ONE ROW NIM

Introduction:

Nim is a two-person, perfect-knowledge game of strategy. "Perfect knowledge" means that there are no hidden cards or moves, and no dice to roll, and therefore that both players know everything about their positions.

Nim is an ancient game in which players take turns picking up stones from distinct piles. (The word "nim" means "take" in old English.) There are variations of the game depending on how many piles of stones there are at the beginning of the game, how many stones a player may take in a turn, and what constitutes a "win." The game is simple, and students need no background knowledge to play the game, yet it serves as a medium for discussing important mathematical ideas: the notion of strategy, division with remainder, modular arithmetic, representing patterns with variables, and mathematical induction.

Procedure for one-row nim:

The game is played with two players and twelve counters (such as beans), which are laid in a single row. (The beans here are numbered, but they will not be numbered in a regular game once students learn how to play the game.)

1	2	3	4	5	6	7	8	9	10	11	12
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Players take turns picking beans in order from the right. In a turn, a player must take at least one bean and can take as many as five beans. The loser is the person who takes the last bean (bean number 1). One way to explain this game is to ask students to think of a player's turn as taking and eating jellybeans, and to imagine that bean number 1 is the "bad" jellybean, with an unpleasant flavor like fish or hot sauce. The loser is the one forced to "eat" the bad jellybean.

The Strategy:

The strategy that will emerge is to choose to go first and to take five beans in the first turn. If the first player does this, then the second player cannot win (unless player 1 makes an error). We say player 1 has a "winning strategy." Another way to think of the game is from the point of view of the loser. The loser is the one who takes bean number 7, and the first player can force the second player into this losing position by picking up five beans in his first turn. In fact, the strategy does not really change if we start 15 or 20 beans. In general, the player who picks up a bean numbered 1, 7, 13, 19, etc. ($6k+1$) is in a losing position. So, by taking every bean up to a $6k + 1$ bean, the first player can force the second player to take a "bad" bean and can then force a win.

Suggested Classroom Procedure:

Phase I:
First play exhibition games on the board between the teacher and a student. Ask the student who should go first, and then let the student win. For teaching the game on the board it is useful to number the beans and have players take beans in order from the right so that the “bad” bean is the one labeled “1.”

The example below shows an exhibition game on the board. Player 1 “takes” 4 beans by placing x’s below the beans he takes, and then player 2 “takes” two beans by making a different mark below the beans he takes. It is convenient to record the players’ turns in rows below the “beans” this way so that students can examine and think about how the game had been played.

1	2	3	4	5	6	7	8	9	10	11	12
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
						/	/	x	x	x	x

If the group is fast, the purpose of the exhibition game is simply to clarify the rules, and the students can the work in pairs, playing the game until they find a winning strategy. If the group needs more time to engage (this is typical with younger students, below sixth grade), play one or two exhibition games, then have pairs of students play each other at the front of the room. Try to get as many students up to the front as possible. Many groups enjoy calling out suggestions for strategies as the game is played. This allows for greater participation, both numerically and emotionally.

Phase II:
At some point, it is usually good, even for younger students, to have them play in pairs 'privately'. If two students in a pair do not seem to be progressing in developing strategy, split the pair up, using students who have more insight. The weaker players will learn from the stronger. If a pair solves the game quickly, add three more beans to the row, making 15 in all. (This is the next stage of the game that will help students move toward a general solution.)

Normally students readily engage in this game, both in groups and in pairs. They find quickly that if the first person takes 5 beans, he can force a win. While some students go from there to a general solution, most do not. The concept of 'winning strategy' is one that must be developed.

Initially, many students are satisfied that they have learned the 'rule' for winning: go first and take five beans. The insight they need is to think of leaving the opponent with 7 beans, rather than of 'taking' 5 beans. You can help a student to see this by asking what they would do if they went first, and there were 11 beans, rather than 12. Typically, they will see right away that they must take 4 beans, and this turns their attention to the number of beans remaining—which is the key to solving the game. You often don't need to verbalize this key for them. They will see it themselves, and sometimes even verbalize it themselves. Making the thinking conscious will be a later step in this activity.

When pairs of students have solved the game with 12 (on any level of solution), we can bring them further by asking them to play the same game with 15 beans. This situation presents difficulties. The insight needed is that if they can force their opponent to take bean number 13, they can then force him to take bean 7, and we already know that this forces a win. The first player can do this if he takes 2 beans.

At this point the leader must judge when the group is ready to talk about the experience. Some faster students can be given 20 beans, just to see if they understand the game in full generality. Even if they do, giving them practice with more beans will enrich the discussion later.

Phase III:
Call the group together to discuss their experiences and what they have learned. Most will know that for 12 beans the first person wins by taking five. We want them to re-think this strategy, to get to the idea that leaving the opponent with 7 beans is a winning position. That is, we want to help them see that if a player takes the seventh bean, no matter what he took before that, and no matter what other bean he takes, he must lose.

There are several ways to do this. One is to play a few games, and circle the seventh bean in red. They will note that taking the seventh bean makes a player lose. Or, you can call their attention to the record of 'exhibition' games, and merely state that the seventh bean was always taken by the loser. (This will be obvious if the games are recorded in rows, as suggested above.)

One effective way to describe the danger of taking bean 7 is to continue the food metaphor: bean 7, like bean 1, is “secretly poisoned” with a bad flavor.

It is a good idea to ask the students to verbalize why this is true, even if they all see it. You can then elicit the idea that the player who forces the other to take the seventh bean always has a winning 'reply' to the move. This is another element of strategy that must be made conscious for some students.

For some groups, this is enough of a lesson for an initial activity. You might choose to go on to another activity, and return to this on another day, perhaps with 15 beans. Because of the intense motivation of winning a game, the experience from a first day usually stays with the students and is available to build on during a second day.

But in fact many groups of students can stay with this one activity. If enough of them have had experience with 15 beans, you can launch a discussion right away. The object of the discussion should be that bean number 13 is also 'secretly poisoned': if a player is forced to take it, then his opponent can force him to take bean 7, and thus to lose.

Once students have this insight, the game becomes abstract: they quickly do not need beans, and can think about the essential elements of the game without the physical props: the number of beans on the table and the allowable moves.

So, for students ready to advance (and here this does not depend on age or background), they can be asked to imagine a row of 100 beans, and to identify the numbers of the 'secretly poisoned' beans. They typically come up with a list:

1, 7, 13, 19, 23, 29...

and a rule: “Add six to get the next poisoned cookie.” This recursion allows them to play a game with relatively few beans. (It is rarely necessary to actually play, however.)

A good question to ask at this point is whether the first player always wins, no matter how may beans are on the table initially. Many students will answer 'yes', then realize that if there are 7, or 13, or 19 beans on the table initially, the first player must lose, not win. This too is an important step in the formulation of the concept of a winning strategy.

But what if they are presented with 1000 beans? Should they go first or second, and how should they play? A story about such a game, say with a very powerful opponent (the 'devil'), might be good motivation. The issue is that they must find a pattern to the poisoned numbers that does not depend on the previous poisoned number. They must solve the recursion.

One way to get them to see how to solve the recursion is to list the bad beans vertically as in the example below and to have a student subtract 1 from each of the poisoned numbers and place it to the left of the list of bad beans:

	Bad Beans
0	1
6	7
12	13
18	19
24	25
30	31
36	37

Students in grades 4 or higher will quickly recognize the 'six times table'. At this point, you can introduce some algebraic notation. (Don't be afraid of this! Many students will understand, and those who don't can pick up the thread of the argument a bit later, without the algebraic notation. At this point in the activity, most students will have the concept of division with remainder, and its application here, in mind. The algebra simply documents what they have thought of.) The six numbers in the times table are all of the form 6K, so the poisoned numbers are of the form 6K+1.

So, for 1000 beans, we must find the next lowest multiple of 6, then add 1 to find the first poisoned bean. Students who don't see the algebra can understand this formulation, and are likely to see that the tool they need is division with remainder.

Now division with remainder means writing 1000 = 6Q + R, where Q is the quotient and R the remainder. But most students will not think of it in this algebraic notation. You will probably have to perform the division as they were taught to:

$$\begin{array}{r} 1\,6\,6, \text{ R. } 4 \\ 6\overline{)\,1\,0\,0\,0} \end{array}$$

then ask them to 'check' the division. This will give them the usual algebraic form: 1000 = 6 x 166 + 4. It is good for them to actually multiply out 6x 166 = 996 to see that this works. Many will have been trained to do this anyway.

You can point to the lists of numbers, to emphasize that 996 is in the list of multiples of 6. Then bring them back to the game: 997 is the first poisoned number, so they want to be the first player, and to take 3 of the 1000 beans.

Extensions: This game is very rich. If students show interest, they can be asked how the game can be changed to be more interesting. Someone will suggest that you vary not just the number of beans on the table, but also the number of beans a player is allowed to take up.

That is, in general, the game can be described as starting with N beans, and players choosing from a set S of moves, which is a set of numbers of beans that can be removed in one turn. (For the initial game, N = 12 and S = {1, 2, 3, 4, 5}).

In full generality, this game has not been solved. So it may not be a good idea to choose S arbitrarily. Some useful variants are:

1) S = {1, 2, 3, 4}. Here the poisoned beans are 5K+1, and it will be easy for students to describe them: they end in a digit 1 or 6. Usually, this is not a good first game, as the students rest on this description and are not motivated to go on to describe the poisoned beans using division with remainder. But as a second game, it is good reinforcement, and offers the insight that the last digit of a number (in decimal notation) is the remainder when the number is divided by 10.

2) S = {1,2,3,4,5}; N = 12 (the initial game), except that the winner is the person who takes the last bean, not the one who avoids it. A 'poisoned cookie' strategy again emerges. This time the poisoned cookies are simply 6, 12, 18....

3) S = {1, 2, 4}, and the object is to take the last bean—the player left with no move at all is the loser. This game is more complicated, and it is more difficult for beginning students to understand that it even has a strategy. The poisoned beans are the multiples of 3.

4) S = {2,3}, except that if there is only one bean left (the poisoned one) the player whose turn it is must take it. Students generally can develop an *ad hoc* winning strategy for a small initial number of beans, but in general the game is difficult.

And if S contains more than three numbers, and does not include the number 1, the general game has not yet been solved. Students enjoy hearing that they have come to a frontier of mathematical knowledge in such a short time.