PAMO 2012 question 1. In circle O, chord AB is not a diameter. A point T is chosen on segment OB. The line through T perpendicular to OB meets AB at C and the circle at D and E. Point S is the orthogonal projection (the foot of the perpendicular) from T onto AB. Prove that $AS \cdot BC = TE \cdot TD$.

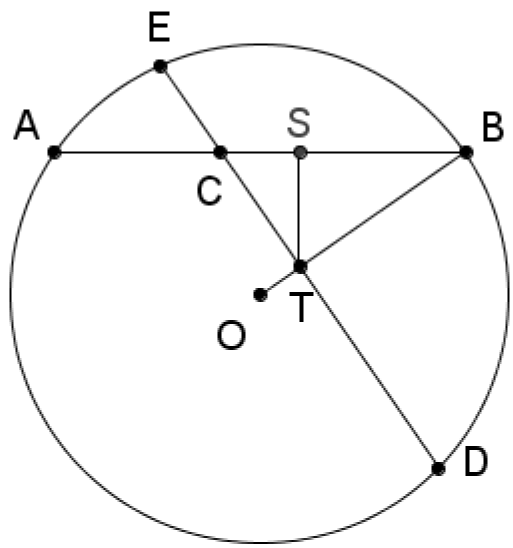

FIGURE 1

Solution. Let's look at what we need to prove. First of all, the expression $TE \cdot TD$ is a red herring. Since $TE = TD$, this expression is just equal to TD^2.

Of more interest is the first expression, $AS \cdot BC$. It's a little strange: we don't have many obvious theorems or results concerning the product of two *overlapping* segments.

Let's try to break this down: We have

$$AS \cdot BC = (AC + CS) \cdot BC = AC \cdot BC + CS \cdot BC. \tag{1}$$

This is more familiar.

OK. Now let's look at the diagram and see if find some more familiar sights. We have two chords intersecting in a circle: AB and DE. And we have a theorem about this: $AC \cdot CB = EC \cdot CD$. And the first product is one we've just seen.

What else in the diagram is familiar? Well, we have right triangle BCT with altitude ST to the hypotenuse. So we can harness the 'three triangle' theorem, which involves products of various segments. Looking good.

The difficulty now is picking out which products to use. Is there anything we've seen already which appears in this 'three triangle' configuration? Yes: in (1) we have the product $CS \cdot BC$, which equals CT^2, by the theorem.

OK. Now we have a game plan. We can relate everything in the equality we must prove to segments along DE:

$$\begin{aligned} AS \cdot BC &= AC \cdot BC + CS \cdot BC \\ &= EC \cdot CD + CT^2. \end{aligned}$$

Let's take a good look at segments along DE. We have:

$$\begin{aligned} EC &= TE - TC = TD - TC; \\ CD &= TD + TC; \\ EC \cdot CD &= (TD - TC)(TD + TC) = TD^2 - TC^2 \end{aligned}$$

So:

$$\begin{aligned} AS \cdot BC = (AC + CS) \cdot BC &= AC \cdot BC + CS \cdot BC \\ &= EC \cdot CD + TC^2 \\ &= TD^2 - TC^2 + TC^2 \\ &= TD^2 \\ &= TD \cdot TE. \end{aligned}$$

And we're done!

PAMO question 2. Find all positive integers m and n such that $n^m - m$ divides $m^2 + 2m$.

Solution. In general, raising to a power produces much larger numbers than squaring. And for a to divide b, it is necessary that $a \leq b$. So there cannot be many ordered pairs (m, n) which satisfy the required relationship. The numbers m, n can't get too big. So it looks like we can 'squeeze them out' case-by-case and using this necessary condition

There are a few exceptions. For example, if $n = 1$, the expression $n^m - m$ just equals $1 - m$. We'll treat this later. The number 1 always seems to require special treatment.

Can $n = 2$? Well, then we must have (as a necessary condition) $2^m - m \leq m^2 + 2m$, or $2^m \leq m^2 + 3m$. This cannot happen very often. Indeed, if $m = 7$, we have $2^7 = 128$, which is already greater than $m^2 + 3m = 70$. So we have the following possibilities for $n = 2$:

m	n^m	$n^m - m$	$m^2 + 2m$
1	2	1	3
2	4	2	8
3	8	5	15
4	16	12	24
5	32	27	35
6	64	58	48

A quick check gives us only the solutions $(1, 2); (2, 2); (3, 2); (4, 2)$.

Now let $n = 3$. We have $3^m \leq m^2 + 3m$, and $m = 4$ is already too big: $n^m - m = 3^4 - 3 = 78$, while $m^2 + 2m = 24$. So we have the following possibilities for $n = 3$:

m	n^m	$n^m - m$	$m^2 + 2m$
1	3	2	3
2	9	7	8
3	27	22	15

We get no new solutions.

For $n = 4,\ 5$, or 6, the value $m = 2$ is already too big, so we must have $m = 1$:

m	$n = n^m$	$n^m - m$	$m^2 + 2m$
1	4	3	3
1	5	4	3
1	6	5	3

We have only the solution (1,4).

It remains to consider the case $n = 1$. In this case, we must have $1 - m$ divides $m^2 + 2m = m(m + 2)$. This is equivalent to $m - 1$ dividing $m(m + 2)$. (A number a is divisible by b iff a is divisible by $-b$.) But $m - 1$ is relatively prime to m, so this must mean that $m - 1$ divides $m + 2$.

Again, this cannot happen very often: the numbers are only 3 apart. And in fact, if a divides b, then a must divide $b + Ka$ for any integer K. Here, this means that if $m - 1$ divides $m + 2$, then $m - 1$ divides $m + 2 + 2(m - 1) = 3m$.

Again, $m - 1$ is relatively prime to m, so $m - 1$ must divide 3. That is, $m - 1 = -3, -1, 1, 3$, and $m = -2, 0, 2, 4$:

m	$n^m - m = 1 - m$	$m^2 + 2m$
-2	3	0
0	1	0
2	-1	6
4	-3	3

Only the value $m = 2$ gives a solution, which is $(2, 1)$.

In summary, we have the following six solutions:

m	n	$n^m - m$	$m^2 + 2m$
1	2	1	3
2	2	2	8
3	2	5	15
4	2	12	24
1	4	3	3
2	1	-1	8