Gumdrop Polyhedra

Equipment:

(1) Spice drops of many colors. They work best if aged a few days, but this is not obligatory

(2) Cocktail toothpicks. The round ones work best.

For the 'soap bubble' extension (a great activity to do out of doors), you need

(3) Dishwashing detergent
(4) Glycerine (available in the facial care department of a drug store). This is optional, but stabilizes the soap bubbles you will be making.
(5) A large bowl or container.

A: Ask students to use three gumdrops of the same color and toothpicks to form an equilateral triangle. Discuss why it is equilateral.

B: Challenge students to add 3 more toothpicks and one more gumdrop—of a different color--to create 4 equilateral triangles. Not every student will see that they must use a three-dimensional shape to do this. But some will, and others will copy them.

The result is a regular tetrahedron: a pyramid with a equilateral triangular base.

At this point, students should get used to the following vocabulary:

Polyhedron (pl. Polyhedra): The three-dimensional analogue of a polygon. (A more precise mathematical definition is not appropriate here.) A polyhedron has three elements: Vertices, Edges, and faces.

In our construction, the vertices are the spice drops and the edges are the toothpicks. The faces are not 'made out of' anything: they are the four equilateral triangles originally asked about.

Be careful not to use the word 'side', which we use for polygons. In three dimensions, this word is ambiguous. It might mean edges, or it might mean faces. We simply do not use it.

Point out that the tetrahedron is a pyramid, but not of the 'Egyptian' sort. It has a triangular, rather than a square, base.

Students should also notice that their tetrahedron is a *regular* figure: its faces, edges and vertices are indistinguishable from each other, except for color. One way to explain this is to think of the polyhedron as a huge space station, made up of four 'pods' (represented by spice drops) where people live and work, connected by corridors (the toothpicks).

You cannot tell which pod you are in just by looking at the corridors and pods. You would have to know the color of the pod, the nature of the furnishings inside, and so on. The geometry of each vertex is identical.

Also have students notice that any two tetrahedra with three vertices of one color and one of another are 'isomorphic'. Except for the specific color used, the colorings are the same. You can demonstrate this by choosing any two tetrahedra of those the students constructed. If you place each with the single-color vertex pointing up, they will look identical except for the particular colors chosen.

Finally, start a chart at the board, tallying the number of faces, edges, and vertices in each polyhedron you build. The chart will look like this initially:

	Faces	Vertices	Edges
regular 4-hedron	4	4	6

B: Have students make another tetrahedron, this time with two gumdrops of one color and two of another. Select two of these, and show that they are also 'isomorphic', by placing identically colored vertices in the same position.

Students may initially think that there is more than one way to color a tetrahedron with pairs of identical gumdrops. It is important to show them that the figure can be rotated so that colorings are identical.

C: Have students take apart one of their tetrahedra and 'add' it to the other. That is, ask them to form a polyhedron made up of two tetrahedra, back-to-back. The resulting polyhedron is made up of six triangles. (Two of the eight triangles in the original two tetrahedra are 'lost' in the construction.) This is a *triangular hexahedron*.

Ask students if the triangular hexahedron is a regular figure. It takes some thought to see that it is not. Some vertices have only 3 edges leading from them, while others have 4. If this were a spaceship, you might prefer a 3-edged pod over a 4-edged pod for some reason.

They can enter a new line in their table:

	Faces	Vertices	Edges
regular 4-hedron	4	4	6
triangular 6-hdron	6	5	9

If there is room, have students leave their hexahedron on the desk. The count itself is good exercise, even if students are looking at the model.

D: Ask if they can think of a regular hexahedron, one in which all the vertices look alike. This is difficult. It turns out that the faces cannot be triangles, but must be squares. Once students see (or are told) that, they quickly realize that a 'regular hexahedron' is simply a cube, and can easily construct it.

But don't let them construct it right away. They are familiar with a cube, and we can get more out of the experience.

Ask them first to fill in the next row of the table, imagining, but not constructing, a cube:

	Faces	Vertices	Edges
regular 4-hedron	4	4	6
triangular 6-hdron	6	5	9
cube	6	8	12

Some students will be surprised to see that cube does not have four of any of its elements.

Then ask them to construct a cube with colored spice drops, so that no two vertices of the same color are connected by an edge. Challenge them to use the fewest number of colors.

The cube can be 'correctly' colored with only two colors. We can think of a bottom square with colors alternating along its perimeter, and a top square with the same two colors alternating. The square are connected by vertical edges so that no two pairs of vertices of the same color, one from the bottom and one from the top, are connected.

We say that the *chromatic number* of the cube is 2, because it can be 'correctly' colored with 2 colors, and no more are necessary. Students can be asked about the chromatic number of a tetrahedron. It is 4, since each vertex is connected to every other. Students may be surprised that the simpler figure of a tetrahedron requires more colors than the more complex figure of a cube. The chromatic number depends on how the vertices are connected, and not simply on how many there are. Students can note that the chromatic number of the triangular hexahedron is also 4: a tetrahedron 'lives' on the top, and you need four colors for its vertices. The two 3-edged vertices, however, can carry the same color.

Looking at a 2-colored cube, students can be asked to visualize the figure formed by the vertices of one color, then of another. These vertices form a regular tetrahedron, and it is good exercise for students to identify this. The edges of the tetrahedron are diagonals of the square faces of the cube, so they are all equal.

An advanced question: these two tetrahedra intersect. What is the solid formed by their intersection. (Have students think about, rather than answer, this question.)

Another advanced question might be to ask students to visualize the six faces of the square extended to form six planes. Into how many regions do these planes divide space? (Answer: 27) The same question for the tetrahedron is more difficult.

E. Have students construct an “Egyptian” (i.e. square-based) pyramid. It has five faces, and so is a (non-regular) pentahedron. Ask them about its chromatic number (answer: 3), and about the number of faces, edges and vertices:

	Faces	Vertices	Edges
regular 4-hedron	4	4	6
triangular 6-hdron	6	5	9
cube	6	8	12
Egyptian pyramid	5	5	8

F: Now ask students to add a second Egyptian pyramid to the first, back-to-back, as they did for the non-regular hexahedra. Ask them the name of the resulting polyhedron. Since it has eight faces, it is called an *octahedron*. (Polyhedra are usually named after the number of faces.)

It is a surprising result that this figure is regular: the vertices are identical, and no matter how you turn it, you have two Egyptian pyramids, one facing up and one facing down. Its chromatic number is 3, and students can count its faces, edges, and vertices:

	Faces	Vertices	Edges
regular 4-hedron	4	4	6
triangular 6-hdron	6	5	9
cube	6	8	12
Egyptian pyramid	5	5	8
Regular 8-hedron	8	6	12

Students can be asked to visualize the figure formed by taking the centroid (middle) of each triangular face and connecting them. Surprisingly, you get a cube. If you do the same thing for a cube, you get an octahedron. These two figures are called *dual figures* for this reason.

Students can check that the figure dual to a tetrahedron is another tetrahedron.

G: Euler's Formula. Students can look at the table they've formed and be asked which are there usually the most of? Answer: edges. But of faces and vertices, it's difficult to tell which there will be more.

Next, squeeze in an extra column in the table, without a label:

	Faces	Vertices	x	Edges
regular 4-hedron	4	4	8	6
triangular 6-hdron	6	5	11	9
cube	6	8	14	12
Egyptian pyramid	5	5	10	8
Regular 8-hedron	8	6	14	12

Students will quickly notice that you got this column by adding the number of faces and the number of vertices, and also that it is two more than the number of edges. This is Euler's formula, an important theorem in what is known as graph theory:

Euler's Formula: For any polyhedron without holes in it, $F + V = E + 2$.

Students can construct a polyhedron of their own and test this formula. But they need guidelines. Typically, they will construct a huge mass of toothpicks and spice drops, which will have holes, internal 'edges', coplanar 'faces' and so forth. It is best to give them guidance. ("Add an Egyptian pyramid to a cube;" "Add a tetrahedron to an octahedron;" etc.)

A proof of Euler's formula is accessible to these students, if the group is interested. See for example http://www.ams.org/samplings/feature-column/fcarc-eulers-formula. (But in fact the proof can be made clearer than this resource demonstrates.)

The formula does not hold for a polyhedron with a hole in it. It does hold, however, for a graph which is a two-dimensional 'map' of the vertices and edges of a polyhedron.

Another fact that can be proved to this group, or just stated, is that there are only five regular polyhedra. There is a regular polygon for any integer N, but not a regular polyhedron. We have already constructed 3 of the regular polyhedra: the tetrahedron, the cube, and the octahedron. The other two are the dodecahedron (12 faces) and the icosahedron (20 faces). The dodecahedron cannot be constructed with spice drops and toothpicks. Its faces are pentagons, and the figure will collapse of its own weight.

The regular icosahedron, however, can be constructed using these materials. A picture of one is easily obtainable on the web. A good exercise for students is to show them a model of a regular icosahedron, and ask them to construct it for themselves, without touching the original model. Laying the model down on a plane will make it hard to see the symmetry of the figure, which is more apparent if it is 'balanced' on one of its vertices. This circumstance makes the exercise more valuable.

G: Soap bubbles

The polyhedron activities can be used without this step, but it is fun to dip the models in soap bubbles and see what happens.

Fill a small pail with water, dishwashing soap, and glycerin. A good mix is about a gallon of water to a pint of dishwashing soap and a cup of glycerine. (The latter is not strictly necessary, but stabilizes the resulting bubbles.)

First hold up a cube, and ask students to predict what the soap bubbles will form. They are likely to think that the bubbles will wrap around the cube. In fact, they form a surface inside the cube, made of planes that intersect at 120 degrees.

What is happening is that the soap bubbles try to find the surface of least area which includes the edges of the cube.

Students can then have fun dipping their own polyhedra in the soapy water and seeing what surface is formed.

Soap bubbles are a serious subject of investigation by mathematicians.