we consider a mortar nonconforming Morley discretization of the fourth order elliptic problems. This discretization method first was

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proposed in [Mar02], there are also error bounds there. A multigrid algorithm for mortar Morley discretization of plate bending problem was discussed in [XLC02] (in a bit different mortar settings).

To our knowledge no domain decomposition methods for solving the discrete problems obtained by this type of discretization was discussed in literature.

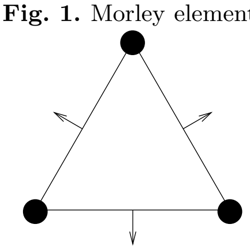
Our method is a substructuring one i.e. we first eliminate the unknowns related to degrees of freedom interior to subdomains (interior in a special sense) and then propose a parallel preconditioner based on Additive Schwarz abstract scheme (ASM) for the derived system of equations, cf. e.g. [TW05]. We introduce local subspaces which form a decomposition of the discrete space. Then the ASM abstract theory allows us to construct a parallel preconditioner and prove the condition number estimates of the preconditioned problem.

In our case we introduce a subdomain based coarse space and edge base spaces. The condition number of the arising preconditioner is proportional to $(1 + \log(H/h))$ where h is the minimum of the local mesh sizes and H is the maximum of the diameters of the subdomain and is independent of the jumps of the coefficients.

2 Discrete space

We first assume that we have a polygonal domain Ω on the plane which is divided into non-overlapping subdomains Ω_k that form a coarse decomposition i.e. $\overline{\Omega} = \bigcup_{k=1}^N \overline{\Omega}_k$ and $\overline{\Omega}_l \cap \overline{\Omega}_k$ is an empty set, a common edge or vertex. We assume a shape regularity of that decomposition in the sens of Section 4 in [BS99] and let $H = \max_k H_k$ for $H_k = \text{diam } \Omega_k$.

Fig. 1. Morley element.



The model differential problem is to find $u^* \in H_0^2(\Omega)$ such that

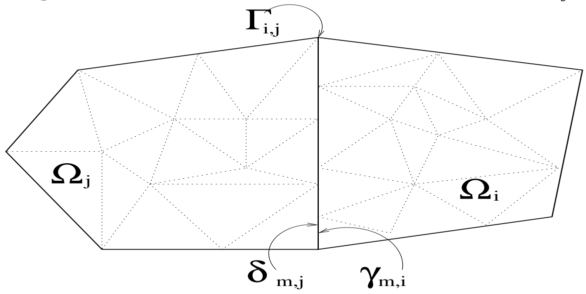
$$a(u^*, v) = f(v) \quad \forall v \in H_0^2(\Omega), \tag{1}$$

where $a(u, v) = \sum_{k=1}^N a_k(u, v)$ for $a_k(u, v) = \rho_k \int_{\Omega_k} \sum_{|\alpha|=2} \partial^\alpha u \; \partial^\alpha v \; dx$. Here $\rho_k > 0$ is a constant, $\alpha = (\alpha_1, \alpha_2), (\alpha_k \geq 0)$ is a multi-index and

$|\alpha| = \alpha_1 + \alpha_2$ is the length of this multi-index. Of course we have that $a_k(u,u)$ is equivalent to $|u|^2_{H^2(\Omega_k)}$. In a subdomain Ω_k we introduce an independent quasiuniform triangulation $T_h(\Omega_k)$ made of triangles with a parameter $h_k = \max_{\tau \in T_h(\Omega_k)} \operatorname{diam}(\tau)$. Note that each interface (a common edge of two substructures) $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$ inherits 1D triangulations $T^i_{h_i}(\Gamma_{ij})$ and $T^j_{h_j}(\Gamma_{ij})$ from the respective triangulations of Ω_i and Ω_j, cf. Figure 2.

In each Ω_k we introduce a nonconforming local Morley finite element space $X_h(\Omega_k)$ formed by piecewise quadratic functions which are continuous at all vertices of all triangles from $T_h(\Omega_k)$, have continuous normal derivatives at the midpoints of all edges of elements from $T_h(\Omega_k)$, and have all respective degrees of freedom related to vertices and midpoints on $\partial\Omega_k \cap \partial\Omega$ equal to zero, cf. Figure 1. We introduce a global space $X_h(\Omega) = \prod_{k=1}^N X_h(\Omega_k)$. We now

Fig. 2. Master and slave sides of of an interface Γ_{ij}.

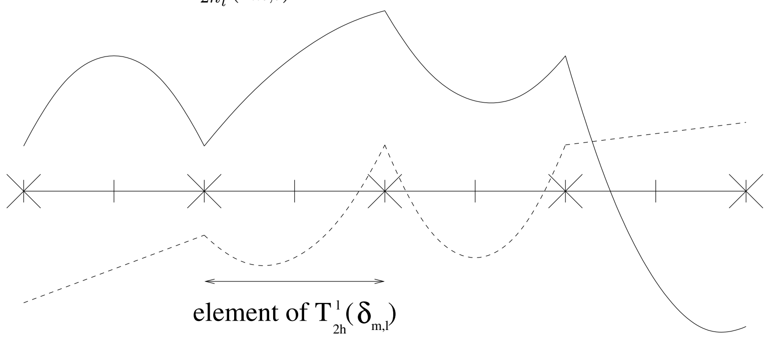


have to choose one side of Γ_{ij} as a master (mortar) one denoted by $\gamma_{m,i}$ associated with Ω_i and the other one as a slave one (nonmortar) denoted by $\delta_{m,j}$ (associated with Ω_j) according to the rule $\rho_i \geq \rho_j$, cf. Figure 2. An important role will play an interface $\Gamma = \bigcup_{k=1}^N \overline{\partial\Omega_k \setminus \partial\Omega}$. We also have to add a technical assumption that $h_i \leq h_j$ due to the proof technique. This assumption is necessary for the proofs of some technical results and is due to the fact that any local Morley finite element function is not sufficiently regular. The other side of $\Gamma_{ij} \subset \partial\Omega_j$ is called slave (nonmortar) and is denoted by $\delta_{m,j}$. Because we assume that $h_i \leq h_j$ and both triangulations are quasiuniform, we can also assume that the two side elements of the slave triangulation $T^j_{h_j}(\delta_{m,j})$, i.e. the ones that touch the ends of $\delta_{m,j}$, are longer than the respective elements of the master (mortar) triangulation $T^i_{h_i}(\gamma_{m,i})$. Let $\gamma_{m,i,h}$ (or $\delta_{m,j,h}$) denotes the set of all midpoints and vertices of $T^i_{h_i}(\gamma_{m,i})$ (or $T^j_{h_j}(\delta_{m,j})$, respectively).

For the simplicity of presentation we also assume that the both 1D triangulations of the interface Γ_{kl}: $T^k_{h_k}(\gamma_{m,k})$ and $T^l_{h_l}(\delta_{m,l})$, have even numbers of the elements.

Let consider $\delta_{m,l}$, then we introduce a coarser $2h_l$ triangulation by subsequently joining together two neighboring elements and get $T^l_{2h_l}(\delta_{m,l})$ - $2h_l$ 1D

Fig. 3. Tangential test space and $2h_l$ interpolant on $T^l_{h_l}(\delta_{m,l})$. Broken line - $v \in M_t^{2h_l}(\delta_{m,l})$, solid line - $I_{2h_l,2}u$, | - the endpoints of elements in $T^l_{h_l}(\delta_{m,l})$, X - the endpoints of elements in $T^l_{2h_l}(\delta_{m,l})$.



triangulation of $\delta_{m,l}$ formed by elements which are the union of two neighboring elements of $T^l_{h_l}(\delta_{m,l})$, cf. Fig. 3. Note that the midpoints of elements of $T^l_{2h_l}(\delta_{m,l})$ are also the vertices of $T_h(\Omega_l)$. Then let $I_{2h_l,2} : C(\Gamma_{kl}) \to C(\Gamma_{kl})$ be a continuous piecewise quadratic interpolant defined on $T^l_{2h_l}(\delta_{m,l})$ and let $M_t^{2h_l}(\delta_{m,l})$ be the space of continuous piecewise quadratic function on $T^l_{2h_l}(\delta_{m,l})$ which are linear on two end elements of $T^l_{2h_l}(\delta_{m,l})$.

We also need another test space related to the trace of normal derivative of finite element functions: $M_n^{h_l}(\delta_{m,l})$ formed by functions piecewise constant on $T^l_{h_l}(\delta_{m,l})$.

The $2h_k$ triangulation of the master $\gamma_{m,k}$: $T^k_{2h_k}(\gamma_{m,k})$, and an operator $I_{2h_l,2}$ - piecewise quadratic interpolant on $T^k_{2h_k}(\gamma_{m,k})$ are defined analogously on the base of elements of $T^k_{h_k}(\gamma_{m,k})$.

Then for each interface $\Gamma_{kl} = \gamma_{m,k} = \delta_{m,l} \subset \Gamma$ we say that $u_k \in X_h(\Omega_k)$ and $u_l \in X_h(\Omega_l)$ satisfy the mortar conditions if

$$\begin{aligned} \int_{\delta_m} (I_{2h_k,2}u_k - I_{2h_l,2}u_l)\phi \, ds = 0 \qquad \forall \phi \in M_t^{2h_l}(\delta_{m,l}) \\ \int_{\delta_m} (\partial_n u_k - \partial_n u_l)\phi \, ds = 0 \qquad \forall \phi \in M_n^{h_l}(\delta_{m,l}). \end{aligned} \tag{2}$$

Here ∂_n is an outer unit normal derivative to Γ_{mk}.

We now introduce a discrete space V^h as the space formed by all functions from $X_h(\Omega)$ which are continuous at the crosspoints (vertices of the subdomains) and satisfy the mortar conditions (2). Our discrete problem is to find $u_h^* \in V^h$ such that

$$a_H(u_h^*, v) = \sum_{k=1}^N a_{h,k}(u, v) = f(v) \quad \forall v \in V^h, \tag{3}$$

where $a_{h,k}(u,v) = \rho_k \sum_{\tau \in T_k(\Omega_k)} \int_\tau \sum_{|\alpha|=2} \partial^\alpha u \, \partial^\alpha v \, dx$. The problem has a unique solution, cf. [Mar02].

3 ASM method

We first eliminate some unknowns in the interiors of subdomains. Because the Morley element is nonconforming thus there are some functions which has all degrees freedom corresponding to respective vertices or midpoints on $\partial\Omega_k$ equal to zero and still the traces onto a master $\gamma_{m,k}$ may be nonzero. Thus we introduce the set Δ_k which consists of all vertices and midpoints that either are on $\partial\Omega_k$ or are interior to Ω_k and are on boundary of any element such that at least one of its edges is not an end element of any $T_h^k(\gamma_{m,k})$ for any master $\gamma_{m,k} \subset \partial\Omega_k$, i.e. it is a set of nodal points either on $\partial\Omega_k$ or interior to Ω_k and such that a nodal basis function corresponding to a degree of freedom of this nodal point may have nonzero traces onto any master $\gamma_{m,k} \subset \partial\Omega_k$. Then let $X_{h,0}(\Omega_k) = \{v \in X_h(\Omega_k) : v(p) = \partial_n v(m) = 0$ for all vertices p and midpoints m in $\Delta_k\}$. We excluded the end elements of $T_h^k(\Gamma_{m,k})$ in the definition of Δ_k because of our condition on the length of the end elements on the interface, see above. The situation is analogous to the case of mortar Crouzeix-Raviart element, cf. [Mar99] where the similar set was introduced.

Each $u \in X_h(\Omega_k)$ is split into two $a_{h,k}$ orthogonal parts: $P_k u$ and discrete biharmonic part of u: $H_k u = u - P_k u$ defined by

$$\begin{cases} a_{h,k}(H_k u, v) = 0 & \text{for all } v \in X_{h,0}(\Omega_k) \\ H_k u(x) = u(x) & \text{for all vertices } x \in \Delta_k \\ (\partial_n H_k u)(m) = \partial_n u(m) & \text{for all midpoints } m \in \Delta_k. \end{cases} \tag{4}$$

Then we define $Pu = (P_1 u, \ldots, P_N u)$ and $Hu = u - Pu$ discrete biharmonic in all subdomains part of u. We also set

$$\tilde{V}_h = HV_h = \{u \in V_h : u \text{ is discrete biharmonic in all } \Omega_k\} \tag{5}$$

Note that each function in $\tilde{V}_h$ is uniquely defined by the values of all degree of freedoms associated with nodal points of $\bigcup_{k=1}^N \Delta_k \setminus (\bigcup_{\delta_{m,j} \subset \Gamma} \delta_{m,j,h})$ as the values of degrees of freedom corresponding to nodes on nonmortar (slave) are set by the mortar conditions (2) and the values of degrees of freedom of nodes interior to subdomains (i.e. not in Δ_k) are set by (4).

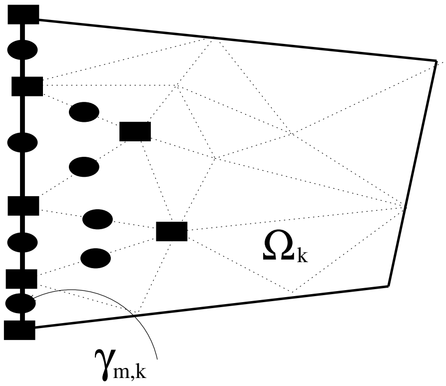
Note that all $P_k u_h^*$ can be precomputed in parallel and it remains to calculate $\tilde{u}_h^* = Hu_h^* \in \tilde{V}_h$ such that

$$a_H(\tilde{u}_h^*, v) = f(v) \quad \forall v \in \tilde{V}_h. \tag{6}$$

4 ASM method

Here we describe our Additive Schwarz method for solving (6). We use an abstract scheme of ASM method, cf. [TW05], i.e. in terms of the decomposition of $\tilde{V}_h$ into subspaces, we also need bilinear forms defined on these subspaces. We first introduce $\Delta_{\gamma_{m,k}} \subset \Delta_k$, cf.[Mar99]: the set of these vertices and mid-

Fig. 4. The set $\Delta_{\gamma_{m,k}}$. The midpoints are denoted by circles and the vertices by squares.



points that are either in $\gamma_{m,k,h}$ or are interior to Ω_k and are on the boundary of elements $e \in T_h(\Omega_k)$ such that at least one edge of this triangle e is contained in $\overline{\gamma}_{m,k} \subset \partial\Omega_k$ and this edge is neither of the two end elements of $T_h^k(\gamma_{kl})$, cf. Figure 4.

Then we introduce $V_{\gamma_{m,k}}$ as the subspace of $\tilde{V}_h$ formed by functions such that the respective degrees of freedom related to all vertices and midpoints in $\bigcup_{l=1}^N \Delta_l \setminus \Delta_{\gamma_{m,k}}$ are equal to zero. In nodal points on slaves and interior to subdomains (not in Δ_k) the values of the respective degrees of freedom are determined by (2) and (4), respectively.

Now we define a coarse space V_0. It is sufficient to define the values of normal derivatives of $u \in V_0$ at midpoints and values of u at vertices in $\bigcup_l \Delta_l = \bigcup_{\gamma_{m,k}\subset\partial\Omega_k} \Delta_{\gamma_{m,k}}$. Note that $\Delta_{\gamma_{m,k}} \cap \Delta_{\gamma_{s,k}} = \emptyset, m \neq s$. Let $V_0 \subset \tilde{V}_h$ be formed by all functions $u \in \tilde{V}_h$ such that for any master (mortar) $\gamma_{s,k} \subset \partial\Omega_k$ there exists linear polynomial p_s for which it holds that

$$\begin{aligned} &u(x) = p_s(x) && \text{for a vertex } x \in \Delta_{\gamma_{m,k}} \\ &(\partial_n u)(m) = \partial_n p_s(m) && \text{for a midpoint } m \in \Delta_{\gamma_{m,k}}. \end{aligned} \tag{7}$$

Because $u \in V_0$ is continuous at the crosspoints thus it is easy to see that the dimension of V_0 is equal to the number of crosspoints (vertices of subdomain not on $\partial\Omega$) and the number of masters $\gamma_m \subset \Gamma$.

Again for the simplicity of presentation we assume that the bilinear forms for all subspaces equal to $a_H(u,u)$.

Then we can define orthogonal projections: $P_0 : V_0 \to \tilde{V}_h$ and $P_m : V_{\gamma_m} \to \tilde{V}_h$ as

$$a_H(P_0 u, v) = a_H(u, v) \quad \forall v \in V_0,$$
$$a_H(P_m u, v) = a_H(u, v) \quad \forall v \in V_{\gamma_m}.$$

Let $P = P_0 + \sum_{k=1}^N P_m$. Next we replace problem (6) by

$$P\tilde{u}_h^* = g, \tag{8}$$

where $g = g_0 + \sum_{\gamma_m \subset \Gamma} g_m$ for $g_0 = P_0\tilde{u}_h^*$ and $g_m = P_m\tilde{u}_h^*$.

We should point out that g_0, g_m can be computed without knowing $\tilde{u}_h^*$. Then we have the following result:

Theorem 1. *For any $u \in \tilde{V}_h$ it holds that*

$$c(1 + \log(H/\underline{h}))^{-2} a_H(u,u) \le a_H(Pu,u) \le C a_H(Pu,u),$$

where C, c are positive constant independent of H and any h_k and $H = \max_k H_k$ and $\underline{h} = \min_k h_k$.

Sketch of the proof.

The proof of this theorem is based on the abstract ASM scheme, cf. e.g. [TW05]. We will give only a brief sketch of the proof here. It is enough to check three key assumptions, cf. Th. 2.7, p. 43 in [TW05]. In our case the assumption II (Strengthened Cauchy-Schwarz Inequalities), cf. Ass. 2.3, p.40 in [TW05], is satisfied with the constant independent of the number of subdomains by the coloring argument and the constant ω in the assumption III (Local Stability), cf. Ass. 2.4, p.40 in [TW05], is equal to one as P_0 and P_m are orthogonal projections. It remains to prove assumption I (Stable Decomposition), cf. Ass. 2.2, p.40 in [TW05], i.e. we have to prove that there exists a positive constant such that for any $u \in \tilde{V}_h$ there are $u_0 \in V_0$ and $u_m \in V_m$ for $\gamma_m \subset \Gamma$ such that $u = u_0 + \sum_{\gamma_m \subset \Gamma} u_m$ and

$$a_H(u_0,u_0) + \sum_{\gamma_m \subset \Gamma} a_H(u_m,u_m) \le C(1 + \log(H/\underline{h}))^2 a_H(u,u). \tag{9}$$

We first define this decomposition. Let $u \in \tilde{V}_h$ and let define $u_0 \in V_0$. It is sufficient to define the values of respective degrees of freedom at each $\Delta_{\gamma_{s,k}}$ associated with each mortar $\gamma_{s,k} \subset \Gamma$. Let a, b be the ends of $\gamma_{s,k} \subset \partial\Omega_k$ and $\overline{u}_{\gamma_{s,k}} = \frac{1}{N_k}\sum_{m \in \gamma_{s,k,h}} \partial_n u(m)$, where the sum is taken over all midpoints on $\gamma_{s,k}$ and N_k is the number of those midpoints on $\gamma_{s,k}$. Then for any mortar $\gamma_{s,k} \subset \partial\Omega_k$ we introduce linear polynomial p_s such that

$$p_s(a) = u(a) \qquad p_s(b) = u(b) \qquad \partial_n p_s = \overline{u}_{\gamma_{s,k}}.$$

Note that the linear polynomial p_s is properly defined by these three conditions. Then we define $u_0 \in V_0$ by setting the values of respective degrees of freedom associated with the vertices and midpoints in $\Delta_{s,k}$ as

$$\begin{aligned} u_0(x) &= p_s(x) \quad \text{for } x \text{ a vertex in } \Delta_{\gamma_{s,k}} \\ \partial_n u_0(m) &= \partial_n p_s(m) \quad \text{for } m \text{ a midpoint in } \Delta_{\gamma_{s,k}} \end{aligned}$$

Thus u_0 is properly defined. Next we define $u_s \in V_{\gamma_{s,k}}$. Again it is sufficient to determine the values of respective degrees of freedom at nodal points in $\Delta_{\gamma_{s,k}}$ for all masters $\gamma_{s,k} \subset \Gamma$. Let $w = u - u_0$ and let:

$$u_s(x) = w(x) \quad \text{for } x \text{ a vertex in } \varDelta_{\gamma_{s,k}}$$
$$\partial_n u_s(m) = \partial_n w(m) \quad \text{for } m \text{ a midpoint in } \varDelta_{\gamma_{s,k}}$$

and $u_s(x) = \partial_n u_s(m) = 0$ for all vertices x and midpoints m in $\bigcup_{k=1}^N \varDelta_n \setminus \varDelta_{\gamma_{s,k}}$. It is obvious that we have $u = u_0 + \sum_{\gamma_{m,k}} u_m = u_0 + w = u$.

Then, using a local equivalence operator introduced in [BS99], some technical tools (modified) from [Mar99] and following the lines of proofs of [Mar05] we can prove (9).

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