1

Balancing Domain Decomposition Methods for Mortar Coupling Stokes-Darcy Systems

Juan Galvis[1] and Marcus Sarkis[2]

[1] Instituto Nacional de Matemática Pura e Aplicada, Estrada Dona Castorina 110, CEP 22460320, Rio de Janeiro, Brazil. jugal@fluid.impa.br
[2] Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brazil, and Worcester Polytechnic Institute, Worcester, MA 01609, USA. msarkis@fluid.impa.br Rio de Janeiro, Brazil. msarkis@fluid.impa.br

1.1 Introduction and Problem Setting

We consider Stokes equations in the fluid region Ω_f and Darcy equations for the filtration velocity in the porous medium Ω_p, and coupled at the interface Γ with adequate transmission conditions. Such problem appears in several applications like well-reservoir coupling in petroleum engineering, transport of substances across groundwater and surface water, and (bio)fluid-organ interactions. There are some works that address numerical analysis issues such as: inf-sup and approximation results associated to the continuous and discrete formulations Stokes-Darcy systems [LSY03, Gal04, GS04, GS05] and Stokes-Laplacian systems [QVZ02, DQ03], mortar discretizations analysis [RY05, GS05], preconditioning analysis for Stokes-Laplacian systems [DQ04, D04, D04b]. Here we are interested on preconditionings for *Stokes-Mortar-Darcy* with *flux boundary conditions*, therefore the global system as well as the local systems require flux compatibilities. Here we propose two preconditioners based on balancing domain decomposition methods [Man93, PW02, DP03]: in the first one the energy of the preconditioner is controlled by the Stokes system while the second one is controlled by the Darcy system. The second is more interesting because it is scalable for the parameters faced in practice.

Let Ω_f, $\Omega_p \subset \Re^n$ be polyhedral subdomains, $\Omega = \text{int}(\overline{\Omega}_f \cup \overline{\Omega}_p)$ and $\Gamma = \text{int}(\partial\Omega_f \cup \partial\Omega_p)$, with outward unit normal vectors on $\partial\Omega_j$ denoted by $\boldsymbol{\eta}_j$, $j = f, p$. The tangent vectors of Γ are denoted by $\boldsymbol{\tau}_1$ $(n = 2)$, or $\boldsymbol{\tau}_l$, $l = 1, 2$ $(n = 3)$. Define $\Gamma_j := \partial\Omega_j \setminus \Gamma$, $j = f, p$. Fluid velocities are denoted by $\boldsymbol{u}_j : \Omega_j \to \Re^n$, $j = f, p$. Pressures are $p_j : \Omega_j \to \Re$, $j = f, p$. We have:

$$\begin{array}{cc} \text{Stokes equations} & \text{Darcy equations} \\ \left\{ \begin{array}{rl} -\nabla\cdot T(\boldsymbol{u}_f, p_f) = \boldsymbol{f}_f & \text{in } \Omega_f \\ \nabla\cdot\boldsymbol{u}_f = g_f & \text{in } \Omega_f \\ \boldsymbol{u}_f = \boldsymbol{h}_f & \text{on } \Gamma_f \end{array} \right. & \left\{ \begin{array}{rl} \boldsymbol{u}_p = -\frac{\kappa}{\mu}\nabla p_p & \text{in } \Omega_p \\ \nabla\cdot\boldsymbol{u}_p = g_p & \text{in } \Omega_p \\ \boldsymbol{u}_p\cdot\boldsymbol{\eta}_p = h_p & \text{on } \Gamma_p \end{array} \right. \end{array} \tag{1.1}$$

here $T(\boldsymbol{v},p) := -pI + 2\mu \boldsymbol{D}\boldsymbol{v}$ where μ is the viscosity and $\boldsymbol{D}\boldsymbol{v} := \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ is the linearized strain tensor. κ represents the rock permeability and μ the fluid viscosity. For simplicity on the analysis we assume that κ is a real positive constant. We also impose the compatibility condition (see [GS05])

$$\langle g_f, 1\rangle_{\Omega_f} + \langle g_p, 1\rangle_{\Omega_p} - \langle \boldsymbol{h}_f \cdot \boldsymbol{\eta}_f, 1\rangle_{\Gamma_f} - \langle h_p, 1\rangle_{\Gamma_p} = 0,$$

and the following interface matching conditions across Γ (see [LSY03, DQ03, QVZ02, DQ04] and references therein):

1. **Conservation of mass across Γ:** $\boldsymbol{u}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{u}_p \cdot \boldsymbol{\eta}_p = 0$ on Γ.

2. **Balance of normal forces across Γ:** $p_f - 2\mu \boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\eta}_f = p_p$ on Γ.

3. **Beavers-Joseph-Saffman condition:** This condition is a kind of empirical law that gives an expression for the component of Σ in the tangential direction of $\boldsymbol{\tau}$. It is expressed by:

$$\boldsymbol{u}_f \cdot \boldsymbol{\tau}_j = -\frac{\sqrt{\kappa}}{\alpha_f} 2\boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\tau}_j \quad j = 1, d-1; \text{ on } \Gamma. \tag{1.2}$$

1.2 Weak Formulations and Discretization.

Without loss of generality we consider the case where $\boldsymbol{h}_f = 0$, $h_p = 0$, and $\alpha_f = \infty$ (here we use the energy of α_f-harmonic Stokes and hamonic Laplacian extensions are equivalents idependent of α_f; see [GS05].

The problem is formulated as: *Find* $(\boldsymbol{u}, p, \lambda) \in \boldsymbol{X} \times M \times \Lambda$ *satisfying, for all* $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X} \times M \times \Lambda$:

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) + b_\Gamma(\boldsymbol{v},\lambda) = \ell(\boldsymbol{v}) \\ b(\boldsymbol{u},q) \qquad\qquad\qquad\qquad\quad = g(q) \\ b_\Gamma(\boldsymbol{u},\mu) \qquad\qquad\qquad\quad\;\; = 0, \end{cases} \tag{1.3}$$

where $\boldsymbol{X} = \boldsymbol{X}_f \times \boldsymbol{X}_f := H_0^1(\Omega_f, \Gamma_f)^2 \times \boldsymbol{H}_0(\text{div}, \Omega_p, \Gamma_p)$; $M := L_0^2(\Omega) \subset L^2(\Omega_f) \times L^2(\Omega_p)$. Here $H_0^1(\Omega_f, \Gamma_f)$ denotes the subspace of $H^1(\Omega_f)$ of functions that vanish on Γ_f. Analogously, $\boldsymbol{H}_0(\text{div}, \Omega_p, \Gamma_p)$ denotes the subspace of $\boldsymbol{H}(\text{div}, \Omega_p)$ of functions that its normal trace restricted to Γ_p is zero. The Lagrange multiplier space is $\Lambda := H^{1/2}(\Gamma)$. Also

$$a(\boldsymbol{u},\boldsymbol{v}) := a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + a_p(\boldsymbol{u}_p, \boldsymbol{v}_p), \qquad b(\boldsymbol{v},p) := b_f(\boldsymbol{v}_f, p_f) + b_p(\boldsymbol{v}_p, p_p),$$

and $b_\Gamma(\boldsymbol{v},\mu) := \langle \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f, \mu\rangle_\Gamma + \langle \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p, \mu\rangle_\Gamma$, $\boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X}, \mu \in \Lambda$, where $\langle \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p, \mu\rangle_\Gamma := \langle \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p, E_{\boldsymbol{\eta}_p}(\mu)\rangle_{\partial\Omega_p}$. Here $E_{\boldsymbol{\eta}_p}$ is any continuous lift-in. The bilinear forms a_j, b_j are associated to Stokes equations, $j = f$, and Darcy law, $j = p$. The bilinear for a_f includes conditions 2 and 3 above. The bilinear form b_Γ is the weak version of condition 1 above. For the analysis of this weak formulation and the well-posedness of the problem see [GS05].

From now on we assume that Ω_i, $i = f, p$, are *two dimensional* polygonal subdomains. Let $\mathcal{T}_i^{h_i}$ be a triangulation of Ω_i, $i = f, p$. We do not assume that they match at the interface Γ. For the fluid region, let $\boldsymbol{X}_f^{h_f}$ and $M_f^{h_f}$ be $P2/P1$ triangular Taylor-Hood finite elements and denote $\mathring{M}_f^{h_f} = M_f^{h_f} \cap L_0^2(\Omega_f)$. For the porous region, let $\boldsymbol{X}_p^{h_p}$ and $M_p^{h_p}$ be the lowest order Raviart-Thomas finite elements based on triangles and denote $\mathring{M}_p^{h_p} = M_p^{h_p} \cap L_0^2(\Omega_p)$. We assume in the definition of the discrete velocities that the boundary conditions are included, i.e., for $\boldsymbol{v}_f^{h_f} \in \boldsymbol{X}_f^{h_f}$ we have $\boldsymbol{v}_f^{h_f} = \boldsymbol{0}$ on Γ_f and for $\boldsymbol{v}_p^{h_p} \in \boldsymbol{X}_p^{h_p}$, $\boldsymbol{v}_p^h \cdot \boldsymbol{\eta}_p = 0$ holds on Γ_p.

We choose piecewise constant Lagrange multipliers space:

$$\Lambda^{h_p} := \left\{ \lambda \,:\, \lambda|_{e_j^p} = \lambda_{e_j^p} \;\text{ is constant in each edge } e_j^p \text{ of } \mathcal{T}_p^{h_p}(\Gamma) \right\},$$

i.e., the mortar is on the fluid region side and the slave on the porous region side, and leads to a nonconforming approximation on Λ^{h_p} since piecewise constant functions do not belong to $H^{1/2}(\Gamma)$. Define $\boldsymbol{X}^h := \boldsymbol{X}_f^{h_f} \times \boldsymbol{X}_p^{h_p}$, and

$$\boldsymbol{Z}_\Gamma^h := \left\{ \boldsymbol{v}^h \in \boldsymbol{X}^h \;:\; (\boldsymbol{v}_f^{h_f} \cdot \boldsymbol{\eta}_f + \boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p, \mu)_\Gamma = 0 \;\; \forall \mu \in \Lambda^{h_p} \right\}. \qquad (1.4)$$

1.3 Matrix and Vector Representations

To simplify notations, from now on we drop the subscript h associated to the discrete variables. We consider the following partition of degrees of freedom:

$$\begin{bmatrix} \boldsymbol{u}_I^i \\ p_I^i \\ u_\Gamma^i \\ \bar{p}^i \end{bmatrix} \begin{array}{l} \text{Interior displacements + tangential velocities at } \Gamma, \\ \text{Interior pressures with zero average in } \Omega_i, \\ \text{Interface normal displacements on } \Gamma, \\ \text{Constant pressure in } \Omega_i, \end{array} \qquad i = f, p.$$

Then, we have the following matrix representation of the coupled problem:

$$\left[\begin{array}{cccc|cccc|c}
A_{II}^f & B_{II}^{fT} & A_{\Gamma I}^{fT} & 0 & 0 & 0 & 0 & 0 & 0 \\
B_{II}^f & 0 & B_{I\Gamma}^f & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{\Gamma I}^f & B_{I\Gamma}^{fT} & A_{\Gamma\Gamma}^f & \bar{B}^{fT} & 0 & 0 & 0 & 0 & B_f^T \\
0 & 0 & \bar{B}^f & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & A_{II}^p & B_{II}^{pT} & A_{\Gamma I}^{pT} & 0 & 0 \\
0 & 0 & 0 & 0 & B_{II}^p & 0 & B_{I\Gamma}^p & 0 & 0 \\
0 & 0 & 0 & 0 & A_{\Gamma I}^p & B_{I\Gamma}^{pT} & A_{\Gamma\Gamma}^p & \bar{B}^{pT} & B_p^T \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{B}^p & 0 & 0 \\
\hline
0 & 0 & B_f & 0 & 0 & 0 & -B_p & 0 & 0
\end{array}\right]
\begin{bmatrix} \boldsymbol{u}_I^f \\ p_{\mathrm{i}}^f \\ u_\Gamma^f \\ \bar{p}^f \\ \boldsymbol{u}_I^p \\ p_{\mathrm{i}}^p \\ u_\Gamma^p \\ \bar{p}^p \\ \lambda \end{bmatrix}$$

and in each subdomain (see [PW02, DP03]) given by:

$$\left[\begin{array}{cc|cc} A^i_{II} & B^{iT}_{II} & A^{iT}_{\Gamma I} & 0 \\ B^i_{II} & 0 & B^i_{I\Gamma} & 0 \\ \hline A^i_{\Gamma I} & B^{iT}_{I\Gamma} & A^i_{\Gamma\Gamma} & \bar{B}^{iT} \\ 0 & 0 & \bar{B}^i & 0 \end{array}\right] = \begin{bmatrix} K^i_{II} & K^{iT}_{\Gamma I} \\ K^i_{\Gamma I} & K^i_{\Gamma\Gamma} \end{bmatrix}. \tag{1.5}$$

The mortar condition 1.4 on Γ (Darcy side as the slave side) is imposed as $u^p_\Gamma = -B_p^{-1} B_f u^f_\Gamma = \Pi u^f_\Gamma$, where $-\Pi$ is the $L^2(\Gamma)$ projection on the space of piecewise constant functions on each e^p_i. We note that that B_p is a diagonal matrix for the lowest order Raviart-Thomas elements.

Now we eliminate $\boldsymbol{u}^i_I$, p^i_I, $i = f, p.$, and λ, to obtain the following (saddle point) Schur complement

$$S \begin{bmatrix} u^f_\Gamma \\ \bar{p}^f \\ \bar{p}^p \end{bmatrix} = \begin{bmatrix} b \\ \bar{b}^f \\ \bar{b}^p \end{bmatrix},$$

which is solvable when $\bar{b}^f + \bar{b}^p = 0$. Here S is given by

$$S := S^f + \tilde{\Pi}^T S^p \tilde{\Pi} = \left[\begin{array}{c|cc} S^f_\Gamma + \Pi^T S^p_\Gamma \Pi & \bar{B}^{fT} & \Pi^T \bar{B}^{pT} \\ \hline \bar{B}^f & 0 & 0 \\ \bar{B}^p \Pi & 0 & 0 \end{array}\right] = \begin{bmatrix} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{bmatrix},$$

where $\tilde{\Pi} := \begin{bmatrix} \Pi & 0 \\ 0 & I_{2\times 2} \end{bmatrix}$ and $S^i := K^i_{\Gamma\Gamma} - K^i_{\Gamma I} \left(K^i_{\Gamma\Gamma}\right)^{-1} K^{iT}_{\Gamma I} = \begin{bmatrix} S^i_\Gamma & \bar{B}^{iT} \\ \bar{B}^i & 0 \end{bmatrix}$.

Define $\boldsymbol{V}_\Gamma := \left\{ \boldsymbol{v} \in \boldsymbol{Z}^h : \boldsymbol{v}_f = \mathcal{SH}(\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f|_\Gamma) \text{ and } \boldsymbol{v}_p = \mathcal{DH}(\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p|_\Gamma)|_\Gamma) \right\}$ and

$$\boldsymbol{M}_0 := \left\{ q \in M^h : q_i = \text{const. in } \Omega_i, i = f, p; \text{ and } \int_{\Omega_f} q_f + \int_{\Omega_p} q_p = 0 \right\}.$$

Here $\mathcal{SH}$ ($\mathcal{DH}$) is the velocity component of the discrete Stokes (Darcy) harmonic extension operator that maps discrete interface normal velocity $\hat{u}^f_\Gamma \in H^{1/2}_{00}(\Gamma)$ ($\hat{u}^p_\Gamma \in (H^{1/2}(\Gamma))'$) to the solution of the problem: find $\boldsymbol{u}_i \in \boldsymbol{X}^{h_i}_f$ and $p_i \in \mathring{M}^{h_i}_i$ such that in Ω_i and $\forall \boldsymbol{v}_i \in \boldsymbol{X}^{h_i}_i$ and $\forall q_i \in \mathring{M}^{h_i}_i$ we have:

$$\begin{cases} a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + b_f(\boldsymbol{v}_f, p_f) = 0 \\ b_f(\boldsymbol{u}_f, q_f) \qquad\qquad\quad = 0 \\ \boldsymbol{u}_f \cdot \boldsymbol{\eta} = \hat{u}^f_\Gamma \text{ on } \Gamma \\ \boldsymbol{u}_f \cdot \boldsymbol{\eta} = 0 \text{ on } \Gamma_f \\ \boldsymbol{u}_f \cdot \boldsymbol{\tau} = 0 \text{ on } \partial\Omega_f \end{cases} \qquad \begin{cases} a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) + b_p(\boldsymbol{v}_p, p_p) = 0 \\ b_p(\boldsymbol{u}_p, q_p) \qquad\qquad\quad = 0 \\ \boldsymbol{u}_p \cdot \boldsymbol{\eta} = \hat{u}^p_\Gamma \text{ on } \Gamma \\ \boldsymbol{u}_p \cdot \boldsymbol{\eta} = 0 \text{ on } \Gamma_f. \end{cases} \tag{1.6}$$

Associated with the coupled problem we introduce the *balanced subspace*:

$$\boldsymbol{V}_{\Gamma,\bar{B}} := \mathrm{Ker}\bar{B} = \left\{ \boldsymbol{v} \in \boldsymbol{V}_\Gamma : \int_\Gamma \boldsymbol{v}^i \cdot \boldsymbol{\eta}_i = 0, i = f,p \text{ and } \boldsymbol{u}_\Gamma^p = \Pi v_\Gamma^f \right\}. \quad (1.7)$$

1.4 Balancing Domain Decomposition Preconditioner I

For the sake of simplicity on the analysis we assume that $\Gamma = \{0\} \times [0,1]$, $\Omega_f = (-1,0) \times (0,1)$ and $\Omega_p = (0,1) \times (0,1)$. We introduce the velocity coarse space on Γ as the span of the $\phi_f^0 = y(y-1)$ (v_0 be its vector representation). Define:

$$R_0 = \begin{bmatrix} v_0^T & 0 \\ 0 & I_{2\times 2} \end{bmatrix}, \quad S_0 = R_0 S R_0^T \quad \text{and} \quad Q_0 = R_0^T S_0^\dagger R_0.$$

Because v_0 is not balanced, S_0 is invertible when pressures restricted to M_0. The low dimensionality of the coarse space and the shape of ϕ_f^0 are keep fixed with respect to mesh paramenters imply stable discrete inf-sup condition for the coarse problem. Denote $\tilde{S}_0 := v_0^T S_\Gamma v_0$ and $\tilde{S} := \bar{B} v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T$. A simple calculation gives $I - Q_0 S = \begin{bmatrix} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{bmatrix}$, where

$$\begin{aligned} \mathcal{P} &:= \left(v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma - v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma \right) + v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} \\ \mathcal{G} &:= \tilde{S}^{-1} \bar{B} - \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma. \end{aligned}$$

Note that $\mathcal{P}$ is projection and that $\bar{B}(I - \mathcal{P}) = 0$, i.e. the image of $I - \mathcal{P}$ is contained on the balanced subspace defined in (1.7); see also [PW02]. Given a residual r, the coarse problem $Q_0 r$ is the solution of a coupled problem with one velocity degree of freedom (v_0) and a constant pressure per subdomain Ω_i, $i = f,p$ with mean zero on Ω. Hence, when v_Γ and u_Γ are balanced functions, the S_Γ-inner product is defined by (see (1.3)):

$$\langle u_\Gamma, v_\Gamma \rangle_{S_\Gamma} := \langle S_\Gamma u_\Gamma, v_\Gamma \rangle = u_\Gamma^T S_\Gamma v_\Gamma$$

coincides with the S-inner product defined by

$$\left\langle \begin{bmatrix} v_\Gamma \\ \bar{q} \end{bmatrix}, \begin{bmatrix} u_\Gamma \\ \bar{p} \end{bmatrix} \right\rangle_S := \begin{bmatrix} v_\Gamma \\ \bar{q} \end{bmatrix}^T S \begin{bmatrix} u_\Gamma \\ \bar{p} \end{bmatrix}.$$

Consider the following BDD preconditioner operator (See [DP03]):

$$S_N^{-1} = Q_0 + (I - Q_0 S) \left(S^f \right)^{-1} (I - S Q_0). \quad (1.8)$$

Also observe that $S_N^{-1} S = Q_0 S + (I - Q_0 S) \left(S^f \right)^{-1} S \left(I - Q_0 S \right)$, and when u_Γ, v_Γ are balanced functions we have:

$$\langle S_N^{-1} S \begin{bmatrix} u_\Gamma \\ \bar{p} \end{bmatrix}, \begin{bmatrix} v_\Gamma \\ \bar{q} \end{bmatrix} \rangle_S = \langle \left(S_\Gamma^f\right)^{-1} S_\Gamma u_\Gamma, v_\Gamma \rangle_{S_\Gamma},$$

and

$$c\langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma} \leq \langle \left(S^f\right)^{-1} S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma} \leq C\langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma}$$

is equivalent to

$$c\langle S_f u_\Gamma^f, u_\Gamma^f \rangle \leq \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \leq C\langle S_f u_\Gamma^f, u_\Gamma^f \rangle. \tag{1.9}$$

Proposition 1 *If u_Γ^f is a balanced function then*

$$\langle S_\Gamma^f u_\Gamma^f, u_\Gamma^f \rangle \leq \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \preceq (1 + \frac{1}{\kappa})\langle S_f u_\Gamma^f, u_\Gamma^f \rangle.$$

Proof. The lower bound follows trivially from S_Γ^f and S_Γ^p beeing positive on the subspace of balanced functions. We next concentrate on the upper bound.

Let v_Γ^f a balanced function and $v_\Gamma^p = \Pi v_\Gamma^f$. Define $\boldsymbol{v}_p = \mathcal{DH} v_\Gamma^p$. Using properties ([Mat89]) of the discrete operator $\mathcal{DH}$ we obtain

$$\langle S_\Gamma^p v_\Gamma^p, v_\Gamma^p \rangle = a_p(\boldsymbol{v}_p, \boldsymbol{v}_p) \asymp \frac{\mu}{\kappa} \|v_\Gamma^p\|^2_{(H^{1/2})'(\Gamma)}.$$

Using the L_2-stability property of mortar projection Π we have

$$\|v_\Gamma^p\|^2_{(H^{1/2})'(\Gamma)} \preceq \|v_\Gamma^p\|^2_{L^2(\Gamma)} = \|v_\Gamma^f\|^2_{L^2(\Gamma)} \preceq \|v_\Gamma^f\|^2_{H^{1/2}_{00}(\Gamma)}.$$

Defining $\boldsymbol{v}_f = \mathcal{SH} v_\Gamma^f$ and using properties of $\mathcal{SH}$ ([PW02],GS05) we have

$$\mu \|v_\Gamma^f\|^2_{H^{1/2}_{00}(\Gamma)} \asymp a_f(\boldsymbol{v}_f, \boldsymbol{v}_f).$$

1.5 Balancing Domain Decomposition Preconditioner II

We note that the previous preconditioner is scalable with respect to mesh parameters, however it deteriorates when the permeability κ gets smaller. In real life applications, permeabilities are in general very small, hence the previous preconditioner becomes irrelevant in practice. In addition, to capture the boundary layer behavior of Navier-Stokes flows near the interface Γ, the size of the fluid mesh h_f needs to be small while the Darcy mesh does not. With those two issues in mind, we were motivated to propose the second preconditioner. Opposed to the former preconditioner, we now control the Stokes energy by the Darcy energy.

We assume that the fluid side discretization on Γ is a *refinement* of the corresponding porous side discretization. For $j = 1, \ldots, M^p$, and on Γ, we introduce normal velocity Stokes functions ϕ_f^j (a bubble $P2$ function) with support on the interval $e_p^j = 0 \times [(j-1)h_p], jh_p]$. Under the assumption of the nested refinement and $P2/P1$ Tatlor-Hood discretization, $\phi_f^j \in \boldsymbol{X}^f|_\Gamma$. Denote $\boldsymbol{X}_f^b$ as the subspace spanned by all ϕ_f^j and $\boldsymbol{X}_n^f$ as subspace spanned by functions of v_Γ^f which has zero average on all edges e_p^j. Note that $\boldsymbol{X}_f^b$ and $\boldsymbol{X}_n^f$ form a direct sum for $\boldsymbol{X}^f|_\Gamma$ and the image $\Pi \boldsymbol{X}_n^f$ is the zero vector. Using this space decomposition we can write

$$S_\Gamma^f = \begin{bmatrix} S_{bb}^f & S_{nb}^{fT} \\ S_{nb}^f & S_{nn}^f \end{bmatrix}$$

and by eliminating the variables associated with the spaces $\boldsymbol{X}_n^f$ we obtain

$$\hat{S}_\Gamma^f = S_{bb}^f - S_{nb}^{fT}(S_{nn}^f)^{-1} S_{nb}^f,$$

and end up again with a Schur complement of the form

$$\hat{S} := \hat{S}^f + \begin{bmatrix} -B_p^{-1}\hat{B}_f & 0 \\ 0 & I_{2\times 2} \end{bmatrix}^T S^p \begin{bmatrix} -B_p^{-1}\hat{B}_f & 0 \\ 0 & I_{2\times 2} \end{bmatrix} = \hat{S}^f + \hat{S}^p,$$

where the matrix $\hat{S}$ is applied to vectors of the form $\begin{bmatrix} u_\Gamma^b & p_0^f & p_0^p \end{bmatrix}^T$. Note that $\hat{B}_f$ and B_p are diagonal matrices of the same dimension and are spectraly equivalent. We introduce the following preconditioner operator

$$\hat{S}_N^{-1} = \hat{Q}_0 + (I - \hat{Q}_0\hat{S})(\hat{S}^p)^{-1}(I - \hat{S}\hat{Q}_0). \tag{1.10}$$

Using the same arguments as before we prove:

Proposition 2 *If u_Γ^b is a balanced function then*

$$\langle \hat{S}_\Gamma^p u_\Gamma^b, u_\Gamma^b \rangle \le \langle \hat{S}_\Gamma u_\Gamma^b, u_\Gamma^b \rangle \preceq (1 + \frac{\kappa}{h_p^2}) \langle \hat{S}_\Gamma^p u_\Gamma^b, u_\Gamma^b \rangle.$$

Proof. Let $v_\Gamma^b = \sum_{j=1}^{M_p} \beta_j \phi_f^j$. And notice that the basis functions ϕ_f^j do not overlap each other on Γ. We have:

$$\|v_\Gamma^b\|_{L^2(\Gamma)}^2 = \sum_{j=1}^{M_p} \beta_j^2 \|\phi_f^j\|_{L^2(\Gamma)}^2 \asymp h_p \sum_{j=1}^{M_p} \beta_j^2,$$

and using $H_{00}^{1/2}$ arguments on intervals e_p^j we have

$$\|v_\Gamma^b\|_{H_{00}^{1/2}(\Gamma)}^2 \preceq \sum_{j=1}^{M_p} \beta_j^2 \|\phi_f^j\|_{H_{00}^{1/2}(e_p^j)}^2 \asymp \sum_{j=1}^{M_p} \beta_j^2.$$

Note that, by considering $\boldsymbol{v}_\Gamma^f = v_\Gamma^b$, we have

$$\langle \hat{S}^f v^b, v^b \rangle \le a_f(\mathcal{SH}\boldsymbol{v}_\Gamma^f, \mathcal{SH}\boldsymbol{v}_\Gamma^f) \asymp \mu \|\boldsymbol{v}_f r_\Gamma\|_{H_{00}^{1/2}(\Gamma)}^2,$$

since the space for discrete Stokes harmonic extension now is richer (includes also $\boldsymbol{X}_n^f$) than in $\mathcal{SH}$, and also the equivalence results between discrete Stokes and Laplacian harmonic extensions. We obtain

$$\langle \hat{S}_\Gamma^f v^b, v^b \rangle \preceq \frac{\mu}{h_p} \| v_\Gamma^b \|_{L^2(\Gamma)}^2 \preceq \frac{\mu}{h_p^2} \mu \| \Pi v_\Gamma^b \|_{(H^{1/2})'(\Gamma)}^2 \asymp \frac{\kappa}{h_p^2} \langle \hat{S}_\Gamma^p v^b, v^b \rangle,$$

where we have used an inverse inequality for piecewise constant functions.

References

[DQ03] M. Discacciati and A. Quarteroni. *Analysis of a Domain Decomp. Method for the Coupling for the Stokes and Darcy Equations.* In F. Brezzi et al (eds), Numerical analysis and advanced applications-Enumath 2001. Springer-Verlag, 2003.

[DQ04] M. Discacciati and A. Quarteroni. Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of stokes and darcy equations. *Comput. Visual. Sci.*, 6(2-3):93–103, 2004.

[D04] M. Discacciati. Iterative Methods for Stokes/Darcy Coupling. *Domain Decomposition Methods in Science and Engineering, Lecture Notes in Computational Science and Engineering, Kornhuber et. al.*, Vol. 40, 2004.

[D04b] M. Discacciati. *Domain decomposition methods for the coupling of surface and groundwater flows.* PhD thesis, Ecole Polytechnique Fédérale, Lausanne (Switzerland), 2004. Thse n. 3117.

[DP03] Maksymilian Dryja and Wlodek Proskurowski. On preconditioners for mortar discretization of elliptic problems. *Numerical Linear Algebra with Applications*, (10):65–82, 2003.

[Gal04] J. Galvis. Finite elements for well-reservoir coupling. Master's thesis, Instituto Nacional de Matemtica Pura e Aplicada, April 2004.

[GS05] J. Galvis and M. Sarkis. In preparation, 2005.

[GS04] Juan Galvis and Marcus Sarkis. Inf-sup for coupling stokes-darcy. In *Proceedings of the XXV Iberian and Latin American Congress on Computational Methods in Engineering, CILAMCE XXV*, November 10-12 2004.

[QVZ02] A. Quarteroni, A. Veneziani, and P. Zunino. Mathematical and numerical modeling for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43:57–74, 2002.

[LSY03] W. Layton, F. Schieweck, and I. Yotov. Coupling fluid flow with porous media flow. *SIAM J. NUMER. ANAL*, 40(6):2195–2218, 2003.

[Mat89] Tarek P. Mathew. *Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretizations of Elliptic Problems.* PhD thesis, Courant Institute of Mathematical Sciences, September 1989. Tech. Rep. 463, Department of Computer Science, Courant Institute.

[Man93] Jan Mandel. Balancing domain decomposition. *Comm. Numer. Meth. Engrg.*, 9:233–241, 1993.

[RY05] B. Riviére and I. Yotov. Locally conservative coupling of Stokes and Darcy flows. *SIAM J. NUMER. ANAL*, 42(5):1959–1977, 2005.

[PW02] Luca F. Pavarino and Olof B. Widlund. Balancing Neumann-Neumann methods for incompressible Stokes equations. *Comm. Pure Appl. Math.*, 55(3):302–335, 2002.