2018-09-06

Quantitative Geometry (ryoung@cims.nyu.edu)
- Seminar-style — Office hours: Tuesdays, 1:30-2:30 or by appt [redacted]

Books: ? Collect emails. Time change?

What is quantitative geometry? Roughly: ~~Many I think of it more as a plit. the~~ ~~afield~~ Measure properties of spaces, understand how these measurements reflect the geometry of a space.
~~I think of it as a~~ So, motivating questions:

1. How can we measure the geometry of a space?
   - scale (how big is a space: diameter, volume, dimension, etc)
   - complexity (how hard is it to describe? simplicial volume, simplices in a triangulation?)
   - connectivity (~~spectrum of Laplacian~~, expansion, spectrum)
   - ~~shape (...)~~

~~1b. How can we measure the geometry?~~
   - ~~other~~ etc. (~~size of the simplest~~ ~~minimal~~ size of "smallest" manifold that has M as boundary, length of ~~shortest~~ closed geodesic, smallest minimal surface, etc.) $\boxed{\text{connections}}$ A lot of these are easier to define than to ~~compute~~ calculate

1b: How can we measure these ~~geometry?~~ Algorithms to ~~compute~~ diameter, find simp. vol, minimal triangulations, etc.)

~~2. How are these invariants connected?~~

2: How do ~~spaces~~ these invariants behave under limits?
   - large scale (GGT)
   - small scale (quantitative differentiation)
   - large complexity (random graphs / surfaces / complexes, etc.)

~~Ex:~~ There's sthing subtle here — often, we're not just interested in the limit, we're interested in ~~what happens close to the limit~~ ~~how we approach the~~ what happens close to the limit.

Let me give an example: Random cubic graphs:
~~Fix~~ ~~Fix $k \geq 3$~~. Let $n > 0$ be even, consider the graph constructed by taking n vertices with 3 half-edges each, connecting the half-edges at random.

Give this the path ~~graph~~ metric where each edge has length 1.
~~Then there's a~~ What's the limit as $n \to \infty$?

~~Now, we can ass~~ There are a couple ways to define this, ~~but I'm going to ignore that~~. But here's a plausible construct:
Pick one vertex to be the base point, O.
When $k=3$, n is large, what does the nbhd of O look like?

~~High Prob~~ ~~With prob $\to 0$ with n,~~
~~(with overwhelming probability)~~ very high
~~O neighbors the~~

O has degree 3; with prob $\to 0$, these 3 neighbors are distinct

" the 6 nbrs of these nbrs are distinct

" 12 ~~3rd~~ verts at dist 3 " "

We only start getting positive probabilities once we start looking at $\sim\sqrt{n}$ vertices (birthday paradox) — so ~~the neighbo~~ a nbhd of radius $\approx \log n$ is a tree.

~~As~~ So $\lim_{n\to\infty} G_n$ is the 3-regular tree T_3

Does this mean that $G_{1000000}$ is tree-like?
Yes and no — locally yes, globally no:
G_n has few symmetries, diameter $\approx \log n$, many paths between two points. But there are still properties that carry over!

Exercise: If $S \subset V(G)$, let
$\partial S = \{ v \in V(G) \setminus S \mid v \text{ adjacent to } S \}$.
Show that $\forall S \subset V(T_3)$ s.t. $|S| < \infty$, $|\partial S| \geq 2|S|$.

Thm (Pinsker, ~~Kolmogo~~): G_n is an expander whp.
I.e. $\exists\, c > 0$ s.t. ~~whp~~
$$\lim_{n\to\infty} P\left[\forall S \subset V(G_n),\ \text{if } |S| < \tfrac{n}{2},\ \text{then } |\partial S| \geq c|S|\right] = 1.$$

~~And that's the sort of thing~~ So: a sequence of spaces that converge to a limit. ~~The limit is simpler, than nicer than than the spaces, but~~ ~~the limit is simpler~~ ~~What~~ ~~That~~ limit is ~~simpler than the original spaces~~
That limit ~~picks up symmetries~~ is nicer/easier than the original, but because it's simpler, it's potentially much different geometrically. How can we use that limit to study the orig space?

3 — How is the geometry of a limit of a ~~space~~ family of spaces related to the geom of the space?

Okay; that's the ~~abstract~~ ~~Pr~~: philosophy — what are we actually doing this semester?

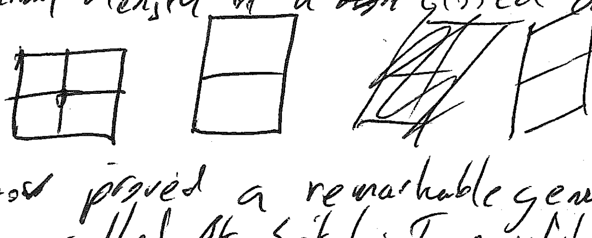
- Filling invariants and asymptotic cones. — ~~This is~~ These are ideas that originated in geometric group theory and geometric measure theory. ~~Geometric group theory studies the geometry of spaces on which large-scale geometry of groups~~ —

~~The idea~~ GGT studies the large-scale geometry of groups — ~~as~~ One way to do that is the isoperimetric problem — ~~there~~ ~~a~~ if you have a closed curve in some ~~space~~ ~~assoc to~~ the simply-connected space (say, curve in $\mathbb{R}^2$) assoc to the group how hard is it to ~~homotope the curve to a po~~ fill the curve by a disc? ~~And how does that depend on the geomet the space length~~ Classical isop: round circle, ~~given~~ round area. But that depends on the space — ~~diff~~ sphere, hyp plane diff. Even if we change the metric — say L_1 metric? ~~The idea of~~ The reason this found appl. in GGT is that the asymptotics of the (growth) fn (the Dehn fn) can tell you something about the ~~space~~ lg-scale geom of the space.

~~Asympt: Specifically interested in the relationship between the~~ ~~as~~ The asymptotic cone is another way of describing the large-scale geom of a space — it's ~~the~~ a limit obtained by viewing a space at larger scales, ~~so~~ so what I want to do is draw some connections between filling invts — ~~and geom~~ lg scale geometry and asymp cone — limit ~~at larger~~ as scales $\to \infty$.

— Systolic geometry: ~~There's a remarkable connection between~~ ~~theorem of Gromov~~ ~~Ro Rz~~: ~~A~~ the systole of a space is the ~~a closed curve in the~~ minimal ~~length~~ length of a ~~non~~ closed curve that is homotopically nontrivial.

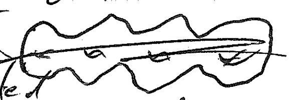
Ex:



T a flat torus

Can show: $sys(T)^2 \leq Area(T)$.

Gromov proved a remarkable generalization to higher dims, more mflds called ~~Gromov's~~ Systolic Inequality. Involves an invt called the filling radius of a mfld — which is connected to the ~~invts~~ filling invts from before. So I want to prove the ineq, talk abt systolic geometry. I'd like (dunno if poss) to talk about how the syst is related to complexity ~~all else equal, a more complicated surface has a smaller syst than a torus of same~~



how this ineq changes as mfld gets more complex. But we'll see. ~~(dunno if we can actually finish with this either poss something on q-diffs)~~

So that's large scale, large cplex. I'd try

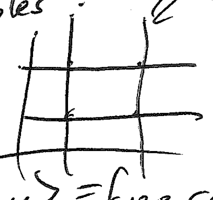
QED, but that may have to wait until some other time.

~~Anyway, let's start.~~ How much time do we have left? Okay, let's see if we can give some actual definitions.

Quick ~~ete~~ intro to geometric group theory (I'll post some other intros on the site)

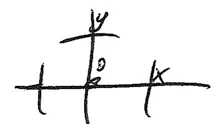
Let G be a group, let $S \subset G$ be a generating set for G (typically finite). The Cayley graph X_S is the graph with $V(X_S) = G$, where g and g' are joined by an edge $\iff g = g's$ for some $s \in S$.

Std examples: $\mathbb{Z}^2 = \langle x, y \mid [x,y] = 0 \rangle$, $S = \{x, y\}$



and we can label/direct edges.

$F_2 = \langle x, y \rangle$ = free group of rank 2:



We put the ~~wo~~ graph metric on X_S - call this the word metric on G corresponding to S. Then ~~G acts on X_S by~~ left-multiplication by $g \in G$ is an action of G on X_S by isometries.

~~But~~ Nintuit: X_S depends on S: if we add a gen, get a diff't graph. But the dependence ought to be small: lengths change by no more than a factor of 2. Even if we add more: const factor change.

The word is ~~quasis~~ that 2 diff't Cayley graphs are quasi-isometric ~~or coarse-Lipschitz~~ (sometimes coarsely bilipschitz equivalent).

Let $k \geq 1, C \geq 0$

Def: A map $f: X \to Y$ is a (k,C)-quasi-isometric embedding if ∀ $x_1, x_2 \in X$

$$k^{-1} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq k\, d_X(x_1, x_2) + C$$

X and Y are ~~QE~~ quasi-isometric (QI) if $\exists f: X \to Y$, $g: Y \to X$ ~~st~~ s.t. f, g are QIEs and ~~$f \circ g$~~ f, g are quasi-inverse: i.e. $\exists D$ s.t. $\forall x \in X$, $y \in Y$, $d(x, g(f(x))) < D$, $d(y, f(g(y))) < D$.

~~Thm: ∀ finite gen sets $S, S' \subset G$, $X_S \sim_{QI} X_{S'}$~~

Ex: $\sim_{QI}$ is an equivalence relation

Ex: If S, S' are finite gen sets of G, then $X_S \sim_{QI} X_{S'}$.

~~But also: Thm (Svarc-Milnor): If X is a geodesic metric space,~~

13. ~~In fact, conseq of a~~

Exit: Often said that GGT is study of props of gps that are invariants under QI. Next time define one of these, Dehn function.

Overflow: This is a special case of ~~something~~ a much broader result.

Def: Let G be a group, X a metric space. We say that G acts geometrically on X if the action is:
- by isometries
- cocompact ($\cancel{G}X$ is compact)
- properly discontinuous: $\forall K \subset X$ cpt, $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. {Like being free, but

{This excludes, say, $\mathbb{Z}$ acting on S^1 by irrational rotation}.

Ex: Cayley graph, deck transformations of the univ. cover of a cpt mfld.

Thm (Svarc-Milnor): If X is a geodesic metric space, $B_r(x)$ is compact $\forall x \in X$, $r>0$, and G acts geometrically on X, then G is ~~finitely~~ generated by some finite set S, and $X_S \sim_{QI} X$.

Pf depending on time: ~~Sketch: Idea of the proof is that to~~ Take $x \in X$, $r>0$ s.t. $GB_r(x) \supset X$. Let $S = \{g \in G \mid gB_{2r}(x) \cap B_{2r}(x) \neq \emptyset\}$.

Maps are not hard: ~~Let~~ let $f: X_S \to X$
$f(y) = gx$, where $g \in G$ is a vertex closest to y. ~~Let $g: X \to X_S$ be a map s.t.~~

Let $g: X \to G \subset X_S$ be a map s.t. $\forall y \in X$,
$y \in g(y) B_{2r}(x)$. Check: quasi-inverses.

Claim: These preserve the metric up to constants. ~~One dir not hard~~ ETS that both maps are coarsely Lip — i.e., $\exists C>0$ s.t.

$$d(f(x_1), f(x_2)) \leq d(x_1, x_2) \cdot C + C$$
$$d(g(y_1), g(y_2)) \leq d(y_1, y_2) \cdot C + C.$$

Discretization!

And exit.

2018-09-13

Last time: Svarc-Milnor thm:

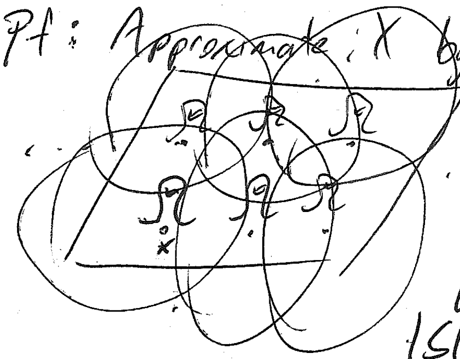
thm: If X is a geodesic metric space s.t. $B_r(x)$ is compact $\forall x \in X, r>0$ (proper) and if G acts geometrically (cocompactly, properly disc, by isoms) on X, then ~~G is fin~~ there is a finite generating set $S \subset G$ s.t. $X_S \sim X$. ~~We say that G is QI to X.~~

Then: — $\forall$ finite generating sets $T, \subset G$, G acts geom on X_T, X_T so $X_T \sim X_S \sim X$ ~~finite gen~~

If X is a proper ~~geodesic~~ metric space, we say $X \sim_{QI} G$ if $X \sim X_S$ for some (any) finite gen set $S \subset G$.

— If M compact manifold / simplicial complex, then $\tilde{M} \sim \pi_1(M)$.

Pf: Approximate X by a discrete object.



Let $x_0 \in X$, let $r>0$ be s.t. $G \cdot B_r(x_0) = X$.

Describe the geometry of X in terms of how those balls overlap.

1. Define graph based on overlaps:

Let $S = \{ g \in G \mid B_{2r}(x_0) \cap B_{2r}(gx_0) \neq \emptyset \}$

$|S| < \infty$ by prop. disc. $= \{ g \in G \mid d(x_0, gx_0) \leq 4r \}$

2. ~~Compare metric on graph to metric on space.~~ Construct maps

Let $f: X \to X_S$ be a map s.t. $f(y) B_r(x_0) \ni y \; \forall y \in X$.

Let $h: X_S \to X$ be the map s.t. $h(y) = gx_0$, where $g \in G$ is the vertex closest to y.

Check: f and h are quasi-inverse.

3. Prove QI.

It's enough to show that $\forall x, x' \in X$, $d_{X_S}(f(x), f(x')) < C d_X(x,x') + C$

and $\forall g, g' \in X_S$, $d_X(h(g), h(g')) \leq C d_{X_S}(g, g') + C$

Start with ~~One is easy~~: $d_X(h(g), h(g')) = d_X(gx_0, g'x_0)$

Let $l = d_{X_S}(g, g')$ = length of shortest path from g to g'

~~= length of shortest way to write~~

Then ~~least number of~~ $\exists\, s_1, \dots, s_l \in S$ s.t. $gs_1 \cdots s_l = g'$ $\Leftrightarrow$ $(g')^{-1} g = s_1 \cdots s_l$.

~~$d_X(gx_0, gs_1 \cdots s_l x_0)$~~ By ~~def~~ S_1 But $d(~~s~~x_0, ~~s~~x_0)$

Let's turn that path in X_S into a path in X:

~~Let $w_i = s_1 \cdots s_i$ then $d(gx_0, g'x_0) \le d(gx_0, gw_0 x_0, gw$~~

Let $w_i = g s_1 \cdots s_i x_0$. Then $w_0 = g x_0$, $w_\ell = g' x_0$, and $d(gx_0, g'x_0) = \cancel{d(w_0, w_\ell)}$

$$d(w_n, w_{n+1}) = d(g s_1 \cdots s_n x_0, g s_1 \cdots s_{n+1} x_0)$$

$$= d(x_0, s_{n+1} x_0) \le 4r$$

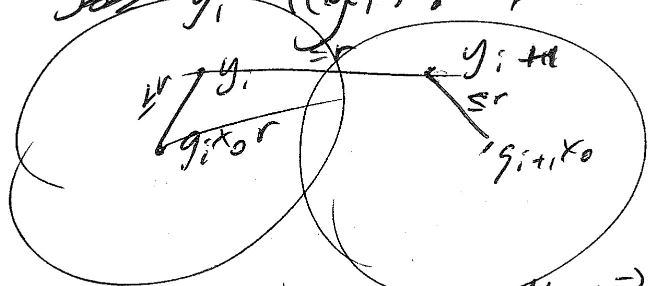
So $d_X(w_0, w_\ell) \le \ell \cdot 4r = 4r\, d_{X_S}(g, g')$.

Conversely: turn a path in X into a path in X_S: ~~Let $n =$~~

First? Let $x, x' \in X$. ~~Let $\gamma: [0, 1] \to X$ be a unit-speed geodesic from x to x'.~~ ~~Let $y_i = \gamma(ir)$~~ γ be a geod from x to x'. Let $n = \lceil \frac{d(x,x')}{r} \rceil$. Let ~~$x_0 = x$~~ $y_0, \ldots, y_n = x'$

be ~~evenly~~ evenly spaced pts on ~~a geod~~ γ.

Let $g_i = f(y_i)$. ~~Then~~ Then $d(g_i x_0, g_{i+1} x_0) \le 3r$



$d(x_0, g_i^{-1} g_{i+1} x_0) \le 3r$

$\Rightarrow g_i^{-1} g_{i+1} \in S$.

So g_i ~~and~~ $g_{i+1} = g_i (g_i^{-1} g_{i+1})$ $\Rightarrow$ g_i and g_{i+1} are adjacent.

So $d_{X_S}(f(x), f(x')) \le n \le \frac{d(x,x')}{r} + 1$

~~(Leave other direction as an exercise)~~

For the other direction, we do the same in reverse — Let $g, g' \in G$. Take a path. Connect the dots to construct a ~~geodesic~~ path in X.

~~Dehn~~ So: we can go back and forth between comb. and geom. distances. ~~distances~~ Now I want to do the same thing in a more complicated setup. Dehn functions: Geometry:

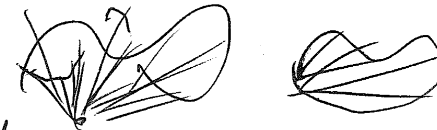
Def: Suppose X is a simply connected Riemannian manifold ~~or simplicial complex. If $\gamma: S^1 \to X$ is a Lipschitz closed curve~~ ~~piecewise smooth~~ ~~Lipschitz closed curve~~ or simplicial complex. If $\gamma: S^1 \to X$ is a Lipschitz closed curve, define $FA(\gamma) = \inf \text{area}(\beta)$ where β ranges over maps $\beta: D^2 \to X$ s.t. $\partial\beta = \gamma$.

~~(Thm: Rademacher) Lipschitz maps are differentiable a.e.~~

Now: This is an invt of a curve, want an invt of a space:

$$FA_X(L) = \sup_{\substack{\gamma: S^1 \to X \\ \ell(\gamma) \le L}} FA(\gamma).$$

Ex: $X = \mathbb{R}^2$. But also $X = \mathbb{R}^n$ — area of the smallest disc = area enclosed, of the smallest null-homotopy.



Ex: X = hyperbolic plane
Same straight lines give linear bound.

~~Combinatorial version~~: Can also do same combinatorially
~~Let $G = \langle S | R \rangle$. If w is a word in S (formal product of generators),~~
~~then w represents the trivial element identity $w =_G 1$~~
~~iff there's a sequence of~~

Let $G = \langle S | R \rangle$ (introduce notation if necessary)
If w is a word in S (formal product of generators, element of free group $F(S)$).
then w represents the identity element $w =_G 1$ iff there's a sequence of moves reducing w to the trivial word, where each move is:
- free insertion / reduction: $w_1 s s^{-1} w_2 \leftrightarrow w_1 w_2$ for some $s \in S$.
- applying a relator: $w_1 r^{\pm 1} w_2 \longleftrightarrow w_1 w_2$ for some $r \in R$

Ex: $G = \mathbb{Z}^2 = \langle x, y \mid [x,y] \rangle$
$= \langle x, y \mid xyx^{-1}y^{-1} \rangle$

$yx \rightarrow (xyx^{-1}y^{-1}) yx \rightarrow xy$
(likewise, $y^{-1}x \rightarrow xy^{-1}$, etc.)

$xxyyx^{-1}x^{-1}y^{-1}y^{-1}$.

Equivalently, $w =_G 1$ iff $\exists r_i \in R, f_i \in F(S)$ s.t. $w =_{F(S)} \prod_{i=1}^{n} f_i r_i f_i^{-1}$.

(~~need~~ clearly, this is reducible in n steps. Exer: show that a reduction corresponds to a way to express w like this).

~~Thm (Gromov): Let X be a Riem~~

Def: If $w \in F(S)$, $w =_G 1$, the filling area of w is
$\delta(w)$ = smallest # of applications of relators to reduce w to a trivial word
$= \min\{n \mid w =_{F(S)} \prod_{i=1}^{n} f_i r_i f_i^{-1}\}$ $\quad \delta_G(L) = \max_{\substack{w \in F(S) \\ \ell(w) \leq L}} \delta(w)$

Ex: $\mathbb{Z}^2$: If $w \in F(x,y)$, then $w =_{\mathbb{Z}^2} 1$ iff # of x's = # of x^{-1}'s and # of y's = # of (y^{-1})'s.

If so, ~~not hard~~ to ~~show~~ say $w \overset{?}{=} xyx^{-1}yyx^{-1}y^{-1}y^{-1}x$.
then we can reduce by collecting x's on the left: after 8 steps: $xx^{-1}x^{-1}x\,y^{-1}yyy^{-1}$, which freely reduces.
If $\ell(w)=n$, this takes $\le n^2$ steps, so $\delta_{\mathbb{Z}^2}(n) \le n^2$.

And same for more complicated groups:

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| \, x,y,z \in \mathbb{Z} \right\} \cong \langle X,Y,Z \mid [X,Y]=Z, [X,Z]=[Y,Z]=1 \rangle$$

Then $XYX^{-1}Y^{-1}=Z$ $\quad XY=ZYX$ - we can commute X,Y, but it produces a Z. So: Sim $X^{-1}(XY)X^{-1}=$ ~~...~~ $YX^{-1}=X^{-1}YZ$

$$XYYX^{-1}Y^{-1}XZ \overset{?}{=}_{\mathbb{H}} 1 ?$$

Collect X's

$$XX^{-1}YZYZY^{-1}XZ$$

$$XX^{-1}YYY^{-1}XZ^3$$

$$XX^{-1}XYZ^{-1}YZ^{-1}Y^{-1}ZZ^3 \rightsquigarrow XYZ^2 \neq_{\mathbb{H}} 1.$$

~~In~~ ~~In general, if w~~ In general, if $\ell(w)=n$, formalize.
~~In general, this takes we can do this in n stages~~
~~If $w=_{\mathbb{H}}1$,~~ then ~~this takes n stages,~~ we can
1 - shuffle an X left ($\le n$ steps)
2 - shuffle $\le n$ Z's right ($\le n^2$ steps)
3 - repeat for each X ($\le n$ times).
4 - freely reduce.

~~Then w~~
This reduces w to a word $w =_{\mathbb{H}} X^aY^bZ^c$.
and $w =_{\mathbb{H}} 1 \iff a=b=c=0$. This takes $\le n^3+n^2$ steps,
so $\delta_{\mathbb{H}}(n) \le n^3+n^2$ (in fact, $\exists c$ s.t. $cn^3 \le \delta_{\mathbb{H}}(n) \le cn^3$ $\forall n \ge 1$).

Relationships:
Thm (Gromov): Let X be a simply connected Riem mfld or simplicial complex, equipped w/ a geodesic metric.
Suppose $B_r(x)$ compact $\forall r>0, \forall x \in X$. Let G be a group acting geometrically on X. Then G is finitely presented, and $\forall$ finite pres. $G = \langle S \mid R \rangle$, $\exists C>0$ s.t.

$$\delta_G(n) \le C\,FA_X(Cn+C) + Cn + C$$

and

$$FA_X(n) \le C\,\delta_G(Cn+C) + Cn + C$$

We write ~~$\delta_G \simeq FA_X$~~ $\delta_G \sim FA_X$. Further, if $G \sim_{QI} H$, then

$\delta_G \sim \delta_H$.

2018-09-20

Last time: Dehn fn, filling area fn, one geom, one combinatorial. And take ~~one~~ a quick opportunity to see how much this varies between groups/spaces:

Ex: $BS(2,1) = \langle a, b \mid aba^{-1} = b^2 \rangle$:

$$a^n b a^{-n} \overset{1 \text{ step}}{\rightsquigarrow} a^{n-1} b^2 a^{-(n-1)} \overset{\text{free}}{\rightsquigarrow} a^{n-1} b a^{-1} a b a^{-(n-1)} \overset{2 \text{ steps}}{\rightsquigarrow} a^{n-2} b^4 a^{-(n-2)}$$

$\underset{2^n - 1 \text{ steps.}}{\rightsquigarrow} b^{2^n}$ So $a^n b a^{-n} b a^n b^{-1} a^{-n} b^{-1} \overset{2(2^n-1) \text{ steps}}{\rightsquigarrow} \varepsilon$.

$\delta_{BS(2,1)}(n) \sim 2^n$. Likewise, Baumslag-Gersten gp:

$$G = \langle a, t \mid (t^{-1} a^{-1} t) a (t^{-1} a t) = a^2 \rangle$$

has DF larger than any tower of exponentials: idea:

$$(t a^n t^{-1}) a (t a^{-n} t^{-1}) = a^{2^n}$$

$$(t a^{2^n} t^{-1}) a (t a^{-2^n} t^{-1}) = a^{2^{2^n}} \cdots$$

Ex: Thm (Novikov-Boone): There is a finitely presented group $G = \langle S \mid R \rangle$ s.t. there is no algorithm to ~~compute~~ compute whether a word $w \in S^*$ represents 1 or not (undecidable word problem).

Cor: There is a group s.t. δ_G is larger than any computable fn.

Pf: ~~Let G have undecidable WP~~ Suppose that $f: \mathbb{N} \to \mathbb{N}$ is a computable fn s.t. $\delta_G(n) \le f(n)\ \forall n$. Then, here's an algorithm. We claim that G has decidable WP.

1. Let $w \in S^*$, let $l = l(w)$.
2. Generate every possible sequence of $f(l)$ applications of relations starting at w.
3. If one of these is ε, then $w =_G 1$. Otherwise, $w \neq_G 1$.

~~But this contradicts the undecidability!~~

So conversely, undecidable WP $\Rightarrow$ uncomputable δ_G.

Exercise: Show that, conversely, if G has decidable word problem, then it has computable DF.

But this is combinatorics - what about geometry?

The growth rate of DF, FA is a geometric invariant.

Def: If $f, g: \mathbb{N} \to \mathbb{R}^{\geq 0}$ or $\mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, we say $f \lesssim g$ if $\exists C > 0$ s.t.

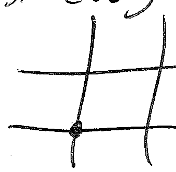
$f(n) \le C g(Cn+C) + Cn + C$. ~~This is a partial order~~. If $f \preceq g$ & $g \preceq f$, we say $f \sim g$. This is an ER.

Thm (Gromov): Let X be a simply-connected Riemannian manifold/simplicial complex. Suppose that G acts geometrically on X. Then G is finitely presented, and $\forall$ finite pres. $G = \langle S | R\rangle$, $FA_X \sim \delta_{\langle S|R\rangle}$. Let $\delta_G = \delta_{\langle S|R\rangle}$ – this is well-defined up to $\sim$.

Thm (Gromov): If $G \sim H$, then $\delta_G \sim \delta_H$.

A lot of pieces to this – I want to prove some, maybe not all.

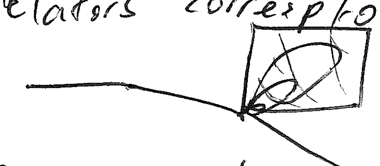
First: ~~Special~~ A special case: If $G = \langle S|R\rangle$, the Cayley complex K_G is the 2-complex s.t. $K_G^{(1)} = X_G$ and K_G has one 2-cell for every translate of a relator. That is, $\forall g \in G$,

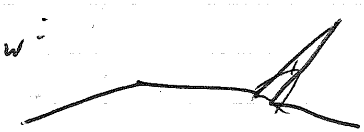


$r \in R$, $\exists$ a closed curve $g\lambda_r$ based at g, labeled by r, ~~and K_G has a 2-cell~~ K_G has one 2-cell $\delta_{g,r}$ with $\partial \delta_{g,r} = g\lambda_r$.

Lemma: Let $w \in S^*$, $w =_G 1$. Then $\exists c > 0$ s.t.
$$c^{-1}\delta_G(w) \le FA_{K_G}(\lambda_w) \le c\,\delta_G(w).$$

Pf: Comb $\Rightarrow$ geom: Free insertion/reduction ~~and~~ and applying relators corresp to htpies of λ_w:





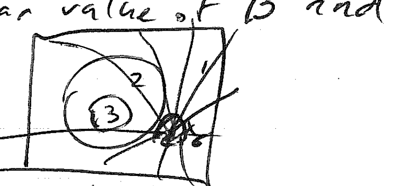
So $FA(\lambda_w) \le c\,\delta_G(w)$.

Geom $\Rightarrow$ comb: A little harder: piecewise smooth

Let $\beta: D^2 \to K_G$ be a filling of λ_w. Since K_G is a 2-cplx,

~~Then Then~~ Then $Area(\beta) = \int_{K_G} |\beta^{-1}(x)|\,dx = \sum_{\sigma \in F(K_G)} \int_\sigma |\beta^{-1}(x)|\,dx$.

$\forall \sigma$, choose x_σ so that x_σ is a regular value of β and $|\beta^{-1}(x_\sigma)| \le \dfrac{\int_\sigma |\beta^{-1}(x)|\,dx}{Area(\sigma)}$



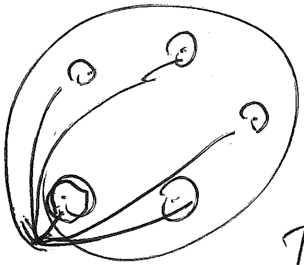
~~Let U_σ be~~ Let U_σ be a small nbhd of x_σ

~~There's a small nbhd of x_σ~~ where ~~the~~ β has constant degree

Compose β with a map sending U_σ to all of σ, $\sigma \setminus U_\sigma$ to $\partial\sigma$.

This map decreases Area(β)

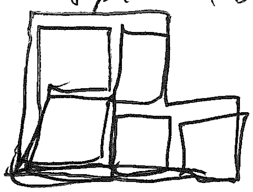
Now: D^2 contains finitely many preimages of the x_σ's, each contained in a nbhd $V_{\sigma,i}$ s.t. $\beta: V_{\sigma,i} \xrightarrow{\cong} \sigma$.



The rest of D^2 is sent to ~~$\mathcal{E}_G$~~ $X_G^{(1)} = K_G^{(1)}$

∂D^2 is homotopic to a bouquet of lollipops

The image of this htpy is a htpy from λ_w to a bouquet of lollipops in ~~$\mathcal{E}_G$~~ X_G. Each of these has boundary htpic to grg^{-1} for some $g_n \in S^*$, $r \in R$, so
$$w \underset{F(S)}{=} \prod_{i=1}^{n} g_i r_i g_i^{-1} \text{ where } n < c\,\mathrm{Area}(\mathcal{E}) \;//$$

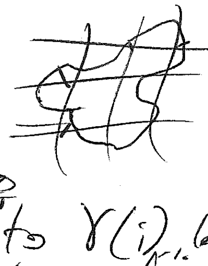


~~Now: Start proving~~ Now let's prove thm. ~~Lemma~~

Lemma: ~~δ_G~~ ~~FA_G~~ $FA_{K_G} \sim \delta_G$.

Pf: From the previous lemma, $\delta_G \lesssim FA_{K_G}$.

~~Fastrovs~~ Claim: $FA_{K_G} \lesssim \delta_G$.



Let $\gamma : [0,l] \to K_G$ be a unit-speed closed curve. ~~Then let g_i~~ let $n = \lceil l \rceil$

$\forall i = 0, \dots, n-1$ let g_i be the vertex closest to $\gamma(i)$. let $g_n = g_0$.

Let ~~w be a word~~ $\lambda = \lambda_w$ be the curve consisting of geodesics in X_G from g_0 to g_1 to $\cdots$ to $g_n = g_0$.

~~Then~~ ~~$\ell(w)$~~ Each segment has bdd length, so $\exists c$ s.t.

$\ell(w) \leq cl$, and $FA(\lambda) \leq c\,\delta(w) \leq c\,\delta_G(c(l+1))$.

Now, compare γ, ~~λ~~ λ: ~~The area~~ The region between γ, λ

can be broken into n closed curves, each of length $\leq c + 2 + 1$, ~~so this region has~~ fitting area C.

So there's a homotopy from γ to λ with area $\leq Cn$

Therefore, $FA(\gamma) \leq Cn + FA(\lambda) = Cl + C + C\delta_G(cl + c)$ //

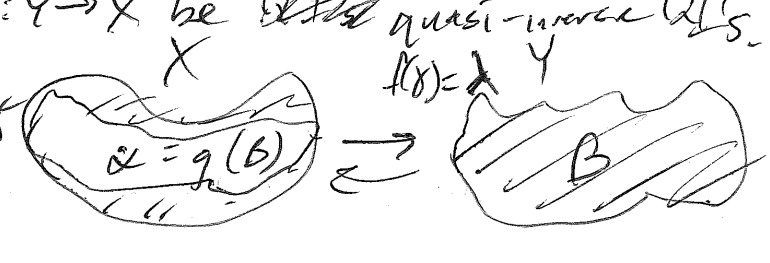
So this proves the thm when $X = K_G$. In general?

~~By~~ By Svarc-Milnor, if G acts geometrically on X, then ~~G~~ $X \sim_{QI} K_G$. We claim this is enough:

Thm: If X, Y are simply-connected simplicial complexes with ~~bdd degree~~, ~~G acts geom on X, then~~ and $X \sim_{QI} Y$, then $FA_X(n) < \infty$, $FA_Y(n) < \infty$ $\forall n$, $F\delta_X \sim \delta_Y$.

Pf: Let $f : X \to Y$, $g : Y \to X$ be ~~QIs~~ quasi-inverse QIs.

Idea: Same as before:



Problem: ~~We can Pb~~ f, g need not be continuous

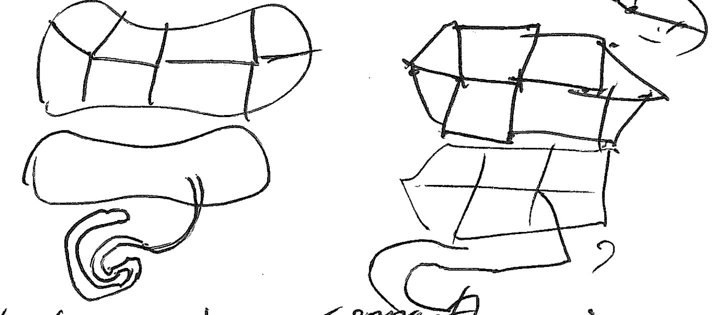
~~— β might be complicated~~

(but we've worked with that before: 

— β might be complicated —

but the problem is

— <u>How do we discretize a disc?</u>



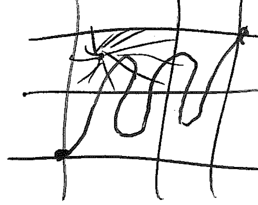
Thm (Federer–Fleming): Let $(M, \partial M)$ be a compact k-manifold with (possibly empty) boundary. Let $m: (M, \partial M) \to (X, X^{(k-1)})$ be a Lipschitz map. Let X be a simp. cplex, $\exists c>0$. Then there is an ~~htpy~~ approx $\bar{m}: (M, \partial M) \to (X^{(k)}, X^{(k-1)})$ ~~and a hom.~~ such that $\mathrm{vol}^k(\bar{m}) < c\, \mathrm{vol}^k(m)$,

$\partial m = \partial \bar{m}$, and m is htpic to $\bar{m}$ rel ~~∂M~~

by a htpy $h_t: (M, \partial M) \to (X, X^{(k-1)})$ ~~s.t. that fixes ∂m~~

- $\mathrm{vol}^{k+1}(h_t) \leq c\, \mathrm{vol}^k(m)$
- $h_t(x) = m(x)\ \forall x \in \partial M, t \in [0,1]$

Furthermore, we may suppose that $\bar{m}$ is admissible: $\exists$ disjoint open balls $B_1, \dots, B_n \subset M$ s.t. each $\bar{m}: B_i \cong \delta_i$ for some k-cells $\delta_1, \dots, \delta_n$ and $\bar{m}(M \setminus \bigcup B_i) \subset X^{(k-1)}$.

Pf: 

Now we can prove the bounds.

Let $\bar{f}: X^{(2)} \to Y^{(2)}$ as follows. $\forall v \in V(X)$, let $\bar{f}(v)$ be vertex closest to $f(v)$. $\forall e = (v,w) \in E(X)$, let $\bar{f}(e)$ be ~~edge from~~ geodesic edge path from $\bar{f}(v)$ to $\bar{f}(w)$. $\forall \delta \in F(X)$, ~~$\partial\delta$ is a~~ $\bar{f}(\partial \delta)$ is a closed edge path. Let $\overset{(u,v,w)}{\bar{f}(\delta)}$ be the filling in $Y^{(2)}$ w/ least cells. ~~Since $FA_Y(n)$~~

~~Each~~ $\exists c > 0$ s.t. $\ell(\bar{f}(e)) \leq c\ \forall e \in E(X)$

Then area$(\bar{f}(\delta)) \leq FA(3c)\ \forall \delta \in F(X)$

Finish if possible.