# NOTES AND CORRESPONDENCE

## A Shallow-Water Model that Prevents Nonlinear Steepening of Gravity Waves

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ABSTRACT

The shallow-water model is used as a testbench for understanding many fundamental dynamical problems (e.g. certain wave–mean interaction problems). One sometimes wants to allow large-amplitude gravity waves to propagate significant distances in such models without forming shocks. This paper presents a simple, and apparently unique, modification of the standard shallow-water model that prevents gravity wave shock formation, but which, at the same time, introduces only minimal changes in other aspects of the behavior. For instance, the presented modification is nondissipative as well as nondispersive, and it preserves the linear structure of the shallow-water equations as well as the nonlinear functional form and material invariance of shallow-water potential vorticity. The modification is derived theoretically and has been tested numerically in several ways in one and two dimensions.

### 1. Introduction

The shallow-water system has a long history of use as a paradigm for three-dimensional rotating stratified flow. The restriction to two dimensions not only gives physical simplification but also offers clear computational advantages. A case in point is the study of interactions between small-scale gravity waves and large-scale balanced, or potential-vorticity-controlled, motions in the atmosphere. The importance of such interactions for chemical, climate, and weather predictions is well recognized (e.g. Holton 1982; McIntyre 1993; Holton et al. 1995 and references therein), and yet, for the foreseeable future, a direct numerical simulation of these interactions in general circulation models lies far beyond the reach of even the most powerful supercomputers.

Therefore, there is a need to parametrize efficiently and accurately the influence of gravity waves on the large-scale flow in general circulation models, and this in turn demands a firm theoretical understanding of the underlying wave–mean interactions. This requires the use, both conceptually and numerically, of a hierarchy of simplified flow models in order to assess the importance of different aspects of these interactions and in order to gain a better understanding of the physical phenomena that are involved.

There are a number of reasons why it might be desirable to keep dissipation to a minimum in such simplified experiments. For instance, current gravity wave parametrizations are usually based on the assumption that all the significant effects are associated with dissipating (including breaking) gravity waves. However, this assumption is based mainly on idealized models, involving, for example, zonally symmetric basic states, and/or small-amplitude, slow-modulation approximations. So the question whether nondissipative wave–mean interactions are ever significant needs addressing.

Recently, an attempt was made to investigate this question on a fundamental level using a combination of analytical and numerical techniques (Bühler 1996; Bühler and McIntyre 1998). This necessitated finding a simple fluid model in which nondissipative wave–mean interactions can be studied. Bühler and McIntyre took a two-dimensional model as a first test bench and stepping stone toward more complicated three-dimensional stratified models. The standard shallow-water (SSW) system suggested itself because of its well-known relation to three-dimensional stratified flow viewed in isentropic coordinates (e.g., Andrews et al. 1987).

However, a fundamental problem arises in the SSW system with very weak dissipation because of the robust tendency of SSW gravity waves to steepen up and to form shocks in finite time.[1] The formation of shocks,

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[1] The time for shocks to form is inversely proportional to the wave amplitude and is generally very short. For instance, an SSW gravity wave with a moderate amplitude of relative depth disturbance $a = 0.1$ will break after just one wavelength of propagation; see also (38).

in particular the associated development of arbitrarily small spatial scales, inevitably leads to the importance of dissipative effects, and in a way that is peculiar to the SSW system and unlike the ways in which stratified gravity waves dissipate. In addition to introducing dissipative effects, shock formation also puts a severe strain on numerical integrations unless highly specialized, ''shock-capturing'' numerical schemes are used (e.g., Hirsch 1990).

This dilemma suggests seeking a modification of the SSW model that removes gravity wave shock formation, but which retains the usefulness of the SSW model as a simple model for quasi-horizontal flow along isentropes. A suitable modification should therefore retain SSW features such as its linear wave structure; its material conservation of a potential vorticity (PV), together with the functional form of this PV in terms of vorticity and layer depth; the character of its balanced, PV-controlled flow component; and the simplicity (including numerical simplicity) of its equations. Furthermore, the ability to easily adapt existing shallow-water codes to the modified system would be of great practical importance. It is now shown that there is only one reasonable way to modify the shallow-water model such that these requirements are met.

## 2. Modification of the shallow-water equations

Consider the equations of motion of the unforced and nondissipative SSW system:

$$\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} + c_0^2\nabla h = 0 \qquad (1)$$

$$\frac{Dh}{Dt} + h\nabla\cdot\mathbf{u} = 0, \qquad (2)$$

where $\mathbf{u} = (u, v)$ is the two-dimensional velocity vector and $\hat{\mathbf{z}}$ is a unit vector in the vertical direction; $h$ is the nondimensional layer depth such that $h = 1$ corresponds to the depth of an undisturbed layer; $f$ is the (possibly location-dependent) Coriolis parameter associated with the background rotation; and $c_0$, a constant, is the absolute value of the linear phase speed of high-frequency gravity waves. The independent variables are the time $t$ and the space coordinates $x$ and $y$. The material derivative is defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}. \qquad (3)$$

To remove gravity wave steepening many modifications of these equations are conceivable, but most of them would not meet the requirements set out in the last paragraph of section 1. For instance, one could try to add new terms to the momentum equation (1) that counteract the steepening of gravity waves. One feasible additional term is presumably given by the next term in the formal expansion of the original shallow-water dynamics in powers of the inverse horizontal length scale; see, for example, Salmon (1988). Inclusion of this term allows solitary waves to exist, which is an indication that the new term adds some stabilizing dispersion and nonlinearity to the system.[2] However, this term involves a second material time derivative of the depth field $h$, which would require a major reconstruction of existing numerical codes and which would also introduce second spatial derivatives, which would increase the complexity of the numerical scheme. Also, inclusion of this term would change the linear dispersion relations of the system as well as the definition of PV. Adding other dispersive terms in an ad hoc manner would suffer from the same problems and might also lose PV conservation altogether.

There is, finally, the possibility of modifying the irrotational pressure term in (1). Specifically, consider a change in (1),

$$c_0^2\nabla h \rightarrow c_0^2\nabla F(h), \qquad (4)$$

with a suitably chosen nondimensional function $F(h)$. It is clear that a modification of the shallow-water system that respects mass continuity (2) and the functional form and material invariance of PV must be of the form (4), apart from $F(\cdot)$ being a nonlocal function of $h$, or even a function of $\mathbf{u}$, which is considered no further.

The change (4) is very easily effected in existing shallow-water codes, whether they use $(u, v, h)$ or $(\nabla \times \mathbf{u}, \nabla\cdot\mathbf{u}, h)$ as dependent variables. Because the change introduces no higher derivatives into the system, the computational requirements for the modified system are essentially the same as for the standard system. Also, the linearized modified shallow-water system can be made to coincide with the linearized SSW system by requiring that

$$F'(1) = 1, \qquad (5)$$

where the prime indicates differentiation. This is a convenient requirement, which also ensures that the change (4) introduces no linear instabilities into the system. It is now shown that one and only one form of $F(h)$ eliminates gravity wave steepening.

## 3. One-dimensional simple waves

Consider first the nonrotating case (i.e., $f = 0$ in Eq. 1), for which SSW gravity waves exhibit strong steepening. The numerical examples below will allow nonzero $f$ and hence will provide an independent check on the importance of background rotation.

[2] The presence of nonzero $f$ alone adds some dispersion to the shallow-water system, and there is evidence that this is enough in some cases to allow nonlinear gravity waves to propagate large distances without forming shocks (e.g., Bühler 1993). However, this particular kind of dispersion acts strongest on the largest scales and is therefore unlikely to be of much use to prevent the steepening of small-scale gravity waves.

The one-dimensional nonrotating shallow-water equations modified according to (4) are

$$u_t + uu_x + c_0^2 F'(h) h_x = 0 \tag{6}$$

$$h_t + uh_x + hu_x = 0, \tag{7}$$

where suffixes denote partial differentiation. Adding $l$ times the second equation to the first results in

$$u_t + u_x[u + lh] + l\left\{h_t + h_x\left[u + \frac{c_0^2}{l}F'(h)\right]\right\} = 0. \tag{8}$$

This equation is in characteristic form (e.g., Whitham 1974, p. 161ff) if the quantities in the two square brackets agree, that is, if

$$l = \pm c_0\sqrt{\frac{F'(h)}{h}}. \tag{9}$$

If (9) holds, then (8) can be written as

$$\frac{d^\pm}{dt}u \pm c_0\sqrt{\frac{F'(h)}{h}}\frac{d^\pm}{dt}h = 0, \tag{10}$$

where

$$\frac{d^\pm}{dt} \equiv \frac{\partial}{\partial t} + C^\pm\frac{\partial}{\partial x} \quad \text{and} \quad C^\pm \equiv u \pm c_0\sqrt{hF'(h)}. \tag{11}$$

The time derivative in (11) gives the change rate following a characteristic with characteristic speed $C^\pm$. There are two characteristic speeds and therefore two families of characteristics, corresponding to whether the upper or lower sign is taken. It will be assumed that $C^+ > 0$ and $C^- < 0$, which is true if the magnitude of the fluid velocity $u$ is everywhere less than the relative nonlinear wave speed $c_0\sqrt{hF'(h)}$.

Equation (10) can be formally integrated along the characteristics,[3] which yields

$$\frac{d^\pm}{dt}R^\pm = 0 \quad \text{for} \tag{12}$$

$$R^\pm \equiv u \pm \int c_0\sqrt{\frac{F'(h)}{h}}\,dh. \tag{13}$$

Here (13), the definition of the so-called Riemann invariants of the system, determines each of the two $R^\pm$ up to a constant of integration. Clearly, knowledge of $u$ and $h$ implies knowledge of $R^+$ and $R^-$, and vice versa.

The invariance $R^\pm$ along the corresponding characteristic curves allows a complete solution of the equations in the case of a so-called simple wave, for example, a wave that is supposed to be generated at $x = 0$ and then to propagate to the right into fluid that was initially at rest. In this situation, all left-going characteristics with speed $C^-$ originate from fluid that was at rest at $t = 0$. Hence $R^-$, which is constant following left-going characteristics, has to have the same value everywhere and, in particular, has to have the value that corresponds to fluid at rest. This establishes a relation between $u$ and $h$ [taking the lower sign in (13)] that holds everywhere in the fluid, which effectively reduces the number of dependent variables from two to one. Further, because $R^+$ is constant following right-going characteristics and $R^-$ is constant everywhere, the physical state (i.e., $u$ and $h$) must be constant on each right-going characteristic. Hence the characteristic speed $C^+$, which depends on $u$ and $h$, must be constant on right-going characteristics as well, which means, for instance, that right-going characteristics appear as straight lines in an $(x, t)$ diagram.

The steepening of the gravity waves can now be understood in terms of the dependence of $C^+$ on the physical state. For simple waves in standard shallow water $F(h) = h$, and hence [cf. (13) and (11)]

$$R^- = u - 2c_0\sqrt{h} + K = -2c_0 + K \tag{14}$$

$$\Rightarrow u = c_0(2\sqrt{h} - 2), \tag{15}$$

and with (11)

$$\Rightarrow C^+ = c_0(3\sqrt{h} - 2). \tag{16}$$

The second equality in (14) derives from evaluating $R^-$ at $t = 0$, that is, for a fluid at rest, where $u = 0$ and $h = 1$. Equation (15) is the relation between $u$ and $h$ in simple waves in the standard shallow-water system, and (16) shows that the nonlinear wave speed $C^+$ is equal to $c_0$ only for a fluid at rest and that it grows with $h$. This implies that compression regions of gravity waves travel faster than rarefaction regions and hence that the gravity wave steepens up and eventually forms a shock.

The task is now to devise a suitable $F(h)$ that allows compression and rarefaction regions to travel with the same speed. In general, $C^+ = \Lambda(R^+, R^-)$ for some function $\Lambda(\cdot, \cdot)$. In a simple wave $R^-$ is constant everywhere, but $R^+$ varies from one right-going characteristic to another. Hence if $C^+$ is to be constant everywhere in a simple wave, then it must be independent of $R^+$, that is,

$$C^+ \overset{!}{=} \Lambda(R^-) \tag{17}$$

for some function $\Lambda(\cdot)$. Equation (17) is the pivotal element in this search for a suitable modification of the standard shallow-water system. Using (11) and (13), one obtains

$$C^+ = R^- + c_0\left[\sqrt{hF'(h)} + \int\sqrt{\frac{F'(h)}{h}}\,dh\right], \tag{18}$$

which implies that (17) holds if and only if the quantity in parentheses is a constant. (It cannot possibly be a function of $R^-$, because it has no dependence on $u$.) Differentiating this quantity with respect to $h$ one ob-

[3] It is here where $f \neq 0$ would complicate the situation, because in that case this formal integration would not be possible.

tains the following second-order ordinary differential equation for $F(h)$:

$$\left[\sqrt{hF'(h)}\right]' + \sqrt{\frac{F'(h)}{h}} = 0, \tag{19}$$

whose general solution is

$$F(h) = A + \frac{B}{h^2}. \tag{20}$$

The value of the constant $B$ is fixed by (5), that is, by the requirement that the linearized system be the same as the standard shallow-water system:

$$F'(1) = 1 \Rightarrow B = -1/2. \tag{21}$$

The value of the physically meaningless additive constant $A$ can be fixed, for convenience, by requiring that

$$F(1) = 1 \Rightarrow A = 3/2, \tag{22}$$

and therefore

$$F(h) = \frac{3}{2} - \frac{1}{2h^2} \tag{23}$$

is obtained as the unique solution to the problem. No other choice of $F(h)$ will allow nonsteepening gravity waves to exist while preserving the linear properties of the shallow-water system (if one discounts physically meaningless changes of the additive constant 3/2).

Repeating the steps that lead to (14) and (16) in the case of the SSW system now produces modified shallow water:

$$R^- = u + \frac{c_0}{h} + K = +c_0 + K \tag{24}$$

$$\Rightarrow u = c_0\frac{h-1}{h} \tag{25}$$

$$\Rightarrow C^+ = u + \frac{c_0}{h} = c_0. \tag{26}$$

Equation (25) is the relation between $u$ and $h$ in simple waves in the modified shallow-water system, and Eq. (26) shows that the nonlinear wave speed is constant and equal to the linear wave speed.[4]

[4] This calculation has shown that simple waves in the modified shallow-water system have no tendency to form shocks. It can further be shown, by the standard method of considering the conservation of mass and momentum across a jump in $u$ and $h$, that the modified shallow-water system in general admits no discontinuous solutions other than those that correspond to a discontinuous simple wave.

## 4. Modified equations of motion, quasigeostrophic approximation, and energy conservation

Some aspects of the modified shallow-water (MSW) equations that result from the choice of $F(h)$ according to (23) are now discussed. The continuity equation (2) remains unchanged, but the modified momentum equation (1) is

$$\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -c_0^2\nabla F(h)$$

$$= -c_0^2\nabla\left(\frac{3}{2} - \frac{1}{2h^2}\right) = -\frac{c_0^2}{h^3}\nabla h. \tag{27}$$

Because the modified pressure term is still irrotational, the PV is unchanged, that is,

$$\nabla \times \mathbf{u} \equiv v_x - u_y,$$

$$q \equiv (\nabla \times \mathbf{u} + f)/h \Rightarrow \frac{Dq}{Dt} = 0 \tag{28}$$

still holds in the MSW system. In this connection it might be interesting to note whether or not the MSW system has the same quasigeostrophic (QG) approximation as the SSW system. The usual QG approximation can be understood in terms of (28) by assuming that the velocity field is approximately described by a streamfunction, $\Psi$ (i.e., $u = -\Psi_y$ and $v = +\Psi_x$), and by using the divergence of the momentum equation to derive an approximate relationship between the height field $h$ and the stream function $\Psi$. This last step results in

$$h \approx \begin{cases} 1 + \dfrac{f}{c_0^2}\Psi & \text{standard SW} \\ \left(1 - 2\dfrac{f}{c_0^2}\Psi\right)^{-1/2} & \text{modified SW.} \end{cases} \tag{29}$$

These approximate relationships differ in their nonlinear terms, but their linearized forms are identical. Because the usual QG approximation uses only a linear approximation of $1/h$ in terms of $\Psi$, this implies that the QG approximation of the MSW system is identical to that of the SSW system, which is in accordance with the fact that by design the two systems have the same linearization.

The energy of the MSW system differs from the SSW energy in the expression for the nonkinetic (potential) energy. The standard law of energy conservation is

$$\frac{d}{dt}\int\int\left(\frac{|\mathbf{u}|^2}{2} + e(h)\right)h\,dx\,dy = -\oint p(h)\mathbf{u}\cdot\mathbf{n}\,ds, \tag{30}$$

where $e(h)$ is the nonkinetic energy density per unit mass, $p(h)$ is the nonadvective flux of momentum such that $c_0^2\nabla F = h^{-1}\nabla p$ in (27), and the area integral on the left is taken over an arbitrary material area, that is, an arbitrary area that moves with flow. The contour integral on the right is taken around the boundary of that material

TABLE 1. Nonkinetic energy density $e(h)$ and nonadvective momentum flux $p(h)$ in the SSW and MSW systems.

| | Standard SW | Modified SW |
|---|---|---|
| Nonkinetic energy density $e(h)$ | $\beta - \alpha/h + c_0^2 h/2$ | $\beta - \alpha/h + c_0^2/(2h^2)$ |
| Nonadvective momentum flux $p(h)$ | $\alpha + c_0^2 h^2/2$ | $\alpha - c_0^2/h$ |

area, with $\mathbf{n}$ as the outward-pointing unit normal vector on that contour and $ds$ as its line element. It is straightforward to show[5] from these definitions that the most general forms of $e(h)$ and $p(h)$ in SSW and MSW are as given in Table 1. In these expressions the constants of integration $\alpha$ and $\beta$ are arbitrary, although the usual derivation of the SSW system motivates a choice, $\alpha = \beta = 0$, in the SSW system. Finally, it can be shown that the expressions for $e(h)$ can be used in a derivation of both SSW and MSW as Hamiltonian systems (see, e.g., Salmon 1988 for a summary of Hamiltonian fluid mechanics).

It can be noted in passing that the nonlinear differences between the respective MSW and SSW forms of the irrotational pressure term $c_0^2\boldsymbol{\nabla}F = h^{-1}\boldsymbol{\nabla}p$ can have some implications for leading-order wave–mean interaction effects. The linear properties of MSW coincide with SSW and hence $O(a^2)$ quantities that can be consistently evaluated from linear, $O(a)$ solutions alone (e.g., wave action and Stokes drift) are identical in both systems. Here $a$ is a suitably defined nondimensional wave amplitude. However, other $O(a^2)$ quantities can be different. For instance, it can be shown (Bühler and McIntyre 1998) that certain wave-induced changes in the mean density (which are familiar from analogous problems in acoustics; e.g., McIntyre 1981) are different in the two systems.

## 5. Numerical tests

The modified system is tested here for simple waves using a one-dimensional finite-difference model of the shallow-water system, based on the two-dimensional model of Ford (1994). Further testing using a range of two-dimensional wave–mean interaction problems was carried out by Bühler and McIntyre (1998), who found no unexpected or undesirable behavior of the MSW system. The model used here includes constant background rotation, which allows a useful check on how nonzero $f$ influences the simple waves, a question that was not addressed in the theory. The one-dimensional equations that were solved here are (subscripts denote partial derivatives):

$$u_t + uu_x - fv + c_0^2F'(h)h_x = G(x, t) \tag{31}$$

$$v_t + uv_x + fu = 0 \tag{32}$$

$$h_t + uh_x + hu_x = 0, \tag{33}$$

where the body force $G(x, t)$ was applied in a local forcing region. The model domain spans 10 wavelengths of a right-going gravity wave that is forced by the body force $G(x, t)$. The force is centered near the left end of the domain and extends over 1.4 of a wavelength. The exact form of the $G(x, t)$ that was used is

$$G(x, t) \equiv \frac{-ac_0^2}{0.7\pi}\frac{d}{dx}[X(x)\cos(kx - \omega t)], \tag{34}$$

where the envelope $X(x)$ is given by

$$X(x) \equiv \begin{cases} 0 & |x - x_0| \geq 0.7\lambda \\ \cos^2\left(\dfrac{\pi}{2}\dfrac{x - x_0}{0.7\lambda}\right) & |x - x_0| < 0.7\lambda \end{cases} \qquad x_0 = 3\lambda. \tag{35}$$

Here $a$ is the amplitude of the forced wave, $\lambda = 2\pi/k$ its wavelength, $k$ its wavenumber, and $\omega$ its intrinsic frequency, which is related to $k$ by the linear dispersion relation

$$\omega^2 = f^2 + c_0^2k^2. \tag{36}$$

Rayleigh damping is applied over the last two wavelengths at either end of the domain to mimic a radiation boundary condition. Spatially there were 21 grid points for one wavelength, using centered differences for the evaluation of derivatives. Temporally there were 100 leap-frog time steps per wave period. The integration started from rest and went up to 10 periods of the gravity wave.

Two cases were considered. In the first case, the wavelength $\lambda$ of the gravity wave was chosen such that the ratio between $f$ and the intrinsic frequency $\omega$ of the gravity wave was 0.1; hence the wave had comparatively high frequency and background rotation could be expected to be negligible. In the second case, $f/\omega = 0.7$; hence the wave had comparatively low frequency and should have been strongly influenced by rotation. The two cases were integrated for both the standard and the modified system using four different wave amplitudes. The results are collected in Fig. 1.

[5] One approach notes the formal equivalence of the equations (27) and (2) and the equations for the two-dimensional evolution of a notional ideal compressible fluid whose density per unit area is equal to $h$, whose enthalpy per unit mass is equal to $c_0^2F(h)$, and whose entropy is constant everywhere (cf., e.g., section 2 in Landau and Lifshitz 1987). As is well known, the SSW system is then analogous to the flow of a perfect gas with ratio of specific heats equal to 2, whereas the MSW system in the same analogy has ratio of specific heats equal to $-1$. The expressions for $e(h)$ and $p(h)$ in Table 1 then follow directly from the standard perfect gas formulas.

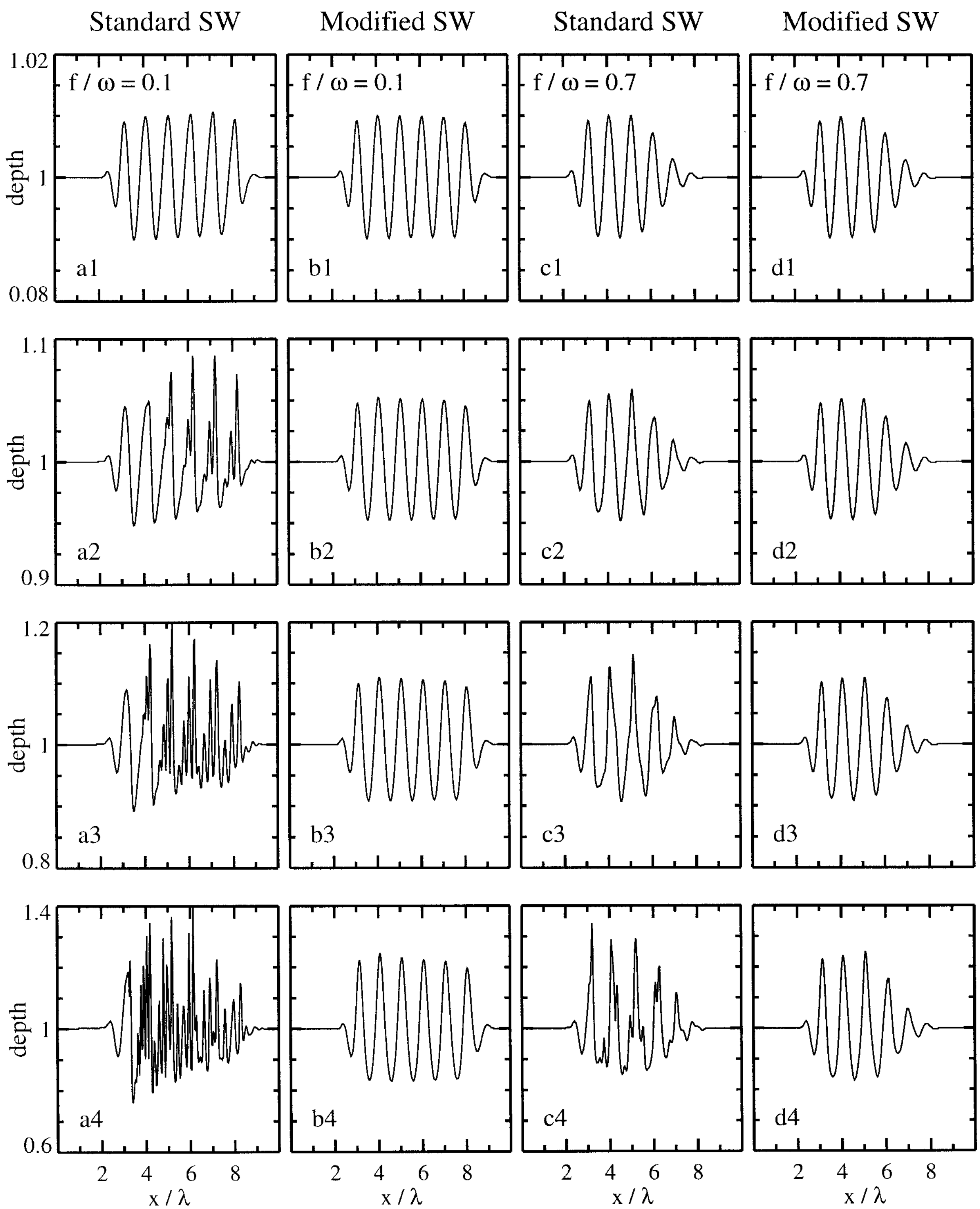


FIG. 1. Propagation of simple right-going gravity waves with different wavelengths $\lambda$ in standard and in modified shallow water. In all cases the layer depth $h$ is plotted at the end of the integration. The wave amplitude increases from top to bottom, and is given by $a$ = (1/100, 1/20, 1/10, 1/5) in the respective rows. Note the changing vertical scale from row to row, which accommodates the different wave amplitudes. The two columns on the left contain high-frequency gravity waves, and the two columns on the right contain low-frequency gravity waves. See text for further details.

The plots show the depth field $h$ at the end of the integration. Each row corresponds to a particular wave amplitude, increasing from top to bottom. The nondimensional wave-forcing amplitudes from top to bottom [which correspond to $\max(|h - 1|)$ in a linear wave] were

$$a = (1/100,\ 1/20,\ 1/10,\ 1/5). \tag{37}$$

Note that the scaling of the vertical axis changes from row to row to compensate for the different amplitudes of the waves. For reference, it is useful to note here the number of wavelengths of propagation before an initially sinusoidal SSW gravity wave forms a shock, which is

$$\text{number of wavelengths before shock forms} = \frac{2}{6\pi a} \approx \frac{1}{9a}. \tag{38}$$

Cases a1–a4 in the first column show the results for the high-frequency gravity wave in the SSW system. At very small amplitude $a = 0.01$ (a1) the wave simply travels down the domain, although even then there is some discernible evidence for nonlinear steepening toward the right of the domain. The criterion (38) allows 10 wavelengths of propagation before shocks form for this wave amplitude, which is not reached in this case. At higher amplitudes (a2–a4), however, shocks do form and render the integration unphysical, as the used numerical model has no mechanism to resolve the small spatial scales associated with the shock. The onset of shock formation is in very good agreement with the theoretical value in (38).

Cases b1–b4 in the second column show the same waves, but this time in the MSW system. As expected, the small-amplitude case (b1) looks very similar to the SSW integration. However, this similarity vanishes at larger wave amplitudes. As the higher-amplitude plots (b2–b4) reveal, there is hardly a change in the appearance of the MSW waves; in particular, there is hardly a sign of nonlinear steepening, even in the last case (b4).

The third column shows the low-frequency cases c1–c4, integrated in the SSW system. The dispersion relation (36) implies reduced group velocity for low-frequency gravity waves, which is why after 10 periods the wave train has not yet spread across the domain. There is evidence in c1–c3 that the low-frequency dispersion delays the onset of shock formation somewhat. This can be qualitatively understood on the basis that shock formation requires assembly of many different spatial scales at the same location, whereas dispersion spreads different spatial scales over a wider region. However, eventually shocks do form, as shown in the lowest plot (c4).

The fourth column shows the low-frequency wave integrated in the MSW system. As before, there is hardly a difference between the various amplitudes, which is a useful check on the assumption made in section 3 that nonzero $f$ does not alter the nonlinear steepening in an essential way.

In summary, these numerical simulations agree very well with the theoretical predictions about the behavior of simple waves in both the SSW and the MSW systems, and they also show that background rotation does not change the main conclusions: background rotation reduces but does not eliminate shock formation in SSW, and it does not reintroduce shocks in MSW.

## 6. Concluding remarks

The presented MSW model eliminates gravity wave steepening while introducing only minimal changes in other features of the SSW model. In particular, the fundamental character of the balanced, PV-controlled flow component has been retained. As noted before, the MSW model is apparently the only shallow-water-type model in which fundamental nondissipative wave–mean interaction problems of the kind investigated by Bühler and McIntyre (1998) can be studied numerically.

The MSW model is also potentially useful in flow studies in which the gravity wave component of the flow is not the focus of attention (e.g., certain vortex dynamics studies). In the appropriate parameter regime the MSW and the SSW models would then produce essentially identical results for the dominant balanced flow component, but the MSW model would prohibit the shock formation of the inevitably present, but weak, gravity waves. This may offer some practical advantage at no extra computational cost.

Another possible application of the MSW model is the fundamental study of higher-order balanced models. For instance, in section 4 it was pointed out that the QG approximations of the MSW and the SSW models differ nonlinearly, and therefore an intercomparison between the two models could highlight to what extent the structure and accuracy of higher-order balanced models depends on details such as the exact form of $F(h)$ in (27).

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## APPENDIX

### Independent Check on Form of *F*(*h*)

The main result, that is, that the only modified shallow-water system in the form of (6) that admits simple waves traveling with unchanged shape has $F(h)$ as given in (23), can be corroborated by a simple independent check. Assuming that $h = h(x - c_0 t)$ and that $u = \phi(h)$, the equations (6) become

$$-c_0\phi' h' + \left(\frac{\phi^2}{2}\right)' h' + c_0^2 F' h' = 0 \quad \text{and} \tag{A1}$$

$$-c_0 h' + \phi h' + h\phi' h' = 0. \tag{A2}$$

The factor $h'$ can be cancelled in both, and the remaining equations can then be easily integrated to give

$$c_0^2 F(h) = A - \frac{B^2}{2h^2} \quad \text{and} \tag{A3}$$

$$\phi(h) = c_0 + \frac{B}{h}. \tag{A4}$$

Requiring that $F'(1) = 1$ and $\phi(1) = 1$ then reproduces the equations (23) (up to the arbitrary value of the additive constant $A$) and the relation between $u$ and $h$ in (25).

It can be noted that in this derivation the form of $F(h)$ arises directly as a consequence of Bernoulli's law and the continuity equation when both are applied in a frame of reference moving with the wave speed $c_0$. Therefore, the same derivation also applies to the standard primitive equations when written in isentropic coordinates (e.g. Andrews et al. 1987), in which the Montgomery streamfunction replaces $F$ and the isentropic density replaces $h$. Traveling wave solutions in that system will hence also be characterized by a relationship between the Montgomery streamfunction and the isentropic density, which is of the same functional form as the dependence of $F$ on $h$ in (23).

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