Numerical analysis
March 10, 2020

By our definition earlier: sensitivity of the $\underset{\underset{\underline{x}}{\uparrow}}{\text{solution}}$ to the $\underset{\underset{\underline{b}}{\uparrow}}{\text{input}}$

Let $\|\underline{b}-\underline{b}'\|$ be small, and $\underline{x} = A^{-1}\underline{b}$, $\underline{x}' = A^{-1}\underline{b}'$.

Then
$$\|\underline{x}-\underline{x}'\| = \|A^{-1}\underline{b} - A^{-1}\underline{b}'\|$$
$$\le \underbrace{\|A^{-1}\|}_{\text{absolute condition number}} \|\underline{b}-\underline{b}'\|$$

But remember, the absolute condition number tells us <u>nothing</u> about the number of correct digits in the answer.

Need the <u>relative</u> condition number:

$$\frac{\|\underline{x}-\underline{x}'\|}{\|\underline{x}\|} \le \|A^{-1}\| \frac{\|\underline{b}-\underline{b}'\|}{\|\underline{x}\|}$$
$$= \|A^{-1}\| \frac{\|\underline{b}-\underline{b}'\|}{\|\underline{b}\|} \underbrace{\frac{\|\underline{b}\|}{\|\underline{x}\|}}_{= \frac{\|A\underline{x}\|}{\|\underline{x}\|} \le \|A\|}$$
$$\Rightarrow \le \underbrace{\|A\| \, \|A^{-1}\|}_{\text{relative condition number.}} \frac{\|\underline{b}-\underline{b}'\|}{\|\underline{b}\|}$$

What else is related to the eigenvalues of $A^t A$?

Recall the singular-value decomposition:

For $\underline{\text{any}}$ matrix A (square or not, invertible or not)

$$A = U S V^t$$

U, V orthogonal $\rightarrow$ $U^t U = I$

S diagonal, $\geq$ entries

If A $\underline{\text{is}}$ invertible, then $A^t A = (V S U^t)(U S V^t)$

$$= (V S^2 V^t)$$

We will return to computing the SVD numerically...

Interpretation of the ℓ^2-condition number: ratio of $\underline{\text{stretching}}$ to $\underline{\text{shrinking}}$.

$$(A^t A)^{-1} = V S^{-2} V^t$$

If $S^2 = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$ $\sigma_1^2 > \dots > \sigma_n^2$,

$$S^{-2} = \begin{pmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{pmatrix}$$

largest

then $\|A\| = \sigma_1$

$\|A^{-1}\| = \frac{1}{\sigma_n}$

$$\Rightarrow \boxed{K(A) = \frac{\sigma_1}{\sigma_n}}$$