Arm Folding, Knot Theory and Topology

A. Folding arms

Most people don't realize that there are two possible ways to fold your arms. Either the left hand or the right hand is on top. Ask students to stand and fold their arms. Then divide them into right-folders and left-folders, without telling them what you are doing. Keeping their hands folded, ask them on what basis you've sorted them. They will see it themselves, especially if you fold your hands in the natural way, and join the correct group.

With a large group, you can sort five or six students, then call others up one by one, asking the group where they belong. If they know the rule, they should just smile, and not blurt it out. They can verify that they know by 'predicting' where you will sort each student.

When all students have folded their hands and noticed the two ways to do so, ask them to fold their hands the 'other' way. This is difficult for some students, and awkward for everyone. We each have a habitual way of folding our hands, and dislike acutely the other way of folding. And it has nothing to do with left- or right-handedness. You can verify this by asking left-handed students to raise their hand. (They will have to unfold their hands to do so!) With most groups, the left-handed students are randomly distributed between the two groups of folders.

When you fold your hands, you are actually forming what mathematicians call a *trefoil* knot: one of the simplest knots. You can demonstrate this by taking any student at all with hands folded, give her a piece of yarn to hold (one end with each hand), then asking her to slowly unfold her hands. The knot will be transferred to the yarn, and students will recognize it easily.

The trefoil knot comes in two forms, left- and right- handed, corresponding to the two ways in which one can fold one's hands. But the two forms do not annihilate each other. If you take two people, each of whom fold their arms opposite ways, and hand an end of each string to each person, and they unfold their arms together you get a different kind of knot. You do not get a simple loop of yarn.

Knots are a serious object of study by mathematicians. For example, various molecules can form knots, including the molecules of DNA that make up life. One basic question mathematicians address, which is not at all simple, is when two knots are the same. (That is, when one knot can be deformed into the other without cutting the knot.) As a special case, we might want to know if a piece of string is knotted at all: whether its form is equivalent to that of a simple loop without knots. (In knot theory, a knot is conceived of as a piece of yarn with the two 'loose' ends joined.)

B. Links: the boat problem

Another object of study is links: when are two pieces of string linked together, like links in a chain.

You can demonstrate this with the 'boat docking' problem. Get two long pieces of yarn, of contrasting colors, each with a large loop at the end. Tie one end of each piece to a desk or chair, which represents a boat. Each 'boat' will tie up at a dock, by throwing its loop over a post on the dock. (You can simulate the post with the leg of an upside-down chair.) Show students how the first boat ties up to the dock, then the second boat. Challenge them to let the first boat go free without taking the loop off of the second boat.

The solution is to pass the lower look through the upper loop, then over the post, then out through the upper loop.

Some students will see how to do this right away, while others will need to be shown how several times, and in slow motion. This ability to visualize the result seems to be independent of computational fluency, of verbal ability, or of other mental skills. So this activity gives different students a chance to excel. This topic is covered specifically in courses on navigation.

If you choose, you can consider a similar problem for three boats. The generalization is immediate and apparent. Even the order in which the loops occur does not present a difficulty: you can remove the 'middle' look or the 'bottom' loop without disturbing the others, in just the same way.

C: Linked Students

Another form of this problem is one that students love. You choose two students (of the same or opposite genders, depending on the group). Tie the left wrist of the first student to the right wrist of the *same* student. It is important to tie this loosely, so that the student can slip out (in case of emergency!). But it is equally important to tell the student that he or she is prohibited from slipping the loop off.

Then tie the left wrist of the second student to the his or her right wrist, but first looping the yarn through the loop formed by the first student and his linked wrists. The students are now linked together in a chain. The challenge is for them to unlink themselves.

Have a pair of scissors handy and watch closely to be sure they don't tangle themselves dangerously. Make sure that the yarn is long enough for each student to 'step through'.

For that is what they will start doing—but it won't help. They will have all kinds of ideas and have fun trying to untangle themselves. Other students can offer suggestions, but don't let more than one or two come near the tangled couple. If you have a manageable group, or help in managing the group, you can tie up two or three couples. They rarely find the solution to this problem by themselves.

But actually, the solution is the same as that for the boat problem. Think of one student's arm as the post, and his other hand as the first boat. The boat has tied up to the post (the second arm) using a link, which is the loop around the student's wrist. The second boat is the second student, and the loop is the one formed by the entire piece of yarn connected her two wrists. It is best to position the students so that the yarn loop is below the yarn loop. This position makes the analogy apparent. Then the yarn loop can be pulled through the wrist look, out and over the post (the first student's hand and arm) and pulled free. Each student remains with his hands 'shackled' by yarn, but the two students are not linked together anymore.

A good way to manage this is to take one pair of students to the side, where others cannot see, and show them the result. They can then testify to the group that (a) they had gotten free without breaking the rules, and (b) properly thought of, this is just the same as the boat problem. But even without saying anything, the appearance of the students, separately shackled, provides a hint to many students, who might have been thinking that the goal is to get the yarn off both students completely.

Once a pair of students has been shown the solution they can help by showing others. Or, they can show how to get back into the linked position, then out of it again.

For a small group of easily managed students, you might try linking three together, or linking three in a cycle of links. But unless you have close control of the students, this activity can easily get out of hand. The activity with pairs of students is enough to get them started thinking topologically.