

## One-piece Chess and Two-Row Nim

These are two mathematically related games that are best played one after the other so that students can see their relationship. Like one-row Nim, they are both simple, two-person, perfect-knowledge games of strategy.

One-piece chess:

This game requires a chessboard and a single counter. A chess board (64 squares) can be photocopied or even drawn. A chessboard is easily drawn by dividing a square into quarters, and then each of those quarters in half vertically (16 squares) and then each of those in half again horizontally (64 squares).

Procedure: The single counter is a rook or castle in chess. But in this game it can move only up or to the right, never down or to the left. And never diagonally.

The rook starts in the bottom left corner. Both players move the same piece, taking turns. They can move it any number of squares up, or any number of squares to the right.

The second player has a winning strategy, by always moving the rook onto the diagonal. This strategy typically emerges from the play. Students see that the seventh diagonal square is a winning position: the player whose turn it is must move off the diagonal, and then the other player can win. The next insight, typically, is that the sixth diagonal square is also a winning position, then the fifth, and so on.

For advanced students (but not for most!), a discussion might ensue about what a 'winning position' is. Formally, a set  $S$  of winning positions is such that (a) the last position in the game is in  $S$ ; (b) from any position outside of  $S$  you can move into  $S$ ; and (c) from a position inside  $S$  you must move outside  $S$ . The set of diagonal squares here satisfies these conditions. Since the first player starts at a position in  $S$ , the second player has a winning strategy, by always moving to a position inside  $S$ .

But it is not necessary to make this generalization for students at this level to have learned from the game. Once they have solved this game, for example for a 'fast' pair of students, you can fold over three rows of the chessboard, and ask them what would happen if they started with a 5x8 chessboard. In this situation, the first player can move onto a 'diagonal' (i.e. a diagonal of the 5x5 subset of squares which includes the last square), so the first player has a winning strategy.

The next activity extends this one.

### III. Two-row nim

Materials: 14 counters (smarties, beans or other objects)

Procedure: The board consists of two rows of seven smarties each. You may take as many smarties as you want on your turn, but only from one row per turn. The object is to be the player that takes the last smartie. It is not essential to the game, but makes it easier to strategize, if you remove smarties from left to right.

The second person has a winning strategy: you simply copy the moves of the first person.

This game is isomorphic to one-piece chess, and this fact is the real reason to play these two games in our context. In the chess game, there are seven possible moves upwards and seven possible moves to the right. We can think of each move as 'taking away' that many moves from the possible moves of an opponent upwards or to the right. We can then keep track of the game with two sets of counters, the 'upwards' counters, and the 'to the right' counters. This is exactly the game of two-row nim.

Students get this idea quickly if you play an exhibition chess game with one of them, and record the moves simultaneously in 'nim' fashion.

If you start with unequal rows, the first player will have a winning strategy—by making the rows equal, then following the symmetric strategy outlined above.

Note: Three-row nim is much more complicated. It can be envisioned as a 3-dimensional array. But it's harder to understand what the winning positions are: the diagonal does not play quite the same role in three dimensions. The game has been solved, and the solution accessible to middle school students. But it's the sort of thing they will memorize as a trick, rather than learn from.