STABILITY OF THE PARAREAL TIME DISCRETIZATION FOR PARABOLIC INVERSE PROBLEM

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Summary. The practical aspect in the parareal algorithm that it consist of using two solvers over different time stepping, the coarse and fine solvers to produce a rapid convergent iterative method for multi processors computations. The coarse solver solve the equation sequentially on the coarse time step while the fine solver use the information from the coarse solution to solve, in parallel, over the fine time steps. In this work we discussed the stability of the parareal-inverse problem algorithm for solving the parabolic inverse problem given by

$$\begin{array}{ll} u_t = u_{xx} + p(t)u + \phi(x,t), & 0 < x < 1,\ 0 < t \le T, \\ u(x,0) = f(x), & 0 \le x \le 1, \\ u(0,t) = g_0(t), & 0 < t \le T, \\ u(1,t) = g_1(t), & 0 < t \le T, \end{array}$$

and subject to the over specification condition at a point x_0 in the spatial domain $u(x_0,t) = E(t)$. We derived the stability amplification factor for the parareal-inverse algorithm and present the stability analysis in accordance to the relation between the coarse and fine time step and the value of $p(t)$. Some model problems are considered to demonstrate the necessary conditions for the stability.

1 Introduction

The parallelization with respect to the time variable is not fairly new approach, the first research article in this area was the article by Nievergelt on the solution of the ordinary differential equation [N64] and the article by Miranker and Liniger [ML97] on the numerical integration of ordinary differential equations.

Recently after the development of the initial algorithm a new form of algorithm has been proposed which consists of discretizing the problem over an interval of time using fine time step and coarse time step in away to allow a combination of accuracy improvement, through an iterative process, and parallelization over slices of coarse time interval. The algorithm has been re setup and then named as *Parareal Algorithm* by Lion's et al. [LMT00], also further modified by Bal and Maday [BM02] to solve unsteady state problem and evidently established the relation between the coarse and fine time step in order to define the time gaining in the parallelization procedure. The stability and the convergence of the algorithm has been further studied by Bal [B03] and mainly concluding that the algorithm replaces a coarse discretization method of order m by a higher order dicsretization method, and also Staff and Ronquist [SR03] presented the necessary condition for the stability of the parareal algorithm. For further detailed views of the method and further applications we refer to Baffico et al. [BBMTZ02], Farhat and Chandersis [FC03], and Maday and Turinici [MT03].

In this article we will emphasize on the stability of the parareal algorithm to solve the following inverse problem for determining a control function $p(t)$ in parabolic equation. Find $u = u(x,t)$ and $p = p(t)$ which satisfy

$$\begin{array}{ll} u_t = u_{xx} + p(t)u + \phi(x,t), & 0 < x < 1,\ 0 < t \leq T, \\ u(x,0) = f(x), & 0 \leq x \leq 1, \\ u(0,t) = g_0(t), & 0 < t \leq T, \\ u(1,t) = g_1(t), & 0 < t \leq T, \end{array} \tag{1}$$

subject to the over specification condition at the point x_0 in the spatial domain

$$u(x_0,t) = E(t), \tag{2}$$

where f, g_0, g_1, E and ϕ are known functions while the functions u and p are unknown, for $-1 < p(t) < 0$ for $t \in [0,T]$. The model problem given by (1) used to describe a heat transfer process with a source parameter present and equation (2) represents the temperature at a given point x_0 in the spatial domain at time t. Thus the purpose of solving this inverse problem is to identify the source control parameter that produces at any given time a desired temperature at a given point x_0 in the spatial domain.

2 Parareal-Inverse Problem Algorithm

The main aspect of the parareal algorithm is to allow a parallelization in time over slices of coarse time interval using coarse time solver in combination of accuracy improvements through an iterative method(predictor-corrector form) using fine and coarse time solvers over each coarse time interval $\Delta t (\Delta t = T/N)$.

In this article the coarse and fine time step solvers will be denoted by $G_{\Delta t}$, and $F_{\delta t}$, respectively, where $\delta t = \frac{\Delta t}{s}$, and s is the number of fine time steps over the coarse interval $[t_n, t_{n+1}] = [t_n, t_n + s\delta t]$, for $n = 0, 1, \ldots N-1$. Through this work we will consider the parareal algorithm scheme in the form presented by Bal [B03] and also later considered by Staff and Ronquist [SR03], given by

$$u_{k+1}^{n+1} = G_{\Delta t}(u_{k+1}^{n}) + F_{s\delta t}(u_k^n) - G_{\Delta t}(u_k^n). \tag{3}$$

The solution algorithm of the inverse problem (1) by implicit type of methods, backward Euler's method, possess an updating of the control function $p(t)$ and $u(x,t)$, or in another words correction steps at each time level prior to proceed to the advanced time level (cf. e.g.[CLW92], [DS05]). On the other hand the solution by the forward Euler's scheme does not require any correction for the control function $p(t)$, but in order to apply the parareal algorithm the updating of the value of $p(t)$ for the fine propagator is required for the advanced fine solution step using (2).

Since the parareal algorithm posses a correction steps over each coarse time interval it was observed that, through the coarse solution propagator, for the correction of the $p(t)$ it is sufficient to perform one iteration only, internally, over the time step $[t_n, t_{n+1}]$ and that is due to the further iterations and correction of the solution by the parareal algorithm. The generic form of the parareal algorithm for the solution of the inverse problem is given as follows.

Algorithm 1 *Parareal- Inverse Problem Algorithm*

1. *Over the domain $\Omega \times [t_n, t_{n+1}]$ and for $k = 1$, consider the coarse propagator i.e.*

$$\frac{u_1^{n+1} - u_1^n}{\Delta t} = (u_{xx})_1^{n+1} + p(t_n)u^{n+1} \quad n = 1, \ldots N-1,$$

the solution u_1^{n+1} denoted by $G_{\Delta t}(u_1^n)$.
$p(t^{n+1})$ correction: Consider the correction of $p(t)$ by the following relation

$$p(t^{n+1}) = \frac{E'(t_{n+1}) - (u_{xx})_1|_{(x_0,t_{n+1})} - \phi(x_0, t_{n+1})}{E(t_{n+1})}.$$

2. *For $k+1 > 1$ and over the domain $\Omega \times [t_n, t_{n+1}]$.*
   a) *Consider the coarse propagator i.e.*

$$\frac{u_{k+1}^{n+1} - u_{k+1}^n}{\Delta t} = (u_{xx})_{k+1}^{n+1} + p(t_n)u_{k+1}^{n+1} \quad n = 1, \ldots N-1,$$

   $p(t^{n+1})$ correction: Consider the correction of $p(t)$ by the following relation

$$p(t_{n+1}) = \frac{E'(t_{n+1}) - (u_{xx})_{k+1}|_{(x_0,t_{n+1})} - f(x_0, t_{n+1})}{E(t_{n+1})},$$

   the solution u_{k+1}^{n+1} is denoted by $G_{\Delta t}(u_{k+1}^n)$.
   b) *Consider the fine propagator solution over $\Omega \times [t_n, t_{n+l}]$, $\quad l = 1, s-1$. Solve for*

$$\frac{u_k^{n+l} - u_k^{n+l-1}}{\delta t} = (u_{xx})_k^{n+l-1} + p(t_{n+l-1})u_k^{n+l-1}.$$

The solution $u_k^{n+s} = u_k^{n+1}$ *is denoted by* $F_{s\delta t}(u_k^n)$, *and*

$$p^{n+l} = \frac{E'(t_{n+l}) - u_{xx,k}|_{(x_0,t_{n+l})} - \phi(x_0, t_{n+l})}{E(t_{n+l})}, \quad \text{for } l = 1, \dots s-1,$$

where $s = \frac{\Delta t}{\delta t}$.

Then the solution u_{k+1}^{n+1} *is given by*

$$u_{k+1}^{n+1} = G_{\Delta t}(u_{k+1}^n) + F_{s\delta t}(u_k^n) - G_{\Delta t}(u_k^n). \tag{4}$$

3 Stability of The Parareal-Inverse Algorithm

Let $u(x,t)$ be the solution of the model problem

$$u_t = u_{xx} + p(t)u(t), \tag{5}$$

subject to the following initial and boundary conditions

$$u(0,t) = 0, u(1,t) = 0 \;\text{ and } u(x_0,t) = u_0, \tag{6}$$

and with the specified condition $u(x_0,t) = E(t)$.

The spatial derivative operator is approximated by the second order central difference approximation given by

$$(u_{xx})_{(x_i,t)} \simeq h^{-2}[u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)] + \mathcal{O}(h^2). \tag{7}$$

For the stability analysis we will consider the Fourier transform of the discrete problem, and over the Fourier domain the problem corresponding to (5) is given by

$$\widehat{u_t} = Q(\xi,t)\widehat{u}(\xi,t), \tag{8}$$

where $Q(\xi,t) = q(\xi) + \widehat{p}(t)$, such that $Q(u) = \widehat{Q(\xi)u}(\xi)$ and $q(\xi) = -2h^{-2}\sin^2(\xi/2)$. Then

$$Q(\xi) = q(\xi) + \widehat{p}(t) = -2h^{-2}\sin^2(\frac{\xi}{2}) + \widehat{p}(t). \tag{9}$$

The forward and backward Euler's schemes are considered to be the fine and coarse solvers for the parareal-inverse algorithm, respectively. The amplification factor of the backward Euler's scheme over the Fourier domain is given by

$$\rho(\xi,t_n)_{G_{\Delta t}} = (1 - Q(\xi,t_n)\Delta t)^{-1} = (1 + (2h^{-2}\sin^2(\frac{\xi}{2}) - \widehat{p}(t_n))\Delta t)^{-1},$$

which is unconditional stable for $p(t) < 0$ [T95], and the corresponding amplification factor for the solution by the forward Euler's scheme over the time interval $[t_n, t_n + s\delta t]$, is given by

$$\rho(\xi, tn)_{F_{s\delta t}} = \prod_{i=1}^{s}(1+Q(\xi, t_{n+i-1})\delta t) = \prod_{i=1}^{s}(1+(-2h^{-2}sin^2(\frac{\xi}{2})+\widehat{p}(t_{n+i-1}))\delta t),$$

and its conditional stable scheme according to the forward Euler's scheme stability condition for any $p(t)$ [T95].

For the stability analysis we will consider the approach by Staff and Ronquist [SR03] and we will present the stability studies for the following cases
case 1- $\Delta t = s\delta t$ $(s > 1)$,
case 2- $\Delta t = \delta t$ $(s = 1)$.

3.1 Case I $\Delta t = s\delta t$ $(s > 1)$

For this case of the stability analysis the coarse time step Δt is divided into s fine subintervals $(s > 1)$ and the iterative solution of (8) by the parareal-inverse algorithm 1 is given by

$$\widehat{u}_{k+1}^{n+1} = (1-Q(\xi, t_n)\Delta t)^{-1}\widehat{u}_{k+1}^{n} + \prod_{i=1}^{s}(1+Q(\xi, t_{n+i-1})\delta t)\widehat{u}_{k}^{n} - (1-Q(\xi, t_n)\Delta t)^{-1}\widehat{u}_{k}^{n}. \tag{10}$$

Following the stability analysis by [SR03] then the stability function, the amplification factor, for (10) is given by

$$\begin{aligned}\rho(\xi, t_n) &= 2(1 - Q(\xi, t_n)\Delta t)^{-1} - \textstyle\prod_{i=1}^{s}(1 + Q(\xi, t_n)\delta t,\\ &= (1 - Q(\xi, t_n)\Delta t)^{-1}\left[2 - (1 - Q(\xi, t_n)\Delta t)\textstyle\prod_{i=1}(1 + Q(\xi, t_{n+i-1})\delta t)\right]\\ &= (1 - Q(\xi, t_n)\Delta t)^{-1}\tau(\xi, t_n)\end{aligned} \tag{11}$$

For the second term,$\tau(\xi, t_n)$, in (11) if we perform the multiplication we then concluding that

$$\tau(\xi, t_n) = \left[2 - (1 - Q(\xi, t_n)\Delta t)\left[1 + \delta t\sum_{i=1}^{s} Q(\xi, t_{n+i-1}) + \mathcal{O}(\delta t^2)\right]\right],$$

therefore

$$\begin{aligned}\tau(\xi, t_n) &= 2 - 1 + Q(\xi, t_n)\Delta t - \delta t(1 - Q(\xi, t_n)\Delta t)\textstyle\sum_{i=1}^{s} Q(\xi, t_{n+i-1}) + \mathcal{O}(\delta t^2),\\ \tau(\xi, t_n) &\simeq 1 + Q(\xi, t_n)\Delta t - \delta t\textstyle\sum_{i=1}^{s}(-2h^{-2}\sin^2(\xi/2) + p(t_{n+i-1}))\\ &\leq 1 - 2r_c\sin^2(\xi/2) + \Delta t p(t_n) + \textstyle\sum_{i=1}^{s}\left(2r_f\sin^2(\xi/2) - \delta t p(t_{n+i-1})\right),\end{aligned}$$

where $r_c = \frac{\Delta t}{h^2}$, $r_f = \frac{\delta t}{h^2}$ corresponding to the coarse and fine propagator respectively, and $\frac{\Delta t}{\delta t} = s$. Hence for $-1 < p(t_n) < 0$ concluding that $|\rho(\xi, t_n)| < |(1 - Q(\xi, t_n)\Delta t)^{-1}||\tau(\xi, t_n)| < 1$. The above concluded conditions for the stability of the first case are summarized in the following theorem.

Theorem 2. *For the inverse model problem (1) solved by the parareal algorithm 1,*

$$u_{k+1}^{n+1} = G_{\Delta t}(u_{k+1}^n) + F_{s\delta t}(u_k^n) - G_{\Delta t}(u_k^n), \tag{12}$$

where $G_{\Delta t}$ and $F_{s\delta t}$ are the coarse and fine solvers respectively, and for $s = \Delta t/\delta t > 1$.

If $r_f = \delta t/h^2$ satisfy the fine solver stability condition and $p(t) \in [-1, 0]$ then the stability function $\rho(\xi, t_n)$, induced by (10) defined by (11), satisfy

$$|\rho(\xi, t_n)| < 1,$$

for all $r_c = \Delta t/h^2$.

3.2 Case II $\boldsymbol{\Delta t = \delta t}$ $\boldsymbol{(s = 1)}$

For the case when $s = 1$ the stability amplification factor is given by

$$\rho(\xi, t_n) = (1 + Q(\xi, t_n)\Delta t) - 2(1 - Q(\xi, t_n)\Delta t)^{-1}.$$

Due to the restricted pages limit the main conclusion will be summarized by the following theorem.

Theorem 3. *For the inverse model problem (1) solved by the parareal algorithm 1*

$$u_{k+1}^{n+1} = G_{\Delta t}(u_{k+1}^n) + F_{\delta t}(u_k^n) - G_{\Delta t}(u_k^n), \tag{13}$$

where $G_{\Delta t}$ and $F_{\delta t}$ are the coarse and fine solvers, respectively. Then

$$|\rho(\xi, t_n)| < 1,$$

for all $\frac{\delta t}{h^2} = \frac{\Delta t}{h^2} < \frac{1}{4}$ and $-1 < p(t) < 0$, where $\rho(\xi, t_n)$ is the amplification factor induced by (10) for $s = 1$ i.e. $\Delta t = \delta t$.

4 Model problem

For the validation of the stability necessary conditions presented in previous section we considered the model problems defined by

$$u_t = u_{xx} + p(t)u + \phi(x, t) \quad \text{over } \Omega = [0, 1] \times (0, 1),$$

with exact solution $u(x, t) = e^{-t^2}(\cos \pi x + \sin \pi x)$, and $\phi(x, t)$ defined in accordance to different definitions of $p(t)$. We considered $p(t) = -1 - t^2 < 0$ and $p(t) = 1 + 2t > 0$ for $t \in (0, 1)$ respectively. The initial, boundary conditions and $E(t) = u(x_0, t)$ at $x_0 = 0.5$ are defined from the exact solution. The stability functions (i.e. the amplification factors) are plotted using polar graphics for different values of the necessary condition.

For the case when $s > 1$ the plots are presented in figure 1 for different values of $p(t)$, r_c and r_f values as well. Figure 1 show how the amplification factor given by (11) exceeded the desired stability bound for $r_f > 0.5$ and also same conclusion for $-1 < p(t) < 0$ and positive values of $p(t)$.

For the case when $s = 1$ the plots of the amplification factor given by $\rho(\xi, t_n)$ in (11) are presented in figure 2. We considered different values for $r = \Delta t/h^2$ and $p(t)$, the plots shows how the stability amplification factor comply with the necessary conditions as stated in theorem 3.

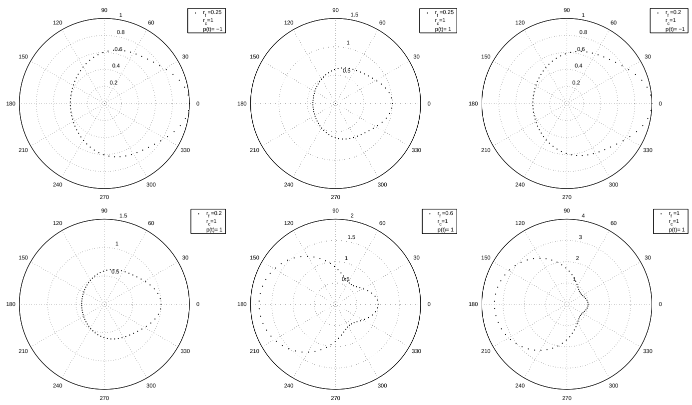

Fig. 1. The stability region for case 1 using different values of r_f, r_c and $p(t)$

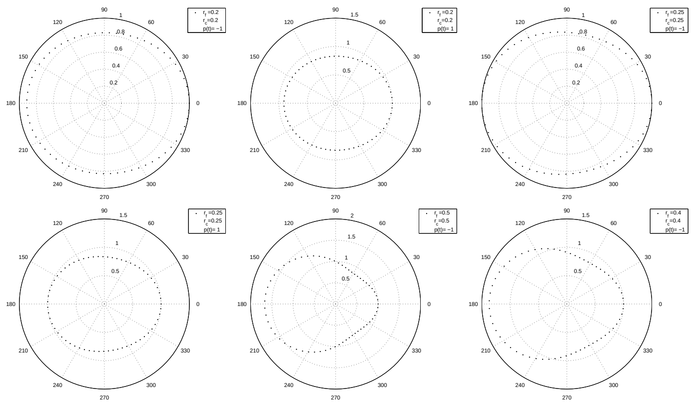

Fig. 2. The stability region for case 2 using different values of the ratio r and $p(t)$

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