Hölder maps to the Heisenberg group

and self-similar solutions to extension problems

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- What else?

Kaufman's construction

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But there is a self-similar map!

The Heisenberg group

Let H be the 3–dimensional nilpotent Lie group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}.$$

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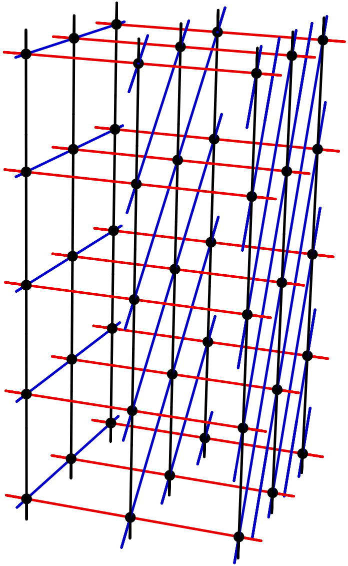
This contains a lattice

$$H^{\mathbb{Z}} = \langle X, Y, Z \mid [X, Y] = Z, \text{ all other pairs commute} \rangle.$$

A lattice in H^3



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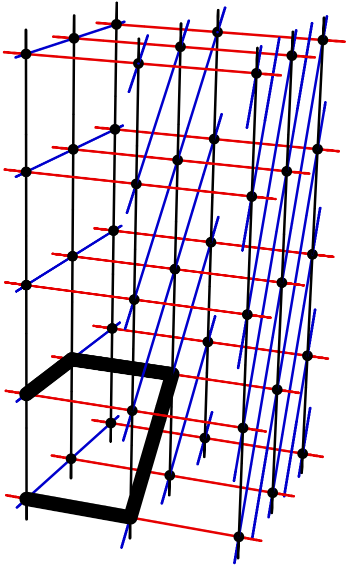

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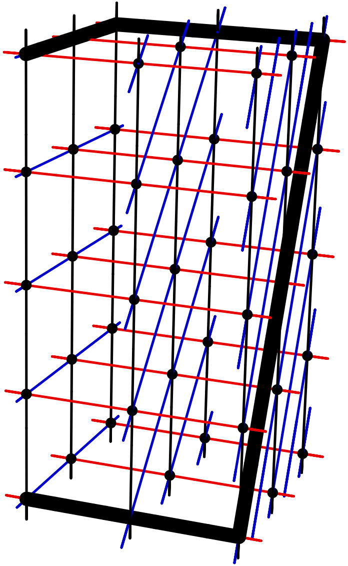

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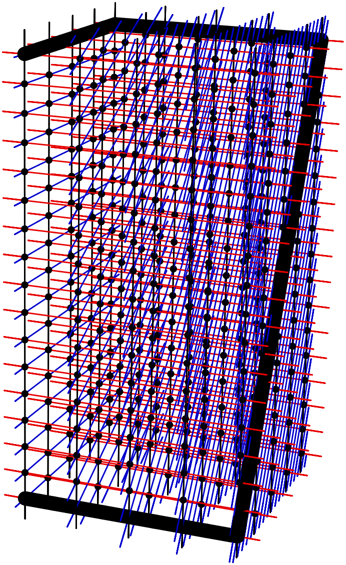

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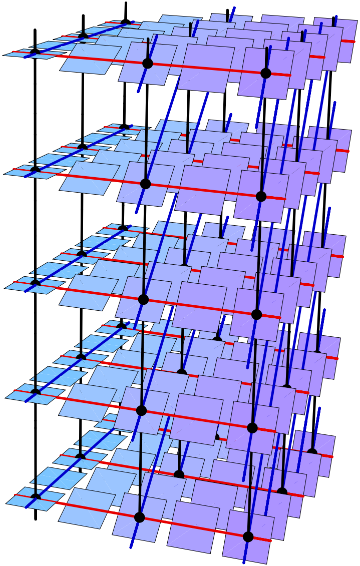
From Cayley graph to sub-riemannian metric



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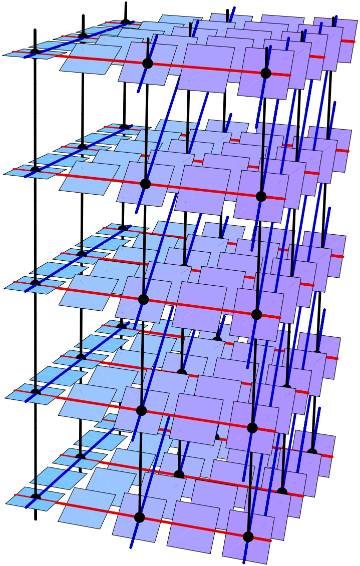

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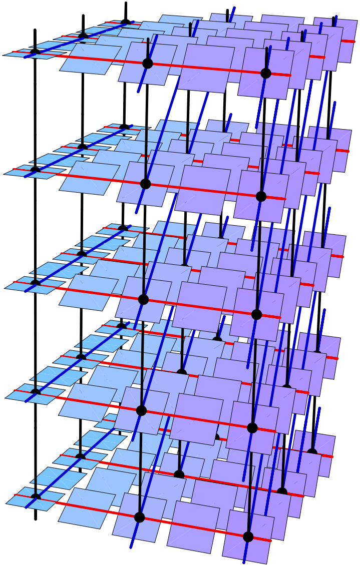

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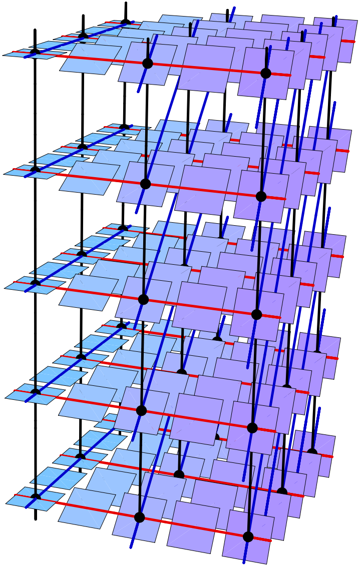

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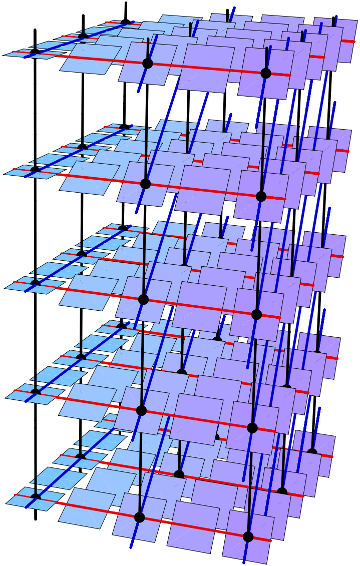

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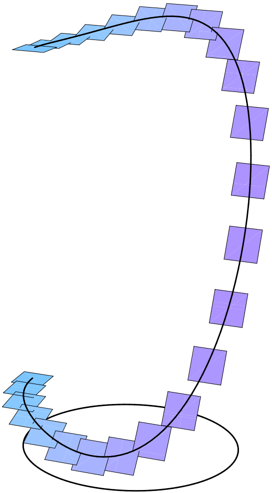
- Non-horizontal curves have Hausdorff dimension 2.

A geodesic in H



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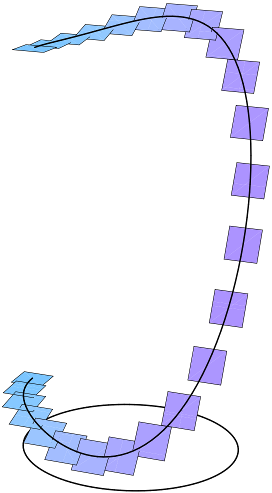

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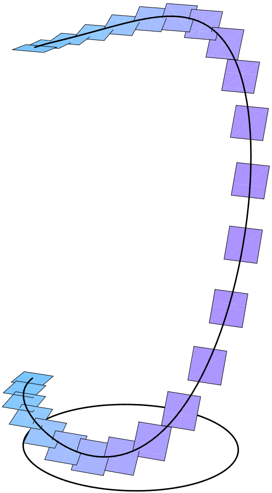

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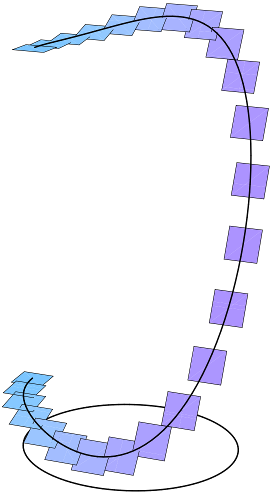

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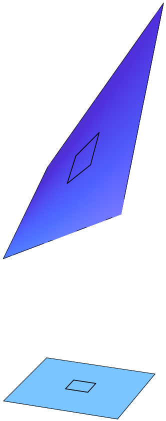
- By the isoperimetric inequality, geodesics are lifts of circular arcs.

A surface in H



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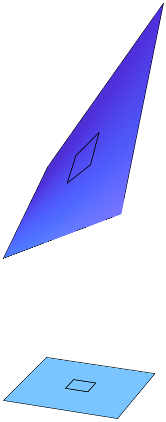

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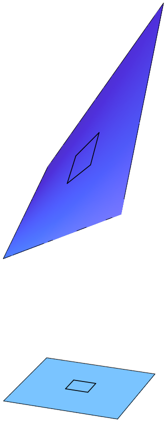

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Let $0 < \alpha \leq 1$. A map $f : X \to Y$ is α–*Hölder* if there is some $L > 0$ such that for all $x_1, x_2 \in X$,

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Question (Gromov)

Let $0 < \alpha \leq 1$. What are the α–Hölder maps from D^2 or D^3 to H?

Hölder maps to H

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What happens when $\frac{1}{2} < \alpha < \frac{2}{3}$?

Hölder maps to H

Theorem (Wenger–Y.)

When $\frac{1}{2} < \alpha < \frac{2}{3}$*, the set of* α*–Hölder maps is dense in* $C_0(D^n, H)$*.*

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When $\frac{1}{2} < \alpha < \frac{2}{3}$, the set of α–Hölder maps is dense in $C_0(D^n, H)$.

Lemma

Let $\gamma : S^1 \to H$ be a Lipschitz closed curve in H and let $\frac{1}{2} < \alpha < \frac{2}{3}$. Then γ extends to a map $\beta : D^2 \to H$ which is α–Hölder.

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We need the following result:

Theorem

There is a $c > 0$ such that for any $n \in \mathbb{N}$, a horizontal closed curve $\gamma : S^1 \to H$ of length L can be subdivided into cn^3 horizontal closed curves of length at most $\frac{L}{n}$.

Maps with signed area zero

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Corollary

Let $\gamma : S^1 \to \mathbb{R}^2$ be a Lipschitz closed curve with $\sigma(\gamma) = 0$ and let $\frac{1}{2} < \alpha < \frac{2}{3}$. Then γ extends to a map $\beta : D^2 \to \mathbb{R}^2$ which is α–Hölder and has null signed area.

Signed-area preserving maps

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- Based on lemma: There is a $c > 0$ such that for any $n \in \mathbb{N}$, a curve $\gamma : S^1 \to \mathbb{R}^2$ of length L can be subdivided into $\gamma_1, \dots, \gamma_{cn^3}$ such that $\ell(\gamma_i) \leq \frac{L}{n}$ and $\sigma(\gamma_i) = \frac{\sigma(\gamma)}{cn^3}$.

Open questions

- What else can this be used for?

Hölder maps from $\mathbb{R}^3$ to H

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