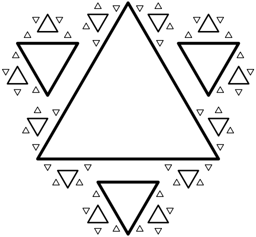
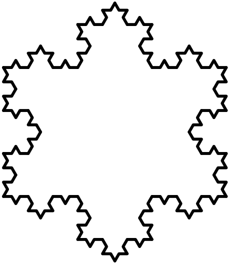
Composing and decomposing functions and surfaces



Robert Young
New York University

July 2022



cims.nyu.edu/~ryoung/slides/slidesICM.pdf

R.Y. was supported by NSF grant DMS 2005609

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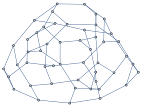
A: Probably too complicated to draw: 

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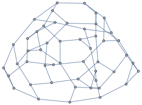
A: Probably too complicated to draw: 

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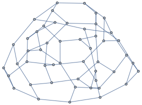
A: Probably too complicated to draw: 

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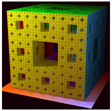
A: You can always add more complexity: 

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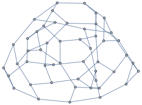
A: Probably too complicated to draw: 

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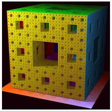
A: You can always add more complexity: 

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Plan

- Warm-up: Lipschitz functions
- Measuring nonorientability
- Applications to metric geometry

Warm-up: Lipschitz functions

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What's the most complex 1–Lipschitz function?

Maybe something like this:

Warm-up: Lipschitz functions

Let $\epsilon > 0$, let

$$1 \gg r_1 \gg r_2 \gg \cdots \gg r_k.$$

Let $f = \sum_{i=1}^{k} \beta_i$, where β_i is a wave with wavelength r_i and amplitude $a_i = \epsilon r_i$.

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As long as $k\epsilon^2 \ll \frac{1}{10}$, f is *mostly* 1–Lipschitz. So there's a 1–Lipschitz function which is ϵ–bumpy at $\approx \epsilon^{-2}$ different scales.

How do you decompose a Lipschitz function?

Let f be 1–Lipschitz on $[0, 1]$. For each i, let f_i be the piecewise-linear approximation of f such that $f_i(k2^{-i}) = f(k2^{-i})$ for all k.

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(See also: Fourier, Littlewood–Paley, Dorronsoro, Jones, David–Semmes, among many others)

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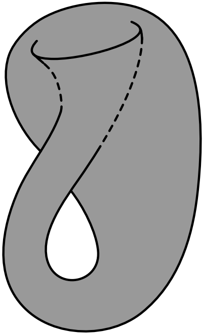

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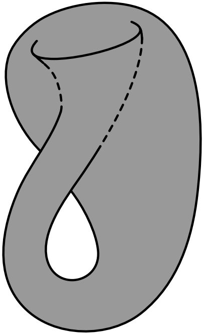

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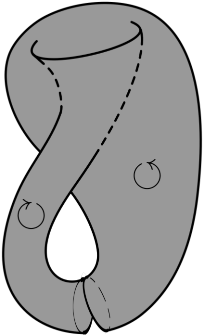

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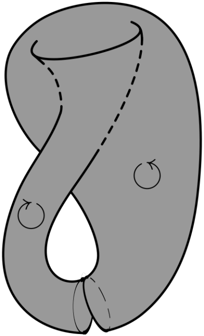
In this case,

$$\text{area}(P) = \text{area}(K) + \text{ area of two discs.}$$



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Quantitative nonorientability for cellular cycles

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What's the most nonorientable surface? How large can $\frac{\mathrm{NO}(A)}{\mathrm{area}(A)}$ be?

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But

$$\mathrm{area}(M) \approx R^2 + \sum_{i=1}^{k} r_i^2 \frac{R^2}{r_i^2} \approx (k+1)R^2,$$

so $\frac{\mathrm{NO}(M)}{\mathrm{area}(M)}$ stays bounded!

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Theorem (Y.)

For every $A \in Z_d(\tau; \mathbb{Z}_2)$*, we have* $\mathrm{NO}(A) \lesssim \text{area } A$.

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For every $A \in Z_d(\tau; \mathbb{Z}_2)$, we have $\mathrm{NO}(A) \lesssim \text{area } A$.

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Corollary (Y.)

If D *is an area-minimizing surface with boundary* T*, then there is an* $\epsilon > 0$ *such that any area-minimizing surface* E *with boundary* $2T$ *satisfies*

$$\text{area}(E) \geq \epsilon\, \text{area}(D).$$

Proof: Decomposing surfaces in $\mathbb{R}^n$

Let $M \in Z_d(\tau; \mathbb{Z}_2)$, let $M_1 = M$.

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Then:

- area(M_i) is a decreasing sequence of integers, so this process terminates.
- area$(M) \approx \sum_i$ area(A_i).
- M_i *is* a quasiminimizer on any set smaller than B_i.

Proof: Uniform rectifiability

Theorem (David–Semmes)

A quasiminimizer in $\mathbb{R}^n$ is uniformly rectifiable.

Definition (David–Semmes)

A set $E \subset \mathbb{R}^k$ is *uniformly rectifiable* if and only if there is a "small" collection of Lipschitz graphs that approximate E on most balls (a *corona decomposition*).

Proof: Conclusion

Therefore:

Proposition

Any mod–2 d–cycle M in $\mathbb{R}^n$ can be written as a sum $M = \sum_i A_i$ of mod–2 d–cycles A_i with uniformly rectifiable support such that $\sum \text{area } A_i \lesssim \text{area } M$.

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So $\text{NO}(M) \leq \sum_i \text{NO}(A_i) \lesssim \sum_i \text{area}(A_i) \lesssim \text{area}(M)$.

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A surface of area A can be decomposed into uniformly rectifiable surfaces of total area $\approx A$, which can be described by Lipschitz graphs of total area $\approx A$.

The Heisenberg group

Let $\mathbb{H}^{2k+1} \subset M_{k+2}$ be the $(2k+1)$–dimensional nilpotent group

$$\mathbb{H}^{2k+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_k & z \\ 0 & 1 & 0 & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & y_k \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_i, y_i, z \in \mathbb{R} \right\}.$$

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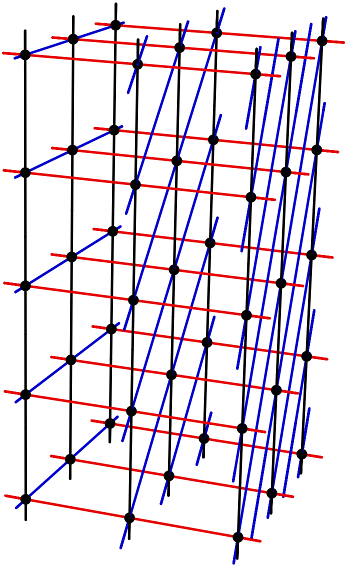
This contains a lattice

$$\mathbb{H}_{\mathbb{Z}}^{2k+1} = \langle X_1, \dots, X_k, Y_1, \dots, Y_k, Z \\ \mid [X_i, Y_i] = Z, \text{ all other pairs commute} \rangle.$$

The Heisenberg group $\mathbb{H}^3_{\mathbb{Z}}$



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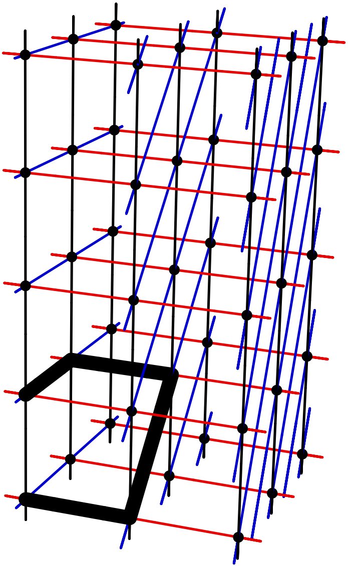

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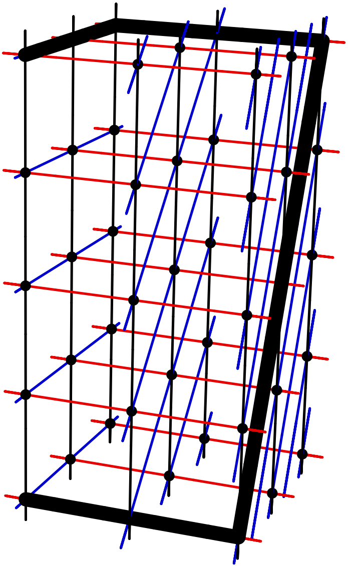

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Cheeger and Kleiner's proof is based on approximating the level sets of functions $\mathbb{H} \to \mathbb{R}$ by planes. Our methods let us decompose these sets into Lipschitz graphs, leading to:

Theorem (Naor–Y.)

Sharp quantitative bounds on Lipschitz maps from $\mathbb{H}$ to L_1.

Applications (with Naor)

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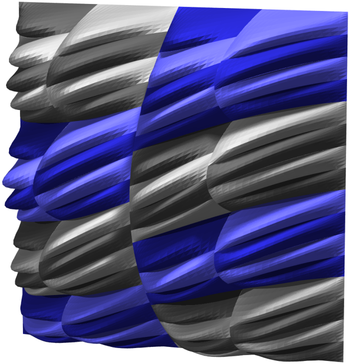
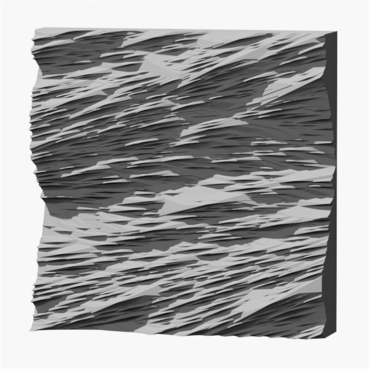
- There is a metric space M that has a bilipschitz embedding into L_1 and L_4, but not L_p for $1 < p < 4$.

Surfaces in $\mathbb{H}$





Some of the most complex Lipschitz graphs in $\mathbb{H}^3$.