The Fifteen Game

Nine cards are laid out on the table: Ace, 2, 3, 4, 5, 6, 7, 8, 9. (Or: the numbers 1 through 9 are written on a chalk board.) The ace here is used as a marker for the number 1. (It has no other value.)

Two players take turns choosing a card. Once the card is chosen, it is inaccessible to the other player. The winner is the first player who collects exactly three cards that add up to exactly fifteen.

Classroom hints: First play some exhibition games. The rules of this game are hard to describe, and the strategy harder to see, than in previous games. Students will think that two cards (say, 9 and 6) constitute a win, because they add up to 15. Or, they will think that once a player has chosen three cards (say, 3, 5, and 6) he has lost, because they don't add up to 15.

As usual, it does not help to try to clarify the rules in advance. They become clear during the play—part of the reason to play exhibition games is to elucidate the rules, and part of the learning for the students is to understand the rules, and then their consequences. (This imitates the structure of axioms and theorems in more serious mathematics.) Have students play each other. It is unlikely that they will develop a reliable strategy.

In fact, there is no winning strategy for either player. Perfect play results in a tie game. Indeed, you will know that students understand the game when they start tying very often.

It is best not to talk much about a strategy for this game, but to come back to it after students have some experience with magic squares.

In fact, this game is isomorphic to tic-tac-toe, played on a magic square. But students can have the joy of discovering this for themselves. Many will comment, as they play, that the strategy is 'blocking', and that it 'feels' like tic-tac-toe. It is best, usually, to share this comment, but not to make it precise. The isomorphism, revealed later, provides precision.

It might be best to play this game a few times—perhaps as a time-filler or transition activity, then wrap it up later (see section VI).

Magic Squares, Part I.

This is a classic topic in recreational mathematics. Here we tap into it only briefly. It is an activity, not a game.

Give groups of students nine cards, ace through nine, of any suit.

I. Ask them to make three rows such that the sum of the rows is the same.

Most students eventually arrive at a solution (there are many solutions). If they need a hint, ask them to lay out three arbitrary rows, and take their sums. They can 'fatten up' a lean sum, and 'slim down' a fat sum, by trading a larger card for a smaller. In trading around like this, they ultimately solve the problem.

Students will complete this task at different rates. As they complete it, you can give them assignment II below.

II. Students now have a 3x3 array of cards in which the rows all have the same sum. Ask them to 'improve' the array, by making the columns also have the same sum.

If they need a hint (and many do), show them that they can keep the row sums the same, but change the column sums, by trading cards within a row. Again, they can make lean columns fatter and fat columns leaner by trading within a row.

As they finish this task, give them assignment III.

III. Ask students to 'improve' their array still more, by making the diagonals have the same sum.

Another hint may be in order: if you take the top row, and make it the bottom row (this is awkward physically, but conceptually easy), you haven't changed row or column sums, but you've changed the diagonal.

You can thus 'sort' the rows. And you can do the same with the columns. Using this (not very obvious) procedure, it never takes more than a few steps to go from the result of assignment II to the completion of assignment III.

As students finish assignment III, have at least three of them put their results on the board for all to see. Choose three different solutions. (There are eight solutions in all.) Try to get, among the three you choose, two solutions that differ by a line reflection, and two that differ by rotation.

Each solution is what is classically called a 'magic square'.

IV. Whole-group instruction:

The topic now becomes geometry, rather than arithmetic. There are eight possible arrangements of the numbers 1 through 9 into a magic square, corresponding to the eight symmetries of a cube.

That is, from any one solution, you can get to any other by rotation (4 rotations, including a rotation through 0 degrees) or line reflection (in the two diagonals and in a horizontal and vertical midline of the square).

Let the students observe the three solutions you've chosen, and ask if they are really all different. They are likely to see for themselves, especially if you've been able to choose two solutions that differ by reflection.

Depending on the group, you can get to different stages of understanding. We've outlined everything below, but feel free to go on to something else when students have had too much.

IV. Relating the solutions: Students quickly get the idea that you can reflect a solution to get another solution. The fact that they can rotate solutions is a bit more difficult, but usually arises by inspecting different solutions.

V: Symmetries of a square: Ask students how many ways you can take one solution and get others. There are seven ways – seven symmetries of a square. Introduce the term 'symmetry': A way to move a figure so that it coincides with itself. Don't forget to count the 'identity' symmetry: rotation by 0 degrees.

To clinch the idea, you can ask students how many symmetries an equilateral triangle has. There are six symmetries: three rotations and three line reflections. But they are harder to spot than the symmetries of a square: the reflections are in oblique lines and are hard to distinguish from rotations.

VI: Combining symmetries: You now have three solutions on the board, say square A, square B, and square C. Have students identify the rotation or reflection that takes A onto B, and also B onto C. Then ask them if they can describe a motion that takes A directly onto C, without going 'through' B. They usually can do this.

This question leads to the algebra of symmetries: you can perform one symmetry, then the next, and the result is a third symmetry. This is called 'composition' of symmetries. For many students, this is enough of an insight. We describe the next step for any students that may be ready for it.

VII. Combining symmetries is not commutative (!). Suppose we have symmetry (a) that takes square A into square B, and symmetry (b) that takes square B into square C:

A ----(a)----->B
B ----(b)----->C

Then the composition a*b = c, takes A directly into C:

A -----(c= a*b)-----> C

But suppose we compose in the other 'direction'? What does b*a do to square A:

A------(b*a)----->???

In general, students will find that the result is not square C, but some other position of the magic square. It makes a difference, usually, which symmetry you apply first, and which you apply second. Composition of symmetries is not *commutative*.

(Note: it is vital, for this step, that (a) and (b) not both be rotations. Rotations will commute with each other. But if at least one of (a) or (b) is a line reflection, the composition of those two symmetries will not commute.)

In advanced work, mathematicians study the *dihedral group* of symmetries of a square. This is the set of eight symmetries, together with the table of operations that they generate. It is well within the abilities of middle school students to generate this table, and to make some interesting deductions and observations. But this is probably not suitable for a summer study situation. We make this note simply to observe that the mathematics here is quite serious.

VI. Magic Square, Part II: Magic squares and tic-tac-toe

Once students have seen a magic square with the numbers 1 through 9, the game 'Fifteen' (activity IV) acquires a new meaning.

Have students play 'Fifteen' again. Then play an exhibition game on the board, and 'keep track' of the moves by placing an x (for one player) and an o (for another player) on the numbers of a magic square. Students will quickly see, usually to their delight, that they've been playing tic-tac-toe.

Their idea of a 'blocking' strategy is exactly what they know from that game. They had been intuiting the isomorphism before it was made precise in this way.

The game tic-tac-toe has many interesting variants. You can play some of these as exhibition games, or ask students to play them as transitional activities:

1. Tic-tac-toe to lose: The usual rules, but the player who puts three markers in a row loses, rather than wins. This is a good way to get students to re-think the rules.

But it is not a good game to play for very long. Students quickly realize that it always ends in a draw, with any serious attention to playing at all.

2. Tic-tac-toe with x's or o's: The usual rules, except that a player, in his turn, can place either an x or an o in any square. The first player to get three in a row of either symbol is the winner.

Again, this game quickly becomes dull, but is good for getting students to re-think rules and strategies. It is important for students to keep track of whose turn is it, or they will begin to argue. It is not possible (as it is in normal tic-tac-toe) to tell whose turn it is from looking at the board alone.

3. You can even combine both variations and play tic-tac-toe to lose, with each player allowed to place either an x or an o.

Games 1-3 are useful to clinch the notions of rules and strategy. The rules of a game are its axioms: the assumptions we start with. The strategies of a game are its theorems: propositions that follow from the rules. It is not necessary to make these concepts conscious for the students to be learning about them. It is enough, on this level, for students to realize that there are rules given, and strategies implicit, but not stated, in the rules.

4. Three dimensional tic-tac-toe. This is a better game than the previous. It is fun to play, and students learn about visualization in three dimensions.

Put three tic-tac-toe boards on the chalkboard. These are the three floors of a building. You can win on any one floor, but also by forming a row of three up the side of the building, or diagonally up one side, or diagonally through the center of the building (the center square of the center board).

Many students have trouble visualizing the winning positions. It is important to play many exhibition games and discuss where the winning positions lie, so that students begin to visualize them. For many groups, it may be appropriate to play only exhibition games, as long as students are paying attention and visualizing the wins.

You can combine this game with variants 1-3. But the straightforward rules, on a three-dimensional board, are usually sufficient to engage students and have them learn mathematics.

Very young students (earlier than 4th grade) will not be able to visualize the board. Typically, they play on the first floor until there is a draw, then go to the second floor, and so on. There seems to be a developmental step necessary for students to visualize three dimensions. Happily, the vast majority of students in fourth grade or higher have made this step.

5. Four-dimensional tic-tac-toe

This game should be played only if and when students are comfortable with and enjoy playing three-dimensional tic-tac-toe.

On the chalkboard, draw a huge tic-tac-toe field. In each square, draw a miniature tic-tac-toe board, nine boards in all (plust the huge one). Players take turns as usual in tic-tac-toe, trying to form a row of three of their counters (X or O's).

Players can win within any singe tiny square, or across three squares which form a row, column or diagonal in the huge tic-tac-toe board. That is, there are eight possible 3-dimensional games contained within this one.

Players can also win with a row of three squares which include the center square of the center tic-tac-toe board, plus two other squares which are symmetric to each other in the center square.

On the one hand, the three- and four-dimensional versions of tic-tac-toe are actually easier to win than the two-dimensional version. If the first player claims the center square, it becomes more and more difficult to block him, as the dimension increases.

However, this is far from apparent to the casual or novice player. Both games are enjoyable even without being solved.

However, and especially in the four-dimensional version, the first player quickly wins. Students find this out, even without having any idea at all of a general strategy.

To add to the enjoyment of the game, try the following variation: even after a player has achieved a single row of x's or o's, the two players continue. They score points each time they complete a row of their marker, until one of them achieves an agreed upon number of points (11 is a good number to agree on!).

It is important that players keep careful track of whose turn it is: it becomes difficult to determine, from the board, who goes next, and arguments can ensue.

The point of this game is to practice (three dimensional!) visualization. It is not expected that students form a strategy or 'solve' the game.

VII. Magic Squares: part III: Linear Algebra

This set of activities can be used after activity V, when students are familiar with magic squares. It is independent of activity VI (tic-tac-toe).

By way of review, give students cards Ace through Nine and have them form a magic square. Put at least one example on the board. (We will not need more than one example.) Review that the 'magic' consists in the fact that the three rows, the three columns, and the two diagonals, all have the same sum.

Then remove the ace from each group's set of cards, and give them a ten instead. Challenge them to form a magic square with the numbers 2 through 10.

Most of the time, students start afresh with this problem, and go through the steps they used to get the original magic square. But sometimes they have the insight that they need only add 1 to each entry in the original square.

If they don't have this insight themselves, put up a copy of a new magic square. Then subtract one from each entry, and let them discover that this gives them one of their 'old' magic squares. The generalization then comes quickly.

To clinch it, you might ask what to do if they want to make a magic square of the numbers 4, 5, 6, 7, 8, 9, 10, 11, 12. They can read to you the entries and you can put them on the board (add 3 to each entry of the original square, or add 2 to each entry of a 'new' (2 through 10) square).

Once they have the idea of getting new squares from old, you can get them to focus on the process, rather than on the squares themselves. Multiply each entry of the original square (1 through 9) by 100. That is, simply place two zeroes after each entry. They will easily see that this new square is still magic.

It is sometimes useful *not* to point out to them that each number has been multiplied by 100. Instead, give them the numbers {3, 6, 9, 12, . . . 27} and ask them to form a magic square. (By this time, it is best to do things by pencil, rather than with cards.). They will eventually see that you can simply multiply the original 1 through 9 square by 3. At that point, you can point out to them that the same process had been used in adjoining two zeroes to each entry earlier: you had multiplied by 100.

The next step, which your group may or may not be ready for, is to ask about a magic square using {4, 7, 10, 13, 16, 19, 22, 25, 28}. If the original square is denoted by S, we can call this square 3S + 1. Then you can give them a square with entries of the form 10S + 2 and challenge them to 'decompose' it: to explain how to get it from the original square.

What is going on here is that 3x3 magic squares form a vector space. If you drop the condition that the entries be distinct numbers, the 3x3 square consisting entirely of 1's (or 2's or 3's or 17's) counts as magic. Then, if S and T are magic squares, you can add them term-by-term to get a magic square S+T. Or, you can scale them: multiply each entry by a number k to get a magic square kS.

The structure of a vector space is one that is basic to much mathematics, and this activity is a springboard to the notion of a linear combination, of a basis, and of much more that students may discover in more advanced work. But for now, the insight that one magic square generates many, many others is more than enough.