Date 2023 09 07

Introduction to Geometric Group Theory

- Office hours by appointment - website - collect emails.
- ~~Possibly~~ optional ~~problem sets, main evaluation is but primarily~~ This is seminar style, may give optional problem sets, but primary resp is to attend lectures, ~~ask~~

~~What is geometric group theory?~~

- ~~How are algebraic properties/structures in groups reflected by their geometry?~~
- ~~What new properties can we introduce based on their geometry?~~
- ~~How can we use geometry to study groups? / What properties embedes / How much can you learn about a group by studying geometry?~~
- ~~What new and interesting geometry arises from studying groups?~~
- ~~What can we say about the geometry props of a sp~~

Goat: ~~Give you an idea of what GGT is, how we can~~

~~What~~ is geometric group theory?

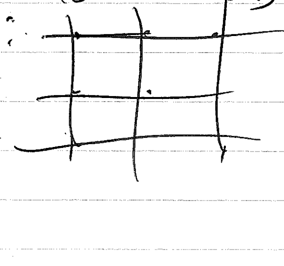
- How is the algebra of a group reflected by its geometry? (Can you tell whether a group is ~~X~~ abelian, nilpotent, solvable by looking at its geometry?)
- ~~What new properties can we introd~~
- How can we study groups geometrically?
  - What new spaces arise by looking at groups?
  - What ~~propert~~ new properties ~~arise?~~ & can we define using the language/viewpoint of geometry?

Goal: Try to answer these questions,

- Try to describe the landscape of groups from POV of geometry.

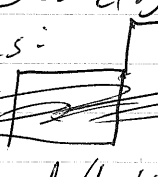
Start: How do we study groups geometrically?

Ex: $\mathbb{Z}^2$:



We can think of $\mathbb{Z}^2$ as the unit grid in the plane. Geometry of this grid?

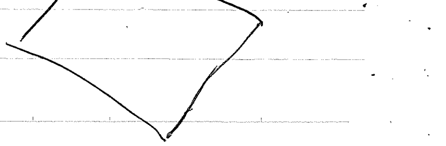
- distances:



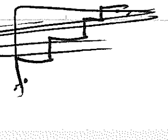
- ~~areas:~~ ~~How much area in a disc?~~

  pth

- spheres and ~~balls~~ balls: size of bdry, size of disc



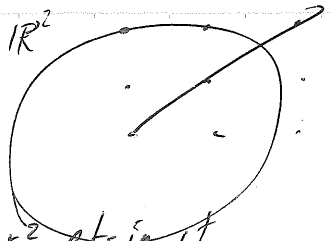
- ~~geodesics~~



- shortest paths:

Or we can embed: $\mathbb{Z}^2 \subset \mathbb{R}^2$

and get similar answers:



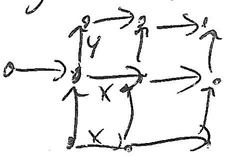
- distance: $\sqrt{|x^2|+y^2}$ $\sim |x|+|y|$
- ~~a~~ spheres:

disc of radius r has $\sim r^2$ pts in it

- sphere of radius r goes close to $\sim r$ points. ~~though the exact number is slightly delicate.~~
- ~~(but there are a lots of different lattices — why pick this one?)~~

How can we generalize?

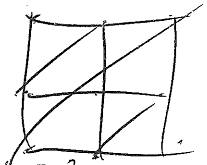
1: The grid is an example of a Cayley graph:
one vertex for every group element, ~~or~~ connect vertices by an edge if they differ by a generator:



generating set ~~xy~~ x, y

~~But~~ Note: this depends on the generating set:

generating set $x, y, x+y = \mathbb{Z}$:



(and distances change).

And we can define this whenever G is a ~~f.g.~~ group equipped w/ gen. set S. Natural to study these graphs.

2: $\mathbb{Z}^2$ is a cocompact lattice in $\mathbb{R}^2$:
- discrete subgroup
- $\mathbb{R}^2/\mathbb{Z}^2$ is compact

(these are important: ~~$\mathbb{Z}^2 \cong \langle 1, \sqrt{2}\rangle$~~ $\mathbb{Z}^2 \to \mathbb{R}$
$(x,y) \mapsto x+\sqrt{2}y$
is injective ~~but image~~, so $\mathbb{Z}^2 \subset \mathbb{R}$, but not discrete subgp;
$\mathbb{Z}^2 \to \mathbb{R}^3$
$(x,y) \mapsto (x,y,0)$ is discrete, not cocompact.)

~~Natural to study these spaces when we can.~~

So, if ~~G~~ Γ is a cocompact lattice in G, a Lie group, it is natural to use ~~study~~ the geometry of G to study Γ.

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- different lattices in same Lie group: ⁘
- lattices in diff Lie groups: ⁘

Problems: 1: These are not unique: # vs #, different Riem metrics on G.
~~But also~~: - $\mathbb{Z} \subset \mathbb{R}$ but also $\mathbb{Z} \subset \mathbb{R} \times S^1$, $\mathbb{Z} \subset \mathbb{R} \times K$ when K is compact/finite.
2: Those spaces don't have the same properties:
for ex: # has multiple geods between two pts; ⁘ does not.
We would like one "geometry" for every group.

So: Idea of GGT: We should view these spaces through a lens where all of these spaces look the same.
We should look for properties that are shared by all these spaces: uniqueness of geodesics don't work, but the size of balls does, the fact that you can cut $\mathbb{R}$, $\mathbb{R} \times S^1$, $\mathbb{R} \times K$ into two pieces does.

So: today: explain this viewpoint.
Def: Let X, Y be metric spaces. Let $L, C > 0$.
A map $f : X \to Y$ is an L-bilipschitz embedding if
$$L^{-1} d(a,b) \le d(f(a), f(b)) \le L\, d(a,b) \quad \forall a,b \in X.$$
f is a bilipschitz equivalence if it is also a bijection.
(Ex: ~~different choices of left invar~~ X, Y two left-inv. metrics on a Lie group G are bilipschitz equivalent).
$f : X \to Y$ is an (L,C) quasi-isometric embedding if
$$L^{-1} d(a,b) - C \le d(f(a), f(b)) \le L\, d(a,b) + C \quad \forall a,b \in X.$$
(bilipschitz when ~~a,b~~ $d(a,b)$ is large).
~~X and Y are quasi-isomet.~~ f is a quasi-isometry if f is also coarsely surjective:
$\forall y \in Y, \exists x \in X$ s.t. $d(y, f(x)) < C$.
X is quasi-isometric to Y if $\exists L, C > 0$ ~~s.t.~~ and $f : X \to Y$ an (L,C)-QI.
Prop: $\sim_{QI}$ is an equivalence relation.
Pf: Exercise. In particular: $\forall L, C\ \exists L', C'$ s.t. if $f : X \to Y$ is an (L,C) QI, then $\exists\, g : Y \to X$ which is an (L', C')-QI s.t.
$\forall x \in X$, $d(g(f(x)), x) < C$
$\forall y \in Y$ $d(f(g(y)), y) < C$.
Then all the examples from before are quasi-isometric:
$f : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$, $f(x,y) = (x,y)$
$g : \mathbb{R}^2 \to \mathbb{Z}^2$ $g(x,y) = (\lfloor x \rfloor, \lfloor y \rfloor)$
Like the prop, these are not inverse, but quasi-inverse.
And same for other examples: # $\sim_{QI}$ #, $\mathbb{R} \sim_{QI} \mathbb{R} \times S^1$
$\sim_{QI} \mathbb{R} \times K$.

(In fact, $K \sim_{QI} *$ for any compact K ~~$\Rightarrow$~~ — quasi-isometry ignores finite objects).

Lemma: ~~Let S, T be two finite~~ ger

~~And this is the lens we need~~ More generally:

Thm (Svarc-Milnor): Let X be a proper geodesic metric space, and let G be a group that acts geometrically on X. Then X is finitely generated, and if G also acts geometrically on Y, then $X \sim_{QI} Y$.

~~Pf~~: (Glosses: X proper = every closed ~~ball~~ metric ball is compact.
G acts geometrically: by isometries,
~~G~~ cocompactly: $G \backslash X$ is compact.
properly discontinuously: $\forall$ compact $K \subset X$
(enough to take K to be metric balls),
$\# \{g \in G \mid gK \cap K \neq \emptyset\} < \infty$.)

Ex: G is a cocompact lattice, X is a Lie group.
~~But also,~~ Ex: G has finitely generating set, acts on Cayley graph
Ex: ~~$G = \pi_1(M)$~~ M compact mfld/complex, So all these are QI.
$G = \pi_1(M)$ acts on $\tilde{M}$.
~~Ex: $G = \mathrm{Isom}(\mathbb{H})$~~. ~~So all the universal covers QI.~~
so the QI class of $\tilde{M}$ depends only on $\pi_1(M)$.

Pf: Let $x \in X$. Let $r = \operatorname{diam} G \backslash X$ so that $G B_r(x) = X$ $\forall x \in X$.
Let $S = \{g \in G \mid g B_{2r}(x) \cap B_{2r}(x) \neq \emptyset\}$.
Then this is finite. We claim that S generates G and $X \sim_{QI} \Gamma$, where Γ is the Cayley graph of G wrt. S.

– S generates G: Let $g \in G$. Let $\gamma: [0, L] \to X$ be a unit-speed geodesic from x to gx. Let $n = \lceil \frac{L}{r} \rceil$, let $x_0 = x, x_1, \dots, x_n = gx$ be evenly-spaced pts along γ, no more than r apart.

Then $\forall i$, $\exists g_i \in G$ s.t. $x_i \in g_i B_r(x)$. — we take $g_0 = e$,
$g_n = g$.

Lemma: Let S, T be two finite gen sets of G. ~~Pf~~ Let Γ_S, Γ_T be the corresp Cayley graphs. Then $\Gamma_S \sim_{QI} \Gamma_T$.

Need a def:

~~Pf: For $g,h \in G$, let d_S be the word metric~~ on G:

d_S: For $g,h \in G$, let $d_S(g,h)$ be the length of shortest path in Cayley graph from g to h. This is the word metric on G, because $d_S(1,g)$ = smallest k s.t.

$g = s_1^{\pm 1} \cdots s_k^{\pm 1}$ where $s_i \in S$.

= ~~short~~ length of shortest word representing g.

and $d_S(g,h) = d_S(1, g^{-1}h)$.

Pf: ~~Let $L = \max_{s \in S} d_T(1,s)$~~ ETS $d_S(1,g) \sim d_T(1,g)$ $\forall g \in G$.

Let $L = \max_{s \in S} d_T(1,s)$, ~~let $k = d_T(1,g)$~~

Let $s_1^{\pm 1} \cdots s_k^{\pm 1} = g$ be shortest word in S representing g.

Each s_i can be written as a word of length $\leq L$ in T.

So g can be written with $\leq kL$ letters of T:

$d_T(1,g) \leq L d_S(1,g)$. Likewise,

$d_S(1,g) \leq L' d_T(1,g)$. //

So all ~~Cayley graphs~~ if G has a finite gen set, then all finite gen sets give QI Cayley graphs.

(What about infinite?)

$\forall i$, $d(x_i, g_i x) \leq r$ and $d(x_i, g_{i+1} x) \leq d(x_i, x_{i+1}) + d(x_{i+1}, g_{i+1} x)$
$\leq 2r$,

so $x \in B_{2r}(g_i x) \cap B_{2r}(g_{i+1} x)$.

Translate by g_i^{-1}

$\Rightarrow B_{2r}(x) \cap B_{2r}(g_i^{-1} g_{i+1} x) \neq \emptyset \Rightarrow g_i^{-1} g_{i+1} \in S$.

So $(g_0^{-1} g_1)(g_1^{-1} g_2) \cdots (g_{n-1}^{-1} g_n) = g_0^{-1} g_n = g$ is a product of elts of S. Claim: $f(g) = gx$ is a QI:

By above, ~~Further~~, $d_S(e,g) \leq n = \left\lceil \frac{L}{r} \right\rceil \leq \frac{d(x,gx)}{r} + 1$

Conversely, $d(x,gx) \leq 4r$, so if $g = s_1 \cdots s_k$, then $d(x,gx) \leq 4kr$.

and coarsely surjective. //