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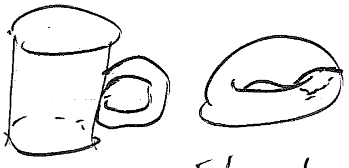
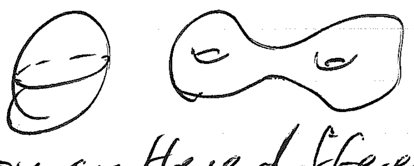
Topology I - Robert Young (ryoung@cims.nyu.edu)
WWH 601. Office hours: W 4-5 or by appt.
Website: cims.nyu.edu/~ryoung/courses/topology.
Books: Munkres, Topology
Hatcher, Algebraic Topology.

Grading: problem sets, ~~midterm~~, final.
COVID: please mask, stay home if you're sick -
~~we'll tr~~ we're assuming that there may be absences this year and we'll try to work around it - just let me know so we can setup cameras, mics, etc. Q's?

Topology is: the study of ~~continuo~~ topological spaces, continuous maps between them, and props preserved by cts maps.
Basic question: When are two topological spaces the same?
Std ex:



but not



But it's sometimes more interesting to ask: How are these different?
1 hole 0 holes 2 holes -
and ~~that's~~ defining what that means, why it's preserved - that's topology. So: goal is to do topology - ~~def~~ define topological invts, understand what happens under maps.

The beginning:
Limits and Continuity. Recall from calculus:
~~Let $(x_i)_{i\in\mathbb{N}}$ a seq. of reals.~~
If $x_i \in \mathbb{R}$, $c \in \mathbb{R}$,
$$\lim_{n\to\infty} x_n = c \iff \forall \varepsilon>0, \exists N>0 \text{ s.t. if } n>N, \text{ then } |x_n - c|<\varepsilon.$$
If $f:\mathbb{R}\to\mathbb{R}$, $x\in\mathbb{R}$, then f is continuous at x
$\iff \forall\varepsilon>0, \exists\delta>0$ s.t. if $|x-y|<\delta$, then $|f(x)-f(y)|<\varepsilon$.

Today:
~~First~~: Generalize this, define topological spaces.
And most basic prop: $\lim x_n = c \Rightarrow \lim f(x_n) = f(c)$.

1. Metric spaces: Let X be a set. A metric on X is a function $d: X\times X\to\mathbb{R}$ s.t. $\forall x,y,z\in X$.
- Non-negativity: $d(x,y)\geq 0$ and $d(x,y)=0 \iff x=y$.
- Symmetry: $d(x,y)=d(y,x)$.
- Triangle inequality: $d(x,z)\leq d(x,y)+d(y,z)$.

Define: if $x_n\in X$, $c\in X$, $\lim_{n\to\infty} x_n = c \iff \forall\varepsilon>0, \exists N>0$ s.t. if $n>N$, then $d(x_n, c)<\varepsilon$.
$f:\mathbb{R}\to\mathbb{R}$ is cts at x if $\forall\varepsilon>0, \exists\delta$ s.t. if $d(x,y)<\delta$, then $d(f(x), f(y))<\varepsilon$.
f is cts if it is cts at x $\forall x\in X$.

Check: if $\lim_{n\to\infty} x_n = c$, then $\lim_{n\to\infty} f(x_n) = f(c)$.

And this works great! Except:
1 - There are lots of different metrics that are topologically the same
Ex: $X=\mathbb{R}^n$ $\quad d_{Euc}(x,y) = \sum_{i=1}^n (x_i-y_i)^2$
$x=(x_1,\dots,x_n)$ $\quad d_1(x,y)=\sum |x_i-y_i|$
$y=(y_1,\dots,y_n)$ $\quad d_\infty(x,y)=\max|x_i-y_i|$.

$$\lim_{i\to\infty} v_i = w \iff \lim_{i\to\infty} v_i = w \iff \lim_{i\to\infty} v_i = w$$
(wrt d_{Euc}) (wrt d_1) (wrt d_∞).

And that's inelegant.
Worse: 2 - Some notions of convergence can't be written as metrics.
Ex: $X=\{$sequences of real #s$\}$.

If $x_i = (x_{i1}, x_{i2}, \dots)$ is a seq of reals for each i,
we say $x_i \to y$ pointwise if $\lim_{i\to\infty} x_{ij} = y_j \ \forall j$.

Ex:
$(0, 0, 0, 0, \dots)$
$(0, 1, 1, 1, \dots)$
$(0, 0, 2, 2, \dots)$
$(0, 0, 0, 3, \dots)$
$\downarrow$
$(0, 0, 0, 0, \dots)$

But pointwise convergence doesn't come from a metric!

(break)

So instead: Open sets. Let (X, d) be a metric space.
For $x \in X$, $r > 0$, let $B(x, r) = \{y \in X \mid d(x,y) < r\}$.
A subset $U \subset X$ is open if $\forall u \in U$, $\exists \varepsilon > 0$ s.t. $B(u, \varepsilon) \subset U$.

Exer: $B(u, \varepsilon)$ is open.
- if U, V are open, then $U \cap V$ is open.
- if U_α is an open set for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha$ is open

And we can def lim, cts in terms of open sets:
Prop: $\lim_{n\to\infty} x_n = c \iff \forall$ open set U s.t. $c \in U$, $\exists N$ s.t. if $n > N$, then $x_n \in U$.
Pf: ($\Rightarrow$) Let U be an open set containing c. Then $\exists \varepsilon > 0$ s.t. $B(c, \varepsilon) \subset U$. Choose N s.t. if $n > N$, then $d(c, x_n) < \varepsilon$.
If $n > N$, then $x_n \in B(c, \varepsilon) \subset U$ //
($\Leftarrow$) Exercise.

Prop: ~~f : (X, d_X) → Y~~ is Let (X, d_X) and (Y, d_Y) be metric spaces.
Then $f : X \to Y$ is cts $\iff \forall$ open set $U \subset Y$, $f^{-1}(U) \subset X$ is open.

Pf ($\Leftarrow$) Let $x \in X$, let $\varepsilon > 0$.
Claim: $\exists \delta > 0$ s.t. $d_X(x, a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon$.
Let $U = B(f(x), \varepsilon) \subset Y$. Then $f^{-1}(U) \subset X$ is open and $x \in f^{-1}(U)$. Let $\delta > 0$ be s.t. $B(x, \delta) \subset f^{-1}(U)$.
Then, $d_X(x, a) < \delta \Rightarrow a \in B(x, \delta) \Rightarrow a \in f^{-1}(U)$
$\Rightarrow f(a) \in U = B(f(x), \varepsilon)$
$\Rightarrow d_Y(f(a), f(x)) < \varepsilon$ //
($\Rightarrow$) Exercise.

So we don't need a metric! Just open sets!

Thus: Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets called open sets s.t.

- $\emptyset, X \in \mathcal{T}$
- $\mathcal{T}$ is closed under arbitrary unions
  i.e., if $\mathcal{A} \subset \mathcal{T}$, then $\bigcup_{U \in \mathcal{A}} U \in \mathcal{T}$.
- $\mathcal{T}$ is closed under finite intersections
  (i.e. if $U_1, U_2, \dots, U_k \in \mathcal{T}$, then $\bigcap_{i=1}^{k} U_i \in \mathcal{T}$.

Ex: (X,d) a metric space, $\mathcal{T} = \{\text{open sets}\}$.
~~(Why finite unions ar~~

Ex: indiscrete topology: $\mathcal{T} = \{\emptyset, X\}$ - coarsest topology
- discrete topology: $\mathcal{T} = P(X)$ - finest topology

Coarse - ~~not very~~ few open sets
Fine - lots of open sets.

Why? ~~Coarse topology is like big drunks - hard to tell diff pts apart.~~
Fine topology: cuts the space into fine pieces.
If X has discrete topology,
$\lim_{n\to\infty} x_n = c \iff \forall U \in \mathcal{T}$ s.t. $c \in U$,
$\exists N > n$ st. $x_n \in U \ \forall n > N$.
But $\{c\} \in \mathcal{T}$, so $\exists N > n$ s.t. $x_n = c \ \forall n > N$.
Coarse top: hard to tell pts apart.
If X has indiscrete topology, (x_n) a seq, $c \in X$.
then $\lim_{n\to\infty} x_n = c$. (if $U \in \mathcal{T}$ s.t. $c \in U$, then
$U = X \Rightarrow x_n \in U \ \forall n$).

So, different ~~depending on~~ topologies change what seqs. converge, what fns are cts.

Exer: Different metrics give same topology: d_{Euc}, d_1, d_∞ all lead to same ~~collection of open set~~ topology on $\mathbb{R}^n$.

In general, listing all open sets is impractical. Instead,
Def: A basis is a ~~coll~~ $\mathcal{B}$ of subsets of X s.t.
- $\forall x, \exists S \in \mathcal{B}$ s.t. $x \in S$.
- if $S, T \in \mathcal{B}$ and $x \in S \cap T$, then $\exists U \in \mathcal{B}$ s.t. $x \in U \subset S \cap T$.

The ~~topology~~ generated by $\mathcal{B}$ is the set

$$\mathcal{T} = \{U \subset X \mid \forall x \in U,\ \exists S \in \mathcal{B} \text{ s.t. } \cancel{x \in \mathcal{B} \subset U}\ x \in S \subset U\}.$$

~~Ex: Let X be a metric space. Then~~
~~$\mathcal{B} = \{B(x,r) \mid x \in X, r > 0\}$~~ is a basis
~~that generates the metric space topology.~~

~~Ex: $X = \mathbb{R}$ $\quad \mathcal{B} = \{[s,t) \mid s < t\}$.~~
~~Ex: $X = \mathbb{R}$~~

Prop: ~~If $\mathcal{T}$ is the basis~~ ~~topology generated by~~ $\mathcal{T}$ is a topology.
~~and $\mathcal{T} = \{$unions of basis elements$\}$.~~

Pf: $\emptyset, X \in \mathcal{T}$ ✓.
~~Let $\mathcal{A} \subset \mathcal{T}$, let $W = \bigcup_{U \in \mathcal{A}} U$.~~

~~If $w \in W$, then $w \in U$ for some $U \in \mathcal{A}$.~~
~~$\Rightarrow w \in U \in W \Rightarrow W \in$~~ Closed under unions

~~Let~~ Suppose $W = \bigcup_{\alpha \in A} U_\alpha$ where $U_\alpha \in \mathcal{T}\ \forall \alpha \in A$

Let $w \in W$. Then $w \in U_\alpha$ for some $\alpha \in A$.
$\Rightarrow w \in U_\alpha \subset W$. $\Rightarrow W \in \mathcal{T}$.

Closed under int:
Suppose $U, V \in \mathcal{T}$. ~~Let x~~ Claim $U \cap V \in \mathcal{T}$.
Let $x \in U \cap V$. Then $x \in S \subset U$, $x \in T \subset V$ for some $S, T \in \mathcal{B}$.
$\Rightarrow x \in S \cap T \Rightarrow \exists R \in \mathcal{B}$ s.t. $x \in R \subset S \cap T \subset U \cap V$.
$\Rightarrow U \cap V$ is open. //

(Overflow)

Prop: $\mathcal{T} = \{$unions of basis elements$\}$.