Date. 2023 09 28

QIs of Hyperbolic Spaces.

Last time: δ-hyperbolicity: Let $\delta > 0$

A geodesic metric space X is δ-hyperbolic if every geodesic triangle is δ-thin: i.e., for any $\gamma_1, \gamma_2, \gamma_3$ connecting v_1, v_2, v_3, $\gamma_i \subset N_\delta(\gamma_{i+1} \cup \gamma_{i+2})$:

Equiv, every geod triangle has a center which is dist $\le \delta$ from all 3 edges.

~~Very~~ Strong condition on geometry, preserved by QIs.

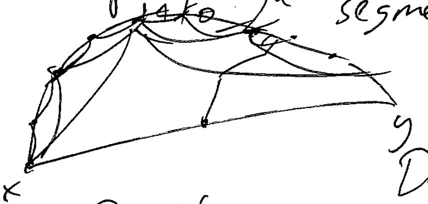
Why?:

~~Prop~~ Lemma: (Exponential divergence):

~~Let γ be a geodesic in X (i.e., an $\mathbb{R} \to X$~~

Let X be δ-hyp. Let $r > 0$, let $\overline{xy}$ be a geodesic of length $2r$, with midpoint m. Any path from x to y outside $B(m, r)$ has length $\ge 2^{\frac{r-1}{\delta}}$. — exponentially large.

Pf: Let γ be such a path. Let $k = \lceil \log_2 \ell(\gamma) \rceil$, and subdivide γ into 2^k segments of length at most 1: let $x_0, \dots, x_{2^k}$ be endpts of these segs. ~~We can draw triangles~~



Claim: one of the x_i's is close to m:

Draw triangles by repeated bisection,

By thinness, m is δ-away from ~~first~~ a pt on first triangle which is δ-away from second, etc. So $\exists\, x_i$ s.t.

$$d(m, x_i) \le \delta k + 1 \Rightarrow r \le \delta k + 1 \Rightarrow \ell(\gamma) \ge 2^{\frac{r-1}{\delta}} \quad //$$

Similar is true for pairs of geodesic rays - either stay close or diverge exp'ly.

Upshot - X is not quite a tree - removing a ball leaves it connected. But it's tree-like: removing a ball creates v. large detours.

Leads to Morse Lemma:

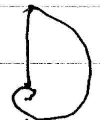
Def: An ~~quasigeodesic~~ (L, C) quasigeodesic in X is a map $\gamma: I \to X$ s.t. γ is an (L, C)- ~~Quasi~~ QI embedding:

$$L^{-1}|s-t| - C \le d(\gamma(s), \gamma(t)) \le L|s-t| + C$$

(Good GGT notion of geodesic - preserved under QIs).

Problem is that ~~these are~~ there can be a lot more QG's than geods — ~~Ex: $\gamma(t) =$~~ Easy ones: if γ is ~~...~~ a geod, $\gamma'(t)$ $d(\gamma(t), \gamma'(t)) < c$ then $\gamma'(t)$ is a QG.

But also QGs that



logarithmic spiral ~~$\gamma(t) = (t+i)^t \in \mathbb{C}$~~ ~~is a Quasigeo~~ ~~for $t \geq 0$~~.

are far from ordinary geods: $\gamma(t) = t e^{i \log(t)}$ is a quasigeod,

ex: Logarithmic spiral is not bdd dist from any straight line.

Morse Lemma ~~says that~~ gives a characterization of QG's in δ-hyp space.

Morse Lemma: Every (L, C)-quasi-geodesic $\gamma : [0, T] \to X$ is D-close to any geodesic $\overline{xy}$ ~~$\overline{xy}$~~ from $\gamma(0)$ to $\gamma(T)$ where ~~...~~
x y

$D = D(L, C, \delta) > 0$.

(i.e., $\gamma \subset N_D(\overline{xy})$ and $\overline{\gamma(0)\gamma(T)} \subset N_\delta(\gamma)$)

Pf: 1: A priori, γ may be discontinuous, but $d(\gamma(i), \gamma(i+1)) \leq L + C$.

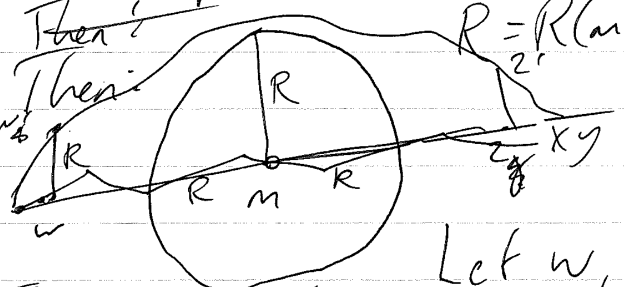
Replace γ by a broken geodesic from $\gamma(0)$ to $\gamma(1)$, to $\gamma(2)$, etc. so γ is Lipschitz (connect-the-dots argument).

In particular, if $z, z' \in \gamma$, then $d(z, z') \leq \ell([z, z']) \leq L' d(z, z') + C'$

2: For $z \in \overline{xy}$, let $R(z) = d(z, \gamma)$. This has a max ~~attd~~ ~~at some pt~~ — let m be a point achieving the max, let $R = R(m)$.

~~Then:~~

Then:



and every pt in $\overline{xy}$ is R-close to γ. Need to bound R.

Let $w, z \in \overline{xy}$ be points s.t. $d(m, w) = d(m, z) = 2R$ on either side.

~~These are R-close to w', z'~~ on γ (If m is close to x or y, we can take $w = x$ or $z = y$.) These are R-close to w', z' on γ, and $d(w', z') \leq 6R$. Since γ is a QG, ~~$\ell([w', z'])$~~ $\exists K = K(L, C)$ s.t. $\ell([w', z']) \leq 6R \cdot K$~~$(L, C)$~~.

Consider the path — this has length $\leq (4 + 6K)R$, stays outside of $B(m, R)$. ~~so~~ $\geq$ By Morse, $(4 + 6K)R \geq 2^{\frac{R-1}{\delta}}$

so $R \leq R^*(L, C, \delta)$,

i.e., $\overline{xy} \subset N_{R^*}(\gamma)$.

3: Claim: $\gamma \subset N_D(\overline{xy})$.

Let $x_0, x_1, x_2, \dots, x_L$ be points on $\overline{xy}$ spaced 1 apart.

$\forall i, \exists y_i \in \gamma$ s.t. $d(x_i, y_i) \le R^*$. Then $d(y_i, y_{i+1}) \le 2R^* + 1$.

By QG, $\ell([y_i, y_{i+1}]) \le \cancel{d(y_i,}\, L'(2R^*+1) + C' \le D(L, C, \delta)$.

Every point on γ lies between y_0 and y_L, so $\gamma \subset N_D(\overline{xy})$ where $D = D(L, C, \delta)$, as desired //.

(break)

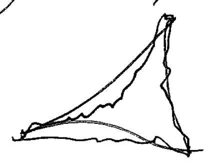
Consequences of Morse Lemma:
QI - invariance.

Thm: Let ~~f: X → Y be a~~ X, Y be geodesic metric spaces and ~~f: X → Y~~ ~~f: X → Y a QI. If X is δ-hyp then~~ ~~∃ ∀δ ≥ 0 ∃δ' if X is δ-hyp, then Y is δ'-hyp.~~

$X \sim_{QI} Y$. If X is δ-hyp, then Y is δ'-hyp where $\delta' = \delta'(L, C, \delta)$.

Pf: Let $f: Y \to X$ be a QI, let $\Delta = \gamma_1, \gamma_2, \gamma_3 \subset Y$ be a geodesic triangle. Then $f(\Delta) = f(\gamma_1), f(\gamma_2), f(\gamma_3)$ is a quasi-geod triangle in X.

By Morse, these are close to a geodesic triangle, which is δ-thin.



So ~~f(γ₁), f(γ₂), f(γ)~~ $f(\Delta)$ is thin $\Rightarrow \Delta$ is thin.

So every triangle in Y is δ'-thin //.

Can we distinguish hyperbolic spaces?

We discussed trees before: all trees are the same? What about hyperbolic spaces? Can we find an invariant that distinguishes $\mathbb{H}^2$ from $\mathbb{H}^3$?

In fact we can: the ideal boundary.

Def: Let X be a geodesic metric space. A geodesic ray is an isom embedding $[0, \infty) \to X$. We say γ, λ are asymptotic if $\exists C$ s.t. $d(\gamma(t), \lambda(t)) < C$ for all t.

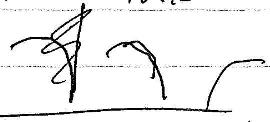
Equiv, $d_{Haus}(\gamma, \lambda) < \infty$.

The ideal boundary of X is the set $\partial_\infty X = \{\text{geod rays}\} / \sim_{asymp}$ where $\sim$ is asymptotic relation.

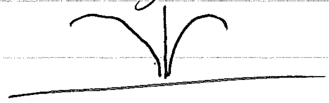
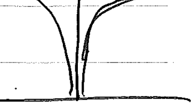
Ex: $\mathbb{R}^n$: asymptotic $\iff$ parallel
so $\partial_\infty \mathbb{R}^n = S^{n-1}$.

Ex: tree:  asymptotic rays eventually coincide
so $\partial_\infty T$ is the set of ends of T. ~~Set~~ is ~~uncountable~~.
(in fact, we can give this a topology of a Cantor set)

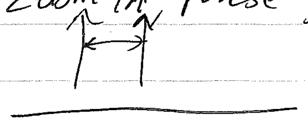
Most important for this:

Ex: $\mathbb{H}^2$  geod rays are ~~sem~~ arcs, lines perp to bdry.

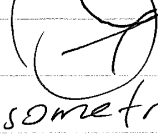
Most rays are inf distance.

~~But~~ What rays are asymptotic?  ~~Takes~~ is isometric to 

– as we zoom in, these get closer together.

And also,  are asymptotic

So $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\} \cong S^1$

~~And~~ In the disc model, explicitly S^1:  More gen $\partial_\infty \mathbb{H}^n = S^{n-1}$

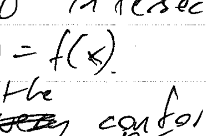
Clearly, Lemma: For any pt p, $x \in \partial_\infty \mathbb{H}$, $\exists!$ geod ray.
For any $y, x \in \partial_\infty \mathbb{H}$, $\exists!$ geod ray.

~~Point~~: Ideal boundary is preserved by isometries:
if $f: X \to Y$ is an isometry and $\gamma \sim \lambda$, then $f(\gamma) \sim f(\lambda)$, so
f induces $f_\infty : \partial_\infty X \to \partial_\infty Y$. ~~In $\mathbb{R}^n$ In $\mathbb{R}^n$, trivial,~~
In $\mathbb{R}^n$, f_∞ is linear part of f. – in $\mathbb{H}^n$, more complex:

~~Thm~~ Prop: ~~For any $f \in$ Isom~~ The map $f \mapsto f_\infty$ is a bijection when $n \geq 3$,
between $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Conf}(S^{n-1})$

Pf: Every isometry of $\mathbb{H}^n$ is a conformal map ~~of~~ $U^n \to U^n$ which extends to the boundary.

Injectivity: ~~Let~~ Suppose $f_\infty = id_{S^{n-1}}$. ~~Let $x \in \mathbb{H}^n$. Draw an ideal triangle centered at x:~~ ~~Then For any geodesic γ,~~
Any geod γ is asymptotic to two points at ∞. – conversely, any two points determine a unique geodesic – so $f(\gamma) = \gamma$ for every geodesic. For $x \in \mathbb{H}^n$, let λ, γ intersect at x – then
$f(\gamma) \cap f(\lambda) = \gamma \cap \lambda = x = f(\gamma \cap \lambda) = f(x)$.

Surjectivity: By theory of conformal maps, ~~every~~ the conformal maps $\mathbb{R}^{n-1} \cup \{\infty\} \to \mathbb{R}^{n-1} \cup \{\infty\}$ ~~is a Möbius f~~ are generated by inversions. These all extend to U^n. 

By Morse, δ-hyp
~~Pf~~ Thm: ~~Let~~ X, Y be geodesic ~~metric spaces. Let f: X → Y be~~

~~Further, we can~~ Ideal boundaries of δ-hyp spaces.

We can define same things for δ-hyp spaces — two rays are asymp. if they stay bdd dist.

~~Thm: Let X, Y be δ-hyp geod. metric spaces,~~ Then, by Morse, a QI $f: X \to Y$ induces $f_\infty: \partial_\infty X \to \partial_\infty Y$. This is a bijection — but we can also make it a homeo. by giving topology.

Lemma: If X is proper (closed metric balls are compact), then:

1 − $\forall x \in X$, $\gamma \in \partial_\infty X$, $\exists$ geod ray from x asymp to γ.

~~2 − $\forall \gamma, \lambda \in \partial_\infty X$, $\gamma \neq \lambda$, $\exists$ geod from γ to λ.~~

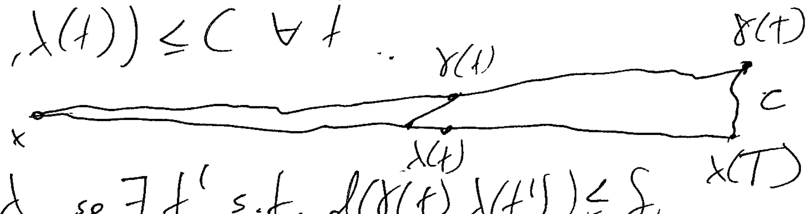
Pf 1: Arzela-Ascoli

~~2: Need to understand λ, γ. By 1, suppose $\lambda(0) = \gamma(0)$.~~

So again, it suffices to consider geodesic rays from a point x.
How can we topologize these?

Lemma: If $\gamma, \lambda: [0,\infty) \to X$, $\gamma(0) = \lambda(0) = x$, $\gamma \sim \lambda$, then $d(\gamma(t), \lambda(t)) \leq 2\delta$ for all t.

Pf: Suppose $d(\gamma(t), \lambda(t)) \leq C$ $\forall t$. ~~Consider~~ Then



$\gamma(t)$ is close to λ, so $\exists t'$ s.t. $d(\gamma(t), \lambda(t')) \leq \delta$.
By triangle, $|t - t'| \leq \delta$, so $d(\gamma(t), \lambda(t)) \leq 2\delta$. //

So $\gamma \not\sim \lambda \Leftrightarrow \exists t$ s.t. $d(\gamma(t), \lambda(t)) > 2\delta$. — we can topologize ~~by~~ according to how ~~long that~~ large that t is.

Let $x \in X$.
~~Def:~~ For every $p \in \partial_\infty X$, pick a rep γ_p s.t. $\gamma(0) = x$
Define d_∞ (depending on x, choice of γ_p's)

$$d_\infty(\gamma_p, q) = e^{-\max\{t \mid d(\gamma_p(t), \gamma_q(t)) \leq 2\delta\}}$$

Not quite a metric (only satisfies Δ-ineq up to consts) but can still define topology. Then:

Thm: If $f: X \to Y$, then $f_\infty: \partial_\infty X \to \partial_\infty Y$ is continuous.
(in fact, quasi-symmetric, to be explained later)