2021-10-07

I want to move now from dealing with general top spaces to <u>Manifolds and Complexes</u>:

Let $n>0$

An n-<u>manifold</u> is a topological space M which is Hausdorff,

- <u>locally euclidean</u>: $\forall x\in M$, $\exists$ a nbhd U and a homeo ~~[illegible]~~ from U to ~~the unit disc in $\mathbb{R}^n$~~ an open subset of $\mathbb{R}^n$.
- <u>second countable</u>: M has a countable basis (e.g., $\mathbb{R}^n$ has the basis $\mathcal{B}=\{B(x,r)\mid x\in\mathbb{Q}^n, r\in\mathbb{Q}, r>0\}$)

And there are lots of these:

$n=1$: $\mathbb{R}$, S^1

$n=2$:  , ...

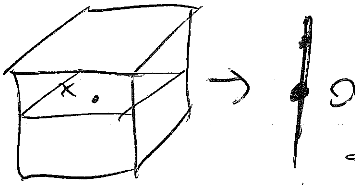
$n=3,4,\dots$ harder to classify.

How to construct?

<u>Subsets of $\mathbb{R}^n$</u>: Let $f:\mathbb{R}^{n+k}\to\mathbb{R}^k$ be a smooth fn. ~~st.~~

~~$\nabla f\neq 0$ $\forall x\in f^{-1}(0)$. Then $f^{-1}(0)$ is a manifold~~

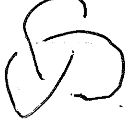
Let $M=f^{-1}(0)$. Suppose that $Df_x:\mathbb{R}^{n+k}\to\mathbb{R}^k$ is surjective $\forall x\in M$. Then M is an n-manifold.



<u>Gluing</u>: 

~~But what does th~~ But what does that mean?

Or: Let $M\subset\mathbb{R}^n$ be an open set. Then M is an n-manifold.

$K=$  $M=\mathbb{R}^3\setminus K$.

iff

Thm (Gordon–Luecke): If K_1,K_2 are smooth knots and $\mathbb{R}^3\setminus K_1\cong\mathbb{R}^3\setminus K_2$, then K_1 and K_2 are equivalent.

How do we glue?

<u>Quotient spaces</u>: Let X be a set, let $\sim$ be an equivalence relation on X. ~~Then (reflexive, sym~~ Then X can be partitioned into equivalence classes of $\sim$ — let $C_y=\{z\in X\mid y\sim z\}$.

Let $X/\sim\;=\{C_y\mid y\in X\}$

Say X is a ~~metric space~~ top. space — can we put a structure on $X/\sim$?

Def: The <u>quotient topology</u> on $X/\sim$ is the finest topology s.t. ~~the~~ $q: X \to X/\sim$, $y \mapsto C_y$ is continuous. That is, $U \subset X/\sim$ is open $\iff q^{-1}(U)$ is open.

$X =$ 

And there are alot of spaces we can construct this way:

Ex: Graphs:

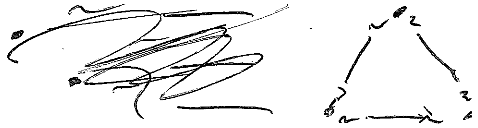
$G = (V, E, i, t)$ $V =$ vertices $i : E \to V$ (initial vert)

$E =$ edges $t : E \to V$ (terminal vert)

$$X(G) = \underline{V \sqcup (E \times [0,1])} / \sim$$

$e \times 0 \sim i(e)$

$e \times 1 \sim t(e)$



This is the 1-d case of

Ex: Complexes ~~: A k-complex is a~~

We can construct a space by following procedure:

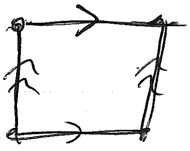
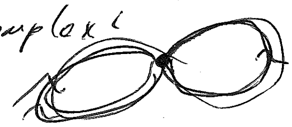
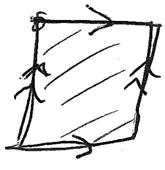
Let X^0 be a ~~coll~~ set of pts with the discrete topology (0-skeleton, vertex set)

For $k > 0$, a k-cell is a closed unit disk in $\mathbb{R}^k$.

If X^{k-1} is defined, let $\{e^k_\alpha\}_{\alpha \in A}$ be a collection of k-cells, let $\varphi^k_\alpha : \partial e^k_\alpha \to X^{k-1}$, define

$$X^k = X^{k-1} \sqcup \bigsqcup_{\alpha \in A} e^k_\alpha / \sim \quad \text{where } x \sim \varphi^k_\alpha(x) \ \forall x \in \partial e^k_\alpha.$$

Any space X^k constructed this way is a ~~k-complex~~ <u>cell complex</u> or CW-complex. If the largest cells have dimension k, it's a <u>k-complex</u>

Ex:  is a complex:  

Ex: Several homeomorphic structs on sphere: