SOME PROBLEMS FROM THE AAAS OLYMPIAD PROGRAM TRAINING SESSION

**Problem 1.** Find all integers $n > 1$ such that $n$ divides $1^n + 2^n + \cdots + (n-1)^n$.

We begin by asking the reader to supply some 'data' about simple cases, then to think about what might be going on.

1. Let $S(n) = \sum_{k=1}^{n-1} k^n = 1^n + 2^n + \cdots + (n-1)^n$.

Compute the first few cases:

| $n$ | $S(n)$ | $S(n)$ | Does n divide S(n)? |
|---|---|---|---|
| 2 | $1^2$ | 1 | *no* |
| 3 | $1^3 + 2^3$ | 9 | *yes* |
| 4 | | | |
| 5 | | | |
| 6 | | | |

2. Before reading further, form an hypothesis. Try to prove your hypothesis. Here are some tools that might help.

4. Lemma: If $n$ is odd, then $k^n + (n-k)^n$ is a multiple of $n$.

Proof: Use the binomail theorem, or have someone explain this to you. It tells you how to write out the expansion of $(n-k)^a$, for different values of $a$. If you don't know this theorem, here are three examples:

$$\begin{aligned}(n-k)^7 =& n^7 - 7n^6k + 21n^5k^2 - 35n^4k^3 + 35n^3k^4 - \\ & - 21n^2k^5 + 7nk^6 - k^7.\end{aligned}$$

$$\begin{aligned}(n-k)^8 =& n^8 - 8n^7k + 28n^6k^2 - 56n^5k^3 + 706n^4k^4 - \\ & - 56n^3k^5 + 28n^2k^6 - 8nk^7 + k^9.\end{aligned}$$

$$\begin{aligned}(n-k)^9 =& n^9 - 9n^8k + 36n^7k^2 - 84n^6k^3 + 126n^5k^4 - \\ & - 126n^4k^5 + 84n^3k^6 - 36n^2k^7 + 9nk^8 - k^9.\end{aligned}$$

We've computed the coefficients here, but their actual value is not important to us. Note that the signs of the terms alternate.

How does this prove our lemma?

5. Make sure you see why this argument does not work for even $n$.

6. How does the lemma above prove the statement in our problem for odd values of $n$?

7. For $n$ even, the discussion gets a bit more complicated. We look first at some numerical examples.

Suppose $n$ contains just one factor of 2, so that $n$ can be written as $2(2a+1)$ (examples are $n = 6,\ 10,\ 14$ and so on). We have $2a+1$ odd terms, and $2a$ even terms. So the sum $S(n)$ of all the terms is odd, and the even number $n$ cannot divide $S(n)$.

Now suppose $n$ contains just two factors of 2, so that $n = 2^2(2a+1)$ (examples are $n = 4,\ 12,\ 20$, etc). Then the even terms of $S(n)$ will have $2^2 = 4$ as a factor (since $n \geq 2$). And the number of odd factors is $\frac{n}{2}$, which is $2(2a+1)$, two more than a multiple of 4. So $S(n)$ will be two more than a multiple of 4. But $n$ is itself a multiple of 4, so any multiple of $n$ will be a multiple of 4, and $S(n)$ cannot be a multiple of $n$.

Suppose now $n$ has a factor of $2^3$ in it; that is, $n = 2^3(2a+1)$ for some $a$ (examples are $n = 8, 24, 40$ and so on. Since $n \geq 3$, the even terms will all be multiples of $2^3$ (the exponent $n$ is at least 3).

And the number of odd factors is $\frac{n}{2}$, which is $2^2(2k+1)$, which is four more than a multiple of 8. So $S(n)$ will four more than a multiple of 8. But $n$ is itself a multiple of 8, so $S(n)$ cannot be a multiple of $n$.

This argument generalizes easily, although the notation again obscures the thought. Here is the generalization:

Suppose $n$ is even. We divide out the largest power of 2 it contains by writing $n = 2^a(2m+1)$. Note that $a < n$ (otherwise $2^a$ would be much too big to divide $n$). Then, in the sum $S(n)$ there are $\frac{n}{2} = 2^{a-1}(2m+1)$ odd terms and one fewer even terms. All the even terms are raised to the $n^{th}$ power, so they are multiples of $2^n$, and so certainly multiples of $2^a$.

We must now show that the sum of the odd terms cannot be a multiple of $2^a$. This is a bit more difficult. A typical odd term is $k^n = k^{2^a(2m+1)} = k^{2^a} \cdot k^{2m+1}$ Now the second factor is odd, so we must show only that $k^{2^a}$ cannot be a multiple of $2^a$.

Let us look at the following algebraic pattern:

$$(k-1)(k+1) = k^2 - 1;$$
$$(k-1)(k+1)(k^2+1) = (k^2-1)(k^2+1) = (k^4-1);$$
$$(k-1)(k+1)(k^2+1)(k^4+1) = (k^2-1)(k^2+1)(k^4-1) = (k^4+1)(k^4-1) = k^8 - 1;$$

We don't need all the detail in the computations above. If we read the equations from right to left, we see that $k^{2^a} - 1$ is always represented as a long string of factors, each of which are even (because a power of $k$ is odd, and one more or one less than a power of $k$ will be even).

In other words, $k^{2^a} - 1$ can be factored into $a+1$ (count them!) even factors, which means that $k^{2^a}1$ is a multiple of $2^{a+1}$, so certainly of $2^a$. Or, we can say that $k^{2^a}$ is one more than a multiple of $2^a$.

Now there are $\frac{n}{2}$ odd terms in $S(n)$, each of which is one more than a multiple of $2^a$. hence the sum of the odd terms is $\frac{n}{2}$ more than a multiple of $2^a$, and so cannot itself be a multiple of $2^a$. This proves our assertion.

*Note:* It is entirely possible to solve this problem without congruences nor Euler's theorem. All that is needed is repeated use of the factorization

$$y^N - x^N = (y-x)\sum_{k=0}^{N-1} x^k y^{N-1-k}$$

(or rather of the fact that $y-x$ divides $y^N - x^N$) and a proof that $2^a \mid k^{2^a} - 1$ for fixed odd $k$ and all integers $a \geq 0$, which is easy to prove by induction (or otherwise immediately seen from the factorization $k^{2^a} - 1 = (k-1)\prod_{b=0}^{a-1}(k^{2^b}+1)$ into $a+1$ even factors).

**Problem 3.** Prove or disprove that there exists a positive integer $N$ having the following properties:

(1) $N$ is divisible by $2^{2012}$;
(2) $N$ contains only the digits 1 and 2.

**Solution to Problem 3.** We give a formal 'official' solution, then some details about how to think about the problem.

*Formal Solution.* Proceeding inductively, we construct a sequence of decimal digits $a_0, a_1, \dots, a_k, \dots$ in $\{1,2\}$ such that $2^{k+1}$ divides the number $A_k = \overline{a_k a_{k-1} \dots a_1 a_0}$ for $k = 0, 1, 2, \dots$.

Set $a_0 = 2$ and, for $k \geq 0$, let

$$a_{k+1} = \begin{cases} 1, & \text{if } 2^{k+2} \text{does not divide} A_k; \\ 2, & \text{if } 2^{k+2} \text{divides} A_k. \end{cases}$$

We prove by induction on $n$ that $2^{n+1} \mid A_n$. This is true for $n = 0$ since $A_0 = a_0 = 2$. Assume it true for $n = k$. Let $q_k = A_k/2^{k+1}$, an integer (by the inductive hypothesis). Note that $2^{k+2} \mid A_k$ if and only if $q_k$ is even, and $a_{k+1}$ is chosen so as to have opposite parity to $q_k$. Now, $A_{k+1} = 10^{k+1}a_{k+1} + A_k = 2^{k+1}(5^{k+1}a_{k+1} + q_k)$. Since $5^{k+1}$ is odd, the choice of $a_{k+1}$ is exactly that which makes the factor $5^{k+1}a_{k+1} + q_k$ even, hence $A_{k+1}$ divisible by $2^{k+1} \cdot 2 = 2^{(k+1)+1}$. This completes the inductive step of the proof.

Now here is some insight into the thinking behind the proof. We start on a very elementary level.

*Lemma I.* For any positive integer $n$, $10^n$ is divisible by $2^n$, and $10^n$ leaves a remainder of 1 when divided by $2^{n+1}$.

*Proof.* (a) The key is thinking of all these numbers as products of primes. We have $10^n = (2 \cdot 5)^n = 2^n \cdot 5^n$, which is clearly divisible by $2^n$. But there are not enough powers of the prime number 2 in the factorization of $10^n$ to make it divisible by $2^{n+1}$. So $10^n$ must leave a remainder, and that remainder can only be 1.

*Lemma II.* For any positive integer $n$, $2 \cdot 10^n$ is divisible by $2^{n+1}$. ft

*Proof.* We have $2 \cdot 10^n = 2 \cdot 2^n \cdot 5^n = 2^{n+1} \cdot 10^n$. By Lemma I, $10^n$ contains $n$ factors of 4, so $10^{n+1}$ is divisible by $2^{n+1}$.

Now to the problem itself. Because $10^n > 2^n$ for $n \geq 1$, we can try 'building up' our number $N$ one digit at a time. We will do this by repeatedly appending digits 1 or 2 to the *left* of our number.

Let us look at some numbers which contain only the digits 1 and 2. Let $n$ be the number of digits in the number we are looking at. We exclude odd numbers, because we want high powers of 2 to divide the number $N$ we construct.

For $n = 1$ we have $2^1 = 2$ divides 2.

For $n = 2$ we have $2^2 = 4$ divides 12. The other possibility, 22, is not divisible by 4.

For $n = 3$ we have four choices: 112, 122, 212, 222. We can quickly see that $2^3 = 8$ divides 112. (In fact $2^4 = 16$ also divides 112, but never mind that.) And in fact we can get the three-digit number 112 by appending a 1 to the two-digit number 12 we've just found. Our plan is working.

For $n = 4$, we can continue our plan, by trying out 1112 and 2112. We find that $2^4 = 16$ divides 2112. (So does $2^5 = 32$, but never mind that.)

For $n = 5$ we try out 12112 and 22112. We find that $2^5 = 32$ divides 22112.

So it looks like we can always append a 1 or a 2 to the left of an $n$-digit number that is divisible by $2^n$ to get an $(n+1)$-digit number that is divisible by $2^{n+1}$.

Using this process, we can build all the way up to $n = 2012$ to get a number $N$ of the required form.

But can we prove that it always works? Well, for some number $n$, suppose $A_n$ is an $n$-digit number divisible by $2^n$. If $A_n$ is divisible by $2^{n+1}$, we use Lemma II, which tells us that $2 \cdot 10^n$ is a multiple of $2^{n+1}$. Then $2 \cdot 10^n + A_n$ is just the number $A_n$ with the digit 2 appended to the left. Since it is the sum of two multiples of $2^{n+1}$, it must itself be a multiple of $2^{n+1}$, and gives us a number of the required form.

It's a bit tougher if $A_n$ is a multiple of $2^n$ but not of $2^{n+1}$. But not much tougher. What is the remainder when $A_n$ is divided by $2^{n+1}$. Well, we can write $A_n = 2^n \cdot k$ for some integer $k$, so $\frac{A_n}{2^{n+1}} = \frac{A_n}{2^n} \cdot \frac{1}{2}$, which is some integer $\frac{A_n}{2}$ divided by 2. The only possible remainder is 1.

Now we look at $A_n + 10^n$. What is its remainder upon division by $2^{n+1}$? We've just seen that $A_n$ leaves a remainder of 1. And $10^n$ also leaves a remainder of 1. So (because $1 + 1 = 2$) $A_n + 10^n$ is in fact divisible by $2^{n+1}$.

These comments motivate the inductive proof given above.

**Problem 4.** Let $x, y, z$ be integers such that $(x-y)^2+(y-z)^2+(z-x)^2 = xyz$. Prove that $x^3 + y^3 + z^3$ is divisible by $x + y + z + 6$.

**Solution 1.** Expanding, we find that

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 2(x^2 + y^2 + z^2 - xy - yz - zx)$$

. Also,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

But we are given that $(x-y)^2 + (y-z)^2 + (z-x)^2 = xyz$. Therefore $x^3 + y^3 + z^3 - 3xyz = \frac{xyz}{2}(x + y + z)$ This means that $x^3 + y^3 + z^3 = \frac{xyz}{2}(x + y + z + 6)$. Notice that at least one of $x, y, z$ is even, because if $x, y, z$ are all odd, the left-hand side of the original equality is even while the right-hand side is odd, a contradiction. Therefore $\frac{xyz}{2}$ is in fact an integers, and $x^3 + y^3 + z^3$ is divisible by $x + y + z + 6$.

**Solution 2.** We note that the condition

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = xyz$$

is *non-homogeneous*: Each term of the left hand side has degree 2, but the right hand side ($xyz$) has degree 3. Similarly, the linear expression $x + y + z + 6$ is inhomogeneous (it includes a mixture of terms of degrees 0 and 1).

This suggests introducing a new variable $w$ to make all expressions homogeneous. To be precise, we will prove the stronger statement (of which the case $w = 1$ settles the original question):

*If $w, x, y, z$ are integers such that $w(x-y)^2 + w(y-z)^2 + w(z-x)^2 = xyz$, then $x^3 + y^3 + z^3$ is divisible by $x + y + z + 6w$.*

Note that $xyz = w(x-y)^2 + w(y-z)^2 + w(z-x)^2 = w\big((x-y)^2 + (y-z)^2 + (z-x)^2\big) = 2w(x^2 + y^2 + z^2 - xy - xz - yz))$. Moreover, $x^3 + y^3 + z^3$ is independent of $w$, but $x + y + z + 6w$ is not. Since $6w = 3(2w)$, this suggests multiplying We conclude that the divisibility required by the problem statement holds. Moreover, if $x + y + z + 6w \neq 0$, the quotient is $x^2 + y^2 + z^2 - xy - xz - yz$.

**Solution.** Let $S(n) = \sum_{k=1}^{n-1} k^n = 1^n + 2^n + \cdots + (n-1)^n$.

It looks like the statement might be true for all odd $n$, and for no even $n$. Let us try to prove this. We will first give a proof using only algebra, then show how the problem relates to some classical results in number theory.

**Solution1.** (A formal solution)

If $n = 2m+1$ is odd then $(n-k)^n \equiv (-k)^n = -k^n \pmod{n}$, so

$$S(n) = \sum_{k=1}^{2m} k^n = \sum_{k=1}^{m} (k^n + (n-k)^n) \equiv \sum_{k=1}^{m} (k^n - k^n) = 0 \pmod{n},$$

and $n \mid S(n)$.

Assume now that $n$ is even, say $n = 2^a(2m+1)$ with $a > 1$. Note that $a < n$, so $2^a \mid 2^n$. If $k = 2l$ is even, then $2^n \mid (2l)^n = k^l$, hence $2^a \mid k^l$. If $k$ is odd, then $k^{2^{a-1}} \equiv 1 \pmod{2^a}$ (by Euler's Theorem or induction on $a$), so $k^n = k^{2^a(2m+1)} = (k^{2^{a-1}})^{2(2m+1)} \equiv 1^{2(2m+1)} = 1 \pmod{2^a}$. Hence,

$$S(n) \equiv \underbrace{1+0+1+0+1+\cdots+0+1}_{n-1 \text{ terms}} = \frac{n}{2} = 2^{a-1}(2m+1) \equiv 2^{a-1} \pmod{2^a}.$$

Hence $2^a | S(n)$.

Since $2^a$ divides $n$, we conclude that $n$ does not divide $S(n)$.