2023 09 14

on the left.

The QI lens:

Last time: ~~Mdn~~ QI's, ~~Milnor~~ Schwarz-Milnor:

Thm: Let X be a proper geodesic metric space and let G be a group that acts geometrically on X. Then X is finitely generated, and if G also acts geometrically on Y, then $X \sim_{QI} Y$.

~~last time:~~

~~(Specifically, for any finite gen. set $S \subset G$, $X \sim_{QI} \Gamma_S$.)~~

Pf: Suffices to show that $\exists$ a finite gen. set S s.t. $X \sim_{QI} \Gamma_S$.

All Cayley graphs are QI, so if G acts on Y, then $Y \sim_{QI} \Gamma_T \sim \Gamma_S$

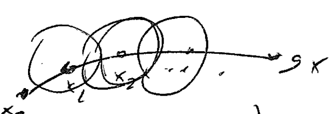
~~Let $x \in X$, $r > 0$ be s.t. $G B_r(x) \supset X$.~~ ~~Note because G acts by isoms, $d(gx,hx)=$~~

Let $r = \operatorname{diam} G \backslash X$, so $G B_r(x) = X$ for any $x \in X$. Let $x \in X$.

Let $S = \{ g \in G \mid \underbrace{g B_{2r}(x) \cap B_{2r}(x) \neq \emptyset}_{\Updownarrow\ d(x,gx) \le 4r} \}$. Then S is finite.

– S generates G: Let $g \in G$. Let $\gamma : [0,L] \to X$ be a unit-speed geodesic from x to gx. We discretize γ: Let $n = \lceil \frac{L}{r} \rceil$, let $x_0, \dots, x_n$ be evenly spaced points along γ, $x_0 = \gamma(0)$, $x_n = \gamma(L)$. Then $d(x_i, x_{i+1}) \le r$.

$\forall i$, $\exists g_i$ s.t. $x_i \in g_i B_r(x)$. (take $g_0 = e$, $g_n = g$.)



Then $\forall i$, $d(g_i x, g_{i+1} x) \le d(g_i x, x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, g_{i+1} x) \le 3r$. ~~so $g \in S$.~~

So $d(x, g_i^{-1} g_{i+1} x) \le 3r \Rightarrow g_i^{-1} g_{i+1} \in S$.

Further, $\underbrace{(g_0^{-1} g_1)}_{\in S} \underbrace{(g_1^{-1} g_2)}_{\in S} \cdots \underbrace{(g_{n-1}^{-1} g_n)}_{\in S} = g_0^{-1} g_n = g$. So S generates G.

~~$-\Gamma_S \sim_{QI} X$. Let $f: G \to X$~~ Further, if d_S is word metric,

$$d_S(1, g) \le n = \left\lceil \frac{d(x,gx)}{r} \right\rceil \le \frac{d(x,gx)}{r} + 1.$$

– $\Gamma_S \sim_{QI} X$: Let $f : G \to X$, $f(g) = gx$. We claim f is a QI.

~~By above, $\forall g, h \in G$,~~ Because G acts by isoms, $d(gx,hx) = d(x, g^{-1}hx)$.

so enough to consider $d(x,gx)$

$$r\, d_S(g,h) - r \le d(gx, hx)$$

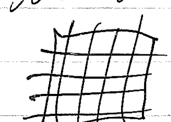
Conversely, $\forall s \in S$, $d(x, sx) \le 4r$. Suppose $g = s_1 \cdots s_k$.

Then we can draw: $x \xrightarrow{4r} s_1 x \xrightarrow{4r} s_1 s_2 x \cdots s_1 s_2 s_3 x \cdots gx$, of length $\le 4rk$.

$\Rightarrow \quad d(x, gx) \leq d_S(1, g) \cdot 4r$. So f is a QI-embedding.
The image of G is Gx and $d(y, Gx) \leq r \;\; \forall y \in X \Rightarrow f$ is coarsely surjective, so f is a QI //.
We thus say that $G \underset{QI}{\sim} H$ if ~~G, H are~~ $\exists$ finite gen. sets $S \subset G$, $T \subset H$ s.t. $\Gamma_S \sim_{QI} \Gamma_T$.

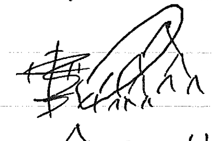
Corollaries: — All finite groups act geometrically on $*$ $\Rightarrow$ all finite are QI.
— Suppose G f.g.; $H \subset G$ is a finite-index subgroup. ~~Then H acts geomet~~ Then H acts geometrically on $\Gamma_S \Rightarrow G \sim_{QI} H$.
— Suppose $N \triangleleft G$ is finite. Then $G/N \sim_{QI}$ ~~to~~ G.

~~What So that So we can associate a geometry to every group.~~
What do these geometries look like? — this ignores finite groups, phenom assoc. w/ finite groups.
Martin Bridson drew the following map: ... but there's still a lot left.
Tour: $|\bullet$ — all finite groups live here.
Rest is infinite groups: complexity roughly left-to-right.
Edges describe two major phenomena in GGT: amenability, hyp/NPC.
Amenability: a little hard to describe — we'll ~~we can~~ talk about it later, but one def. for f.g. groups: these are groups where ~~the Banach-Tarski paradox~~ you can find subsets of the Cayley graph with small boundary:



area $\sim r^2$
bdry $\sim r$ $\quad \frac{r}{r^2} \to 0$ as $r \to \infty$.

Vs. non amenable where you can't: any subset of n verts has at least $n+1$ edges leading out.



Related to ~~a lot of~~ many diff'd props, like Banach-Tarski: 

So: abelian groups. (classification says that every abelian is QI to $\mathbb{Z}^n$) If you extend an amenable group by an amenable group: ~~...~~ you get an amenable gp: i.e. if $N \triangleleft G$ and N, G/N are amenable, then so is G.

Examples: Nilpotent groups (covered in my topics course in the spring)
— abelian with a twist: $\langle X, Y, Z \mid [X,Z]=[Y,Z]=1, [X,Y]=Z \rangle$.
$[X^n, Y^n] = Z^{n^2}$ — so we can write Z^{n^2} as a word of length $4n$.

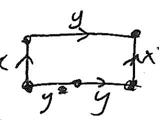
Solvable groups: Geometry starts to get more complex:

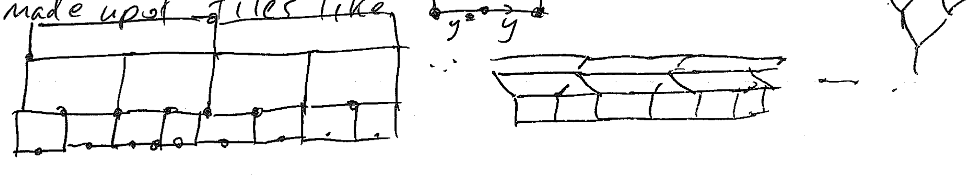
$BS(1,2) = \langle x, y \mid xyx^{-1} = y^2 \rangle$.

Then $x^n y x^{-n} = y^{2^n}$ — exponential distortion.

~~(This also gets into another issue: Computations in groups~~

~~Clearly We have~~: What does this look like geometrically?

Cayley graph is made up of tiles like 



This is metabelian: $1 \to \mathbb{Z}[\tfrac{1}{2}] \to BS(1,2) \to \mathbb{Z} \to 1$.

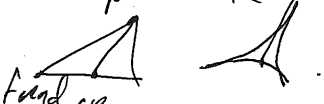
More complex groups exist.

Elementary amenable: can be constructed from ~~sub~~ finite and abelian gps by taking subgroups, quotients, extensions, ~~d~~ direct unions.

Open question: Are all ~~elementary~~ amenable groups elementary amenable?
(Hope for some algebraic good behavior.)

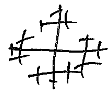
Hyperbolic/NPC:

Recall from diff geo: ~~Ba~~ negatively curved / NPC spaces are nice:

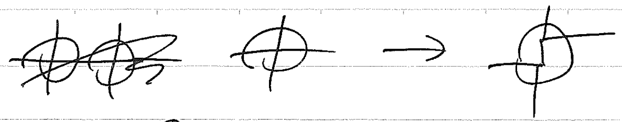
- unique geodesics
- universal cover is homeomorphic to $\mathbb{R}^n$.
- triangle comparison: 
- compact NPC ~~smfld~~ mflds have ∞ fund gp.
- ~~flat torus theorem~~
- splitting theorems
  - if fund gp is a product, then space is a product

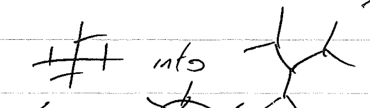
~~Hyperb~~ This side of the map covers groups with similar properties:

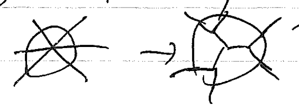
Prototypically: F - free groups.

 free group of rank 2.

- Unique geodesics
- Cayley graph is contractible
- ~~Universal cover is contractible~~.
- triangles are tripods:

Note: All free groups are QI: 

turns  in a QI way.

Same for , etc.

Hyperbolic groups generalize this: include: ~~#~~ groups that act geometrically complete mflds w/ negative curv. hyperbolic plane / hyperb. space.

~~- include: the groups of isometries of hyperbolic space~~

- fundamental groups of negatively curved closed mflds.

Coarse version: - ~~"random groups" $G = \langle a, b, c \mid abc^{-1}ab^{2}a^{2}a^{-2} \ldots$~~

- Can be weakened to δ-hyperbolic, Gromov-hyperbolic $w_1 = 1, w_2 = 1, \ldots, w_n = 1\rangle$

where w_i are random words ~~of length l, then this is~~

~~$P[G$ is h. then G is hyperbolic with high probability~~

then: ~~if n is fixed, $l \to \infty$~~ t

- random group presentations: $\langle a, b, c \mid v_1 = 1, w_2 = 1, \ldots w_n = 1\rangle$

is usually hyperbolic, ~~also~~ has a good coarse version!

This class is ~~Q~~ closed under QI — remarkable thm of Gromov.

Nonpositive curv. is more fragile:

C_0 = CAT(0) groups ~~includes~~

= groups that act geometrically on complete CAT(0) spaces

(incl. complete mflds w/ $K \leq 0$).

Includes ~~a~~ abelian groups, but also a lot of nice Lie groups:

eg. $SL_n(\mathbb{R})$ acts on a nonpositively curved symmetric space

Problem: This is not closed under QI's, doesn't have one good coarse version ~~so~~ so ~~versions~~.

part of this map is devoted to weakenings that ~~are~~ are a little more coarse: semi-hyperbolic, automatic, combable, etc.

that try to capture ~~things like unique geodesics,~~ aspects of NPC,

but there isn't one single notion of nonpositive curvature.

~~So, these are~~ But these are geometrically nice.

If one edge is roughly "algebraically nice", the other is "geometrically nice".

that leaves the middle — where we have much less of a foothold.

~~In fact, groups here are often hard to construct: Lie groups are around the edges, or show that they're infinite:~~

The edges ~~together a lot of groups that are "easy" to understand/construct.~~

- if you ~~construct it out of simpler pieces~~ — might be solvable, ~~a~~
- if ~~you~~ it's a Lie group ~~— left~~ side of ~~map~~.
- ~~if you write down~~ a group presentation ~~→ hyperbolic~~.
- fundamental group of a manifold ~~: why is the fund gp infinite?~~
-

~~Examples:~~ These are ~~two big paradigms for understanding groups~~:

~~Amenable~~: Decompose into simpler groups.

~~— Apply to some~~

~~Hyp/NPC: Look~~ ~~No tools of amenability, no tools from NPC~~.

~~but there are a lot of groups out there.~~

Middle: everything else: A lot of interesting groups here:

- self-similar groups - subdirect products
- anything that doesn't fit in this amenable/hyperbolic binary

But it gets more difficult to construct, more difficult to prove infinite, etc.

~~If it you just start writing down a group, how do you know it's a group,~~

~~how~~ ~~So there are various paths and dots here, but less known~~

~~there could be~~ ~~there are a lot of groups.~~ and there's a lot we don't know.

E.g. there's a ~~lot~~ close relationship

E.g.: these groups are groups where it's usually reasonably easy to show whether an elt is nontrivial: all these extensions, an elt is nontrivial if nonzero in any factor

- ~~ab~~ NPC - an element is nontrivial if there's a geodesic that hits it

Even these other examples ~~typ~~ generally have good algorithms.

But there's ~~are~~ a remarkable thm of Novikov:

Thm (Novikov): There is a finitely-presented group with unsolvable word problem:

i.e. ~~there is a~~ $G = \langle g_1, \dots, g_n \mid r_1 = 1, r_2 = 1, \dots, r_s = 1 \rangle$

and there is no algorithm that takes a product

$$w = g_{i_1}^{\pm 1} \cdots g_{i_k}^{\pm 1}$$ and returns whether $w = 1$.

2020.8.21

Hyperbolic space and Hyperbolicity.

Today: One of the most important geometries in GGT: Hyperbolicity.

First, quick refresher on hyperbolic geometry: For $n \geq 2$
~~The~~ Hyperbolic n-space is the unique ~~n-dimensional~~ complete simply-connected n-manifold with constant sectional curvature -1.

Two common ways to write it:

Disc model: $\mathbb{H}^n = D^n$ with metric $ds^2 = \frac{4\,dx^2}{(1-r^2)^2}$

Upper half-space: $\mathbb{H}^n = U^n = \{(x_1, \dots, x_n) \mid x_n > 0\}$, $ds^2 = \frac{dx^2}{x_n^2}$

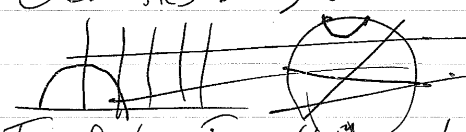
~~Closely connected to conformal geometry:~~ conformal metric on $\mathbb{R}^n$ is a metric $ds^2 = f(x)^2 dx^2$. conformal map: $M \to N$ is a smooth map that preserves angles, acts as scaling + rot'n

These are both conformal models: angles are preserved. Note that scaling factor goes to ∞ near boundary of each, so these are complete.

And these are isometric: if $n = 2$, can take $D^2 = \{z \mid |z| < 1\}$,
$U^2 = \{z \mid \mathrm{Im}(z) > 0\}$

and then $z \mapsto \frac{z-i}{z+i}$ sends U^2 to D^2.

~~Geodesics are circles and lines that are orthogonal to ∂D^n or ∂U^n.~~



~~Isometries:~~

In fact, $\mathrm{Isom}(\mathbb{H}^n) = \{$conformal maps from $D^n \to D^n\}$
$= \{$conformal maps from $U^n \to U^n\}$.

If $n=2$, $\mathrm{Isom}(\mathbb{H}^2) = \{z \mapsto \frac{az+b}{cz+d} \mid ad-bc=1,\ a,b,c,d \in \mathbb{R}\}$
$\cong PSL(2,\mathbb{R})$ (because $\frac{az+b}{cz+d} = \frac{(-a)z-b}{(-c)z-d}$.)

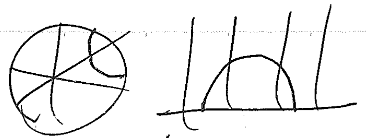
Notable examples: rotations. translations and scalings.

~~(but note that by uniqueness, Isom sp is transitive on frame bundle).~~

~~Key fact: FLTs send circles~~

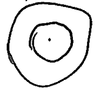
~~These rotations give us~~ By uniqueness, Isom of is transitive on unit tangent bundle, so we should be able to use these to produce all geodesics: clearly, and are geods, what are rest?

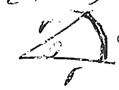
~~Prop:~~ Lemma: $z \mapsto \frac{az+b}{cz+d}$ sends circles and lines to circles and lines
- so all geodesics are circles or lines that are orthogonal to ∂D^n or ∂U^n.



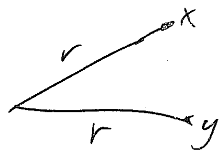
What is the geometry like?

1. Geodesics spread faster than in Euclidean space:

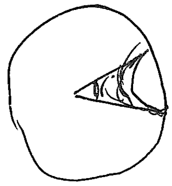
 circumference of circle is $2\pi\cosh r$

which is exp in r, so  $\theta\sinh r$ grows exp.

~~2. There are ultraparallel geodesics: No parallel lines, but:~~

2. triangles:  ~~this distance isn't exp, by triangle ineq.~~

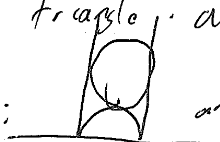
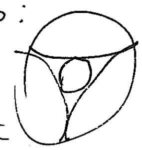
$d(x,y) \le 2r$. How much less?

 what happens as $r \to \infty$?

$\to$  $2r - 2c \le d(x,y) \le 2r$

~~In fact,~~ This is a $\frac{2}{3}$-ideal triangle: 2 verts at ∞.

Ideal triangle: all three to ∞:

Easier to see in half-space:  and see that all are isometric 

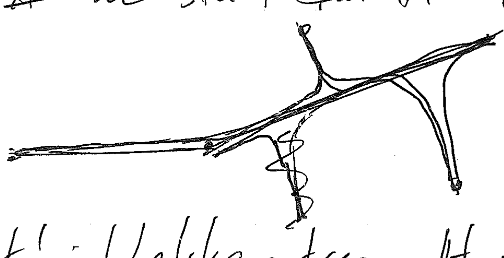
What does it mean that this is a triangle? All geods converge at ∞.

Exercise:
- there's an incircle: calculate its radius
- show that each side is contained in a nbhd of the other two
- show that any triangle can be drawn as a subset of an ideal triangle
- show that any triangle has area $\le \pi$.

3. tree-like structure: $\exists\, \delta > 0$ s.t.

~~Every The~~ Every edge of a triangle is contained in a δ-nbhd of the other 2.

~~So if we start like~~ take Quadrilaterals?



- every collection of n points is δ_n-close to a tree.

But! Unlike a tree, $\mathbb{H}$ is 1-ended: the complement of any ball is connected.

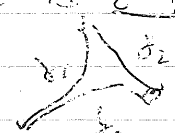
break.

No.

Date.

~~Generalizing and coarsifying: How can we define hype~~ -
Hyperbolicity for groups?: How do we define a notion of a hyperbolic Cayley graph?

Def: Let $\delta > 0$. A ~~space~~ geodesic metric space X is δ-hyperbolic if every geodesic triangle is δ-thin. That is, if $\gamma_1, \gamma_2, \gamma_3$ are three geods

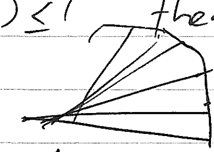


then $\gamma_i \subset N_\delta(\gamma_{i+1} \cup \gamma_{i+2})$ $\forall i$.

(Note: A geodesic is an isometric embedding $I \to X$. (~~i.e.~~ shortest path betw two pts).

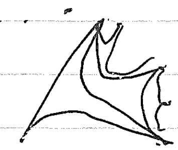
Then: - ~~Geodesics are δ~~ $\forall p, q \in X$, any two geodesics from p to q are δ-close.

- geodesics fellow-travel: if $d(y,z) \le 1$, then any geod $\overline{xy}$ is $(\delta+1)$-close to $\overline{xz}$.



- polygons are thin: if $\gamma_1, \dots, \gamma_n$ form a geodesic n-gon, then $\gamma_i \subset N_{n\delta}(\gamma_{i+1} \cup \dots \cup \gamma_{i+n-1})$ (triangulate poly).

(in fact, can do better:

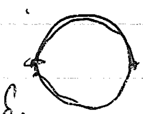


)

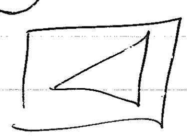
- every triangle has a center: p s.t. $d(p, \gamma_i) \le \delta$ $\forall i = 1, 2, 3$.

- X does not contain:

- ~~large circles~~: isometrically embedded circles bigger than 2δ.



isometrically embedded ~~copies of~~ planes



Lemma: Let X be a δ-hyperbolic geodesic metric space. Let $r > 0$ ~~be large~~. If $\overline{xy}$ is a geodesic of length $2r$, with midpoint m, then every path from x to y outside $B(m,r)$ has length at least $2^{\frac{r-1}{\delta}}$.

Pf: ~~Subdivide~~ ~~Let $L = \ell(\overline{xy})$~~ Let γ be a const-speed path outside $B(m,r)$, ~~$L = \ell(\gamma)$~~

~~Subdivide γ into 2^k segments of length ≤ 2, where~~

Let $k = \lceil \log_2 \ell(\gamma) \rceil$, let $x_i = \gamma(\frac{i}{2^k})$