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Stratified Canonical Forms of Matrix Valued Functions in a Neighborhood of a Transition Point

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1 Introduction

1.1. Let X be a complex analytic manifold or an algebraic variety over a perfect field k of dimension d, let x_o be a point in $X(k)$, and let $g\colon X \to \mathbf{gl}(n, k)$ be a germ of a matrix valued analytic (respectively, polynomial) function on X in an analytic (respectively, Zariski) neighborhood of x_o.

Questions regarding the possibility of reduction of such functions to the Jordan form $j(x)$, or to some other normal form in a neighborhood of a point x_o of X, by conjugating it by a matrix function $u\colon X \to \mathbf{GL}(n, k)$ in the same category; and questions about the existence of a conjugating matrix function u with suitable analytic properties, play an important role in various branches of analysis, geometry, and mathematical physics. In particular, such questions arise frequently and unavoidably in nearly all the main branches of the theory of ordinary ([Gn], [W1]–[W3]) and partial ([CH], [GV], [HYP], [N], [P1]–[P3]) differential equations, dynamical systems and their bifurcations ([CLW]), the perturbation theory of linear operators ([B], [K]), and the quantum field theory ([BT]).

A reduction to canonical forms was used especially widely, systematically, and successfully in the general theory of hyperbolic systems of PDE since the pathbreaking works of Petrowsky ([P1]–[P3]), at least. Indeed, many of the principal general results on the existence and uniqueness of the solutions of the Cauchy (initial value) and mixed (initial-boundary value) problems, on the correctness of these problems, on explicit constructions of the fundamental and other solutions or parametrices for these problems, and on properties and asymptotics of their solutions, etc., for hyperbolic systems, have been obtained using a reduction of the principal symbols of these systems to suitable

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local normal forms in the same category. See the monographs and surveys in [CH], [I], [IV], [GV], [HYP], and the vast literature quoted there.

Notice that the results which provide some sufficient conditions for a *local diagonalization* of a matrix function within some category (smooth (= C^∞), analytic, algebraic, etc.) play an important role in many of the works on hyperbolic systems mentioned above, and in many other subjects. Quite general results on the local diagonalization were contained already in the works of Petrowsky ([P2], [P3]). Variants of his results for various categories in the general or in some special cases were obtained and used later by many authors (see, for example, [B], [BT], [K], [W1]–[W3], etc.).

All the known general results in this direction contain assumptions that some discrete invariants of the matrix functions are locally constant (see Section 2 below for details), and that they are false without them. Since the assumptions of this type are too restrictive for many important potential applications, it is very desirable to find more general methods of dealing with the problem of local diagonalization without these assumptions. In this paper, we shall develop a new approach to this problem based on the use of suitable stratifications of the parameter space X. It will allow us to obtain a stratified version of the local diagonalization without the usual constancy assumptions.

1.2. More precisely, we shall be mainly concerned here with the study of the diagonalization of a matrix valued function $g(x)$ in a sufficiently small neighborhood (analytic or Zariski) U of a point x_o in X. The obvious necessary conditions for the diagonalization of $g(x)$ on U in the analytic or Zariski categories, respectively, are the following:

(i) All the eigenvalues $e_i(x)$ of $g(x)$ are morphisms in the corresponding category; i.e., they are holomorphic functions on U in the analytic case, and they are polynomial functions on U in the algebraic case.

(ii) For any point x of U, there exists a matrix u_x such that

$$u_x g(x) u_x^{-1} = d(x), \tag{1.2.1}$$

where $d(x) = \mathrm{diag}(e_1(x), \dots, e_n(x))$ is the diagonal matrix with the eigenvalues $e_i(x)$ of $g(x)$ on its principal diagonal. (Here, the correspondence $x \to u_x$ is not assumed to be in one of the above categories; it is only a set-theoretic map.)

These necessary conditions are, in general, very far from being sufficient for the local analytic or algebraic diagonalization of $g(x)$. (See [BT, Sections 3, 5] for counterexamples.) However, it is known in the analytic category, at least, that such a local diagonalizing matrix-function $u(x)$ does exist, if the multiplicities of all the eigenvalues $e_i(x)$ of $g(x)$ remain constant in a neighborhood U of x_o (see [P2], [P3], and Sections 2.4–2.6, below). We shall consider below a much more complicated case where these multiplicities do change in a neighborhood of x_o. Following the vast classical literature on ordinary differential equations (see [W1]–[W3] and the literature quoted there), we shall call such a point a *transition* or *turning point*. Transition points arise frequently in various applications. Therefore, the problem of normal forms for matrix functions in their neighborhoods are of significant interest.

In this paper, we shall construct a canonical stratification $(S_r \mid r \in R)$ of a neighborhood U of a transition point x_o by submanifolds S_r which are algebraic in the algebraic case, and are analytic in the analytic case, and are such that the multiplicities of all the eigenvalues of $g(x)$ remain constant on each strata S_r. Here, R is a finite set. We shall show that for a sufficiently small U, and under some additional assumptions on this stratification and on the matrix function $g(x)$, the diagonalization can be achieved in the following stratified sense.

1.2.2. For each $r \in R$, there exists a morphism $u_r\colon\ S_r \to \mathbf{GL}(n)$ in the corresponding category such that

$$u_r(x)g(x)u_r(x)^{-1} = d(x)$$

for each x in S_r.

1.3 General notation and terminology

Throughout the whole of this paper, unless it is explicitly stated otherwise, we shall keep the following notation:

1.3.1. In the *analytic* category: X will be a complex analytic variety over the field $\mathbf{C}$ of complex numbers or a germ of such a variety. For a unification of notation with the algebraic case, we shall sometimes denote the underlying topological space X as $X(\mathbf{C})$.

1.3.2. In the *algebraic* category: X will be a scheme over a perfect field k, $\overline{k}$ an algebraic closure of k.

1.3.3. In both categories: Let $\mathcal{O}_X$ be the structure sheaf of X in the corresponding category. For a point x in X, let $\mathcal{O}_x$ be its local ring on X in the corresponding category, i.e., the stalk of the sheaf $\mathcal{O}_X$ at the point x. For a ring R, let $\mathbf{gl}(n, R)$ (respectively, $G = \mathbf{GL}(n, R)$) be the algebra (respectively, the group) of all (respectively, all invertible) $n \times n$ matrices with coefficients from R.

For a matrix valued function $g\colon\ X \to \mathbf{gl}(n)$, let $C(x)$ (respectively, $Z(x)$) be its centralizer in $\mathbf{gl}(n)$ (respectively, in $\mathbf{GL}(n)$), let $e_1(x), \dots, e_n(x)$ be the set of all eigenvalues of

$g(x)$, let $V_i(x) \subset V = (\bar{k})^n$ be the eigenspace consisting of eigenvectors of $g(x)$ with the eigenvalue $e_i(x)$, and let $m_i(x)$ be the geometric multiplicity of the eigenvalue $e_i(x)$ of $g(x)$, i.e., the dimension of the eigenspace $V_i(x)$. If $g(x)$ is a diagonalizable matrix, $m_i(x)$ coincides with the algebraic multiplicity of the eigenvalue $e_i(x)$, i.e., its multiplicity as a root of the characteristic polynomial of $g(x)$. Denote by $d(x) = \mathrm{diag}(e_1(x), \dots, e_n(x))$ the diagonal matrix with the eigenvalues $e_i(x)$ of $g(x)$ on its principal diagonal in a certain fixed order (each $e_i(x)$ is repeated as many times on the diagonal as its multiplicity).

1.4 Local data

1.4.1. We shall refer to the following situation as the *local case* in the *analytic* category: X is a germ of an analytic variety at a point x_o or its sufficiently small representative. If this germ is nonsingular, we can take as a representative a sufficiently small polycylinder in $\mathbf{C}^n$ with its center at a point x_o.

1.4.2. We shall refer to the following situation as the *local case* in the *algebraic* category: $X = \mathrm{Spec}\, A$ is the spectrum of a localization A of a k-algebra B of finite type over a perfect field k with respect to its prime ideal b and x_o the closed point of X.

1.4.3. In both local categories: Let D be a divisor on X from the same category, $Y = X - D$. Sometimes we shall refer to such a Y as a *deleted neighborhood* of x_o.

1.5. Let $D = \cup_i D_i$ be a decomposition of D into a union of its irreducible components D_i. We say that D is a *divisor with normal crossings* if all its components D_i are nonsingular, and they intersect transversally.

Assume now that X is a nonsingular analytic germ or a smooth local scheme over k and let m_o be the maximal ideal of the local ring $\mathcal{O}_{X,x_o}$ of the point x_o on X. For a nonsingular X, each irreducible component D_i of D can be defined by a single equation f_i in m_o. Then in pure algebraic terms, the condition of transversality of the intersection of the collection of all the irreducible components D_i of D is equivalent to the condition of linear independence of the set of the images $\overline{f_i}$ of all f_i in the vector space m_o/m_o^2 over the residue field $k_o = \mathcal{O}_{X,x_o}/m_o$ of x_o ([SGA1, XIII, 5.1]).

1.6. In the algebraic category, we shall say that a divisor D is *defined over a field* k *or* D *is a* k-*divisor*, if all its irreducible components D_i are defined over k.

2 Diagonalization in a neighborhood of a point: The case of constant multiplicities of eigenvalues

2.1. Everywhere in this section, X will be a reduced, irreducible analytic space over **C** or a reduced, irreducible algebraic variety over a perfect field k.

2.2. Let $g\colon X \to \mathbf{gl}(n)$ be a morphism in the analytic or in the algebraic categories, respectively. We shall say that a matrix valued function $g(x)$ is *pointwise semisimple* if for any x in X, the matrix $g(x)$ is semisimple; i.e., it is diagonalizable in the sense of (1.2.1) with the diagonalizing matrix u_x defined over **C** in the analytic case, and over the algebraic closure $\overline{k(x)}$ of the residue field $k(x)$ of the point x on X in the algebraic case.

In the algebraic case, if all the eigenvalues $e_i(x)$ of $g(x)$ are defined over the field $k(x)$ for all x in X, then the diagonalizing matrices u_x in (1.2.1) can be chosen, in fact, over the field $k(x)$ itself, for all x in X. In particular it is so, if e_i is a k-polynomial function on X.

Consider the following assumptions on g:

2.2.1. The dimension $d(x)$ of the centralizer $Z(x)$ of $g(x)$ is constant in X.

2.2.2. The eigenvalues $e_1(x), \dots, e_n(x)$ are morphisms in the corresponding category; i.e., they are holomorphic functions on X in the analytic category, and they are k-polynomial functions on X in the algebraic category.

For a pointwise semisimple matrix function g, condition (2.2.1) is equivalent to the following condition:

2.2.3. The multiplicities of all the eigenvalues $e_i(x)$ are constant in X.

2.3. For any index i in $[1, n]$, consider the subset V_i of trivial vector bundle $X \times V$:

$$V_i = ((x, V_i(x)) \subset X \times V \mid \text{for all } x \text{ in } X).$$

Denote $g_i = g - e_i\mathrm{Id}\colon \mathcal{O}_X^n \to \mathcal{O}_X^n$, where Id is the identical endomorphism of $\mathcal{O}_X^n$. Then by the definition, $V_i = \mathrm{Ker}(g_i)$. This implies that V_i is a coherent $\mathcal{O}_X$-submodule of $\mathcal{O}_X^n$. We shall call it the *sheaf of eigenvectors* or simply the *eigensheaf corresponding to the eigenvalue* e_i of g. In the case when this sheaf is a vector bundle, we shall call it the *eigenbundle corresponding* to e_i.

2.4. Proposition. Let X be as in Section 2.1. Assume that the geometric multiplicity $m_i(x) = \dim V_i(x)$ is a constant function of x in X. Then $V_i(x)$ is a subbundle of the trivial bundle $I^n = X \times V$ over X, locally trivial in the analytic topology on X in the analytic case, and in the Zariski topology on X in the algebraic case. □

Proof. Consider first the algebraic case; i.e., assume that X is a reduced irreducible algebraic variety over k. Since the question is local, we can assume that $X = \text{Spec } A$ is affine. Therefore, V_i corresponds to a finitely generated A-module N_i ([FAC]).

It is enough to show that N_i is a free A-module in the case when A is an integral local noetherian ring. Let m be the maximal ideal of A, $k' = A/m$ and let K be the fraction field of A. As an A-submodule of torsion free A-module A^n, N_i is torsion free over the integral local ring A, and by assumption, (2.2.3) $\dim_K(N_i \otimes_A K) = \dim_{k'}(N_i \otimes_A k') = m_i$. Let $x'_1, \dots, x'_{m_i}$ be a basis of $N_i \otimes_A k'$ over k', and let x_j be a lift of x'_j to N_i for all $j, 1 \leq j \leq m_i$. Then the elements x_j generate N_i over A by the Nakayama lemma ([AM]), and they generate $N_i \otimes_A K$ over K. The dimension equality above shows that they must be linear independent over K. Since N_i has no torsion, the $x_j, 1 \leq j \leq m_i$, are linear independent over A as well. Hence, N_i is a free A-module.

The proof in the analytic case is absolutely parallel. Indeed, it is enough to show that for each point x of X, the stalk $(V_i)_x$ of the coherent sheaf V_i at x is a free $\mathcal{O}_x$-module ([GPR, Ch. 1, 7.13]; or [GR, Ch. 4, Sec. 4]). Therefore, the main arguments of the proof in the algebraic case given above can be repeated in the analytic case without any changes. ■

2.5. Corollary. Assume that $g(x)$ is pointwise semisimple and satisfies conditions (2.2.2) and (2.2.3) above. Then it is diagonalizable by an analytic (respectively, algebraic) matrix function $u\colon X \to \mathbf{GL}(n)$ if and only if all the locally trivial subbundles V_i of $\mathcal{O}_x^n$ are trivial. In particular, it is so if X is local in the analytic or algebraic category. □

2.6. Remarks. A variant of Corollary 2.5 for the local case in the smooth (C^∞) category was contained in the works of Petrowsky and played a basic role there; see the theorem of Section 1, Chapter III of the new, corrected editions of [P2] and [P3]. (A global version of this result stated in the original (1938) edition of [P2] was wrong for topological reasons, as it was independently pointed out by J. Leray, L. Pontryagin, and J. Schauder. Fortunately, the local result was sufficient for the derivation of the main results of [P2].) See also [GV, Ch. 1, App.] and [IV] for more modern and precise formulations of the local result. Petrowsky's proof of it given in [P2], [P3] can be easily accommodated to the local analytic category as well. Another proof in the local analytic category can be found in [B].

The algebraic case of Corollary 2.5 is, perhaps, known as well, but we cannot give a suitable reference. Proposition 2.4 can be derived from Corollary 2.5 and its variants, at least, for a pointwise semisimple matrix function g. However, we preferred to give here an independent algebra-geometric proof, which is valid in a more general situation and sheds more light on the underlying geometric structures.

3 Purity theorems for vector bundles

3.1. We shall assume in this section that X is a local object in the analytic or algebraic category, as in Section 1.4 and $x_o \in X(k)$.

Let D be a divisor in X in the corresponding category, $Y = X - D$, and $c(D)$ is the number of irreducible components of D. In the algebraic category, we shall always assume that D is a k-divisor, in the sense of Section 1.6. For a vector bundle E on Y, denote by $\mathrm{rk}\, E$ its rank as an $\mathcal{O}_Y$-module. Denote $d = \dim X$ and let $[d/2]$ be the greatest integer not greater than $d/2$.

Under *purity theorems for vector bundles*, we shall mean statements which give some sufficient conditions for a vector bundle E on Y to be trivial. In this paper, we shall use the following purity results.

3.2. Theorem (Gabber [Gb2]). Let X be a nonsingular analytic germ as in Section 1.4.1, let $D = \cup_i D_i$ be a divisor with normal crossings, and let E be an analytic vector bundle on $Y = X - D$. Assume that E can be extended to X as a coherent analytic $\mathcal{O}_X$-module. Then E is free. □

3.3. Theorem. Let $X = \mathrm{Spec}\, A$ be in the local algebraic category as in Section 1.4.2, let D be a k-divisor, and let E be a vector bundle on Y. Assume that X is smooth and that one of the following conditions is satisfied:

(i) (Rao [Ra]) k is an infinite (and perfect as before) field, D is a k-divisor with normal crossings, and $\mathrm{rk}\, E \geq \min(c(D),\ [d/2])$; or

(ii) (Gabber [Gb1]) $\dim X \leq 3$.

Then E is free. □

3.4. Remarks. (1) In the case when $c(D) = 1$, i.e., when D is smooth and irreducible, the statement of Theorem 3.3(i) has been conjectured by Quillen ([Q]) in order to generalize the celebrated conjecture of Serre ([FAC]) on the triviality of all vector bundles on the affine space $\mathbf{A}^n_k$ over a field k to the affine space $\mathbf{A}^n_R$ over an arbitrary regular ring R. For a smooth irreducible divisor D, it has been derived in [Ra] from the remarkable results of Quillen ([Q]) and Suslin ([Su]) which proved the original Serre conjecture. Conversely, this statement for an arbitrary irreducible smooth divisor D implies the Serre conjecture ([Q]).

(2) On the other hand, statement 3.3(ii) follows from the main results of the thesis of Gabber ([Gb1]). Their proofs do not depend on any results of the Serre conjecture, and it is unlikely that statement 3.3(ii) can be derived from them.

(3) Rao conjectured ([Ra]) that the statement of 3.3(i) remains valid without any assumptions on the $\mathrm{rk}\, E$. The following conjecture generalizes this conjecture of Rao and Theorem 3.3.

3.5. Conjecture. Let k be an arbitrary field, let X be a local k-scheme as in Section 1.4.2, let D be a k-divisor, let $Y = X - D$, and let E be a vector bundle on Y. Assume that X is smooth over k and that all the irreducible components D_i of the divisor D are smooth, and that their pairwise intersections are transversal. Let E be a locally free sheaf on Y. Then E is free. □

4 Diagonalization of matrix valued functions on deleted neighborhoods

In this section, we shall preserve the notation and assumptions of Section 3.1. The following two theorems provide a model for the "diagonalization on a stratum," and therefore constitute the basic ingredients of the main results of this paper.

4.1. Theorem. Let X be a small polycylinder in $\mathbf{C}^n$, let D be an analytic divisor of X, and let $g\colon X \to \mathbf{gl}(n, \mathbf{C})$ be a holomorphic pointwise semisimple matrix valued function. Assume also that D is a divisor with normal crossings and that the restriction of g onto $Y = X - D$ satisfies conditions (2.2.2) and (2.2.3). Then g is analytically diagonalizable in Y; i.e., there exists an analytic map $u\colon Y \to \mathbf{GL}(n, \mathbf{C})$ such that for all y in Y, we have an equality

$$u(y)g(y)u(y)^{-1} = d(y).$$ □

Proof. As was mentioned in Section 2.3, the sheaf V_i formed by eigenvectors of $g(x)$ in $\mathcal{O}_X^n$ corresponding to the eigenvalue e_i is coherent over X. By construction, its restriction $V_i \mid Y$ on Y coincides with the sheaf of eigenvectors corresponding to this eigenvalue in $\mathcal{O}_Y^n$. Conditions (2.2.2) and (2.2.3) and Proposition 2.4 imply that $V_i \mid Y$ is, in fact, a vector bundle. Therefore, $V_i \mid Y$ is trivial by Theorem 3.2. The conclusion of Theorem 4.1 follows now from Corollary 2.5. ■

The following theorem is an analogue of Theorem 4.1 in the algebraic category.

4.2. Theorem. Let X be a smooth local algebraic scheme over a perfect field k, as in Section 1.4.2, let D be a k-divisor of X, let $Y = X - D$, and let $g\colon Y \to \mathbf{gl}(n)$ be a polynomial pointwise semisimple matrix valued function. Assume that the eigenvalues e_i of g satisfy conditions (2.2.2) and (2.2.3). Assume also that one of the following conditions on the quadruple $(k,\ X,\ D,\ g)$ are satisfied:

(i) $\dim X \leq 3$, or

(ii) k is an infinite (and perfect) field, D is a k-divisor with normal crossings, and $m_i \geq \min(c(d),\ [d/2])$ for all i.

Then g is diagonalizable on Y by an (algebraic) k-morphism $u\colon Y \to \mathbf{GL}(n)$. □

The proof of this theorem follows directly from Theorem 3.3 and Corollary 2.5.

4.3. Remarks. In contrast to the algebraic case (Theorem 4.2), we need in the analytic case (Theorem 4.1) the assumption that the map g is defined on the whole of X, not just on Y. This imposes some implicit assumptions on the growth of g near D. For applications, it will be interesting to find some other conditions which would imply the result of Theorem 4.1 with some assumptions of different types on g.

5 Construction of a natural stratification of X associated with a matrix function

5.1. In this section, we shall assume that X is a germ of an analytic variety or a local k-scheme, as in Section 1.4. Assume in addition that X is nonsingular and irreducible. Let $X \to \mathbf{gl}(n)$ be a semisimple matrix valued function in the corresponding category which satisfies condition (2.2.2), but does not satisfy (2.2.3); i.e., x_o is a transition point of g.

For any pair (e_i, e_j) of distinct (as functions on X) eigenvalues of g, consider the subvariety $F_{(i,j)}$ of X defined by the equation $e_i = e_j$. It may be empty, of course. In the algebraic case, by assumption (2.2.2), all the eigenvalues e_i are defined over k and, therefore, all the $F_{(i,j)}$ are defined over k as well. Consider the following conditions on the system of the subvarieties $F_{(i,j)}$.

(5.1.1). All the nonempty subvarieties $F_{(i,j)}$ are nonsingular, and their pairwise intersections are transversal.

(5.1.2). All the nonempty subvarieties $F_{(i,j)}$ are nonsingular, and the collection of all of them forms a k-divisor with normal crossings.

Notice that the nonsingularity of the varieties $F_{(i,j)}$ implies their irreducibility. Indeed, each irreducible component of $F_{(i,j)}$ contains the point x_o. This point can be nonsingular on $F_{(i.j)}$ only if this variety has no more than one irreducible component; i.e., it is irreducible. This follows from the well-known fact of local algebra that the local regular ring $\mathcal{O}_{F_{(i,j)},x_o}$ of x_o on $F_{(i,j)}$ is analytically unibranched (see for, example, [ZS, Ch. VIII, Th. 32]).

Notice also that the conditions of (5.1.2) clearly imply those of (5.1.1).

Consider now pairwise intersections $F_{(i_1,j_1;i_2,j_2)}$ of the subvarieties $F_{(i_1,j_1)}$ and $F_{(i_2,j_2)}$. Notice that if (5.1.1) is satisfied, then all the varieties $F_{(i_1,j_1;i_2,j_2)}$ are nonsingular and irreducible. Furthermore, if condition (5.1.1) (respectively, condition (5.1.2)) is satisfied for all $F_{(i,j)}$, then the system of subvarieties

$$(F_{(i_o,j_o;i_k,j_k)} \mid \text{ for all } (i_k, j_k))$$

of $F_{(i_o,j_o)}$ satisfies condition (5.1.1) (respectively, condition (5.1.2)) as well.

5.2. Assume now that the system of all the subvarieties $F_{(i,j)}$ satisfies condition (5.1.1). Consider the sequence of locally closed subvarieties of X constructed as follows:

(i) $X^1 = X - \cup_{(i,j)} F_{(i,j)}$.

(ii) Notice that our constructions and condition (5.1.1) imply that $F_{(i_o,j_o)}$ is again a polycylinder in the analytic case (respectively, it is isomorphic to Spec A_{i_o,j_o} for some smooth local ring A_{i_o,j_o} in the algebraic case). In both cases, this variety is nonsingular and irreducible. So, we can continue the same process with X replaced by $F_{(i,j)}$ for all pairs of indices (i, j). Namely, for a fixed pair of indices (i_o, j_o), denote

$$X^1_{(i_o,j_o)} = F_{(i_o,j_o)} - \cup_{(i,j)\neq(i_o,j_o)} F_{(i_o,j_o;i,j)}.$$

Similarly, the subvarieties $X^1_{(i,j)}$ can be constructed for all other pairs of indices (i, j) with nonempty $F_{(i,j)}$.

(iii) We can continue this process further for each $F_{(i,j;k,l)}$, and construct the variety $X^1_{(i,j;k,l)}$ of the same type, etc.

(iv) As a result, we shall obtain a collection of locally closed subvarieties $X^1, X^1_{(i,j)}$, for all pairs $(i, j), X^1_{(i,j;l,k)}$, for all quadruples of indices $(i, j; l, k), \dots$, and so on. For a simplification of our notation, we rename these varieties as $S_r, r \in R$, where R is a finite set of indices. We shall call the stratification $\mathcal{S} = (S_r \mid r \in R)$ of X obtained in this way the *stratification associated with the matrix function* g.

We shall summarize the main properties of this stratification in the following lemma.

5.3. Lemma. Let X be as in Section 5.1. Then the stratification $\mathcal{S} = (S_r \mid r \in R)$ of X constructed above has the following properties:

(i) Each of the strata S_r has a form $Z_r - T_r$, where the pair (Z_r, T_r) consists of a local nonsingular and irreducible object Z_r in the corresponding category (as in Section 1.4) and its divisor T_r.

(ii) The collection of all the irreducible components of the divisor T_r satisfies the conditions of (5.1.1) (respectively, condition (5.1.2)), for all r, if the initial terms $F_{(i,j)}$ defining the stratification satisfy them. In the algebraic case, each of the irreducible components of each divisor T_r is defined over the field k.

(iii) Let c be the number of nonempty and distinct subvarieties $F_{(i,j)}$, let $c(T_r)$ be the number of the irreducible components of T_r, and let $T_o = \cup_{(i,j)} F_{(i,j)}$. Assume that condition (5.1.1) is satisfied. Then $c = c(T_o) \geq c(T_r)$, for all $r \in R$.

(iv) On each stratum S_r, the multiplicities $m_i(x)$ of all the eigenvalues $e_i(x)$ of $g(x)$, and the dimensions of the centralisers $C(g(x))$ and $Z(g(x))$ of $g(x)$ in $\mathbf{gl}(n)$ and $\mathbf{GL}(n)$, respectively, are constant. Denote by $m_i(r)$ the constant value of $m_i(x)$ on S_r.

(v) If a stratum S_t is in the closure of S_r, then $m_i(t) \geq m_i(r)$. In particular, for the open strata $S_o = X^1$ which was defined by equality 5.2(i) above, $m_i(o) \leq m_i(r)$, for any $r \in R$. □

5.4. Let D be the k-subalgebra of diagonal matrices in $\mathbf{gl}(n, k)$, and let $H_{(i,j)}$ be the hyperplane in D defined by the linear equation $e_i = e_j$. These hyperplanes intersect pairwise transversally, and the collection of them and their intersections define a stratification $(W_s \mid s \in S)$ on D similarly to that described above for X.

Consider the map $d\colon X \to D, x \to d(x) = \mathrm{diag}(e_1(x), \dots e_n(x))$ defined by the set of all the eigenvalues $e_i, 1 \leq i \leq n$, of g taken in some fixed order. Then the varieties $F_{(i,j)}$ considered above are the inverse images of the hyperplanes $H_{(i,j)}$ in D:

$$F_{(i,j)} = d^{-1}(H_{(i,j)}) = X \times {}_2 H_{(i,j)} \tag{5.4.1}$$

(as analytic varieties or schemes, respectively).

The following lemma is useful for a verification of conditions (5.1.1) and (5.1.2). However, it will not be used in this paper.

5.5. Lemma. Assume that the map $d\colon X \to D$ defined in Section 5.4 above is transversal to each of the hyperplanes $H_{(i,j)}$ (in the sense of ([EGA, IV, 17.13.3]). Then all the varieties $F_{(i,j)}$ are nonsingular and irreducible. □

Proof. The nonsingularity of the varieties $F_{(i,j)}$ follows from equalities (5.4.1), the nonsingularity of the hyperplanes $H_{(i,j)}$, and the transversality of the map d to these hyperplanes ([EGA, IV, 17.13.2]). (Notice that the proof of [EGA] works in the analytic category as well.) A proof that the nonsingularity of $F_{(i,j)}$ implies its irreducibility has already been given in Section 5.1. ■

6 Stratified conjugacy in neighborhoods of transition points

In this section, we shall preserve the assumptions and the notation of Section 5. The main goal of this paper is to prove the following two theorems.

6.1. Theorem. In the *local analytic* situation of Section 1.4.1, assume that the matrix function g is pointwise semisimple and satisfies condition (2.2.2), and that the collection of the subvarieties $F_{(i,j)}$, for all pairs (i, j), satisfies condition (5.1.2). Then on each stratum S_r of the stratification described above, there exists an analytic morphism $u_r\colon S_r \to \mathbf{GL}(n)$ which diagonalizes g on S_r:

$$u_r(x)g(x)u_r(x)^{-1} = d(x), \quad \text{for all } x \text{ in } S_r. \tag{6.1.1}$$

□

Proof. For indices i such that the multiplicity m_i is a constant function on X, the eigenbundles V_i are trivial on the whole germ X by Proposition 2.4. For all the rest of the indices i, $m_i(x)$ is a constant function on each stratum S_r by Lemma 5.3(iv), and the restriction $V_i \mid S_r$ of the coherent eigensheaf V_i on S_r is, in fact, a vector bundle by Proposition 2.4, for all r. Under our assumptions, the triviality of these vector bundles follows from Theorem 3.2. Therefore, the matrix function g is diagonalizable on each stratum S_r by Corollary 2.5. ■

6.2. Theorem. In the *local algebraic* situation of Section 1.4.2, assume that the matrix function g is pointwise semisimple and satisfies condition (2.2.2), and that one of the following conditions is satisfied:

(i) $\dim X \leq 3$, and the collection of the subvarieties $F_{(i,j)}$, for all pairs (i, j), satisfies condition (5.1.1); or

(ii) k is an infinite (and perfect) field, the collection of the subvarieties $F_{(i,j)}$, for all pairs (i, j), satisfies condition (5.1.2); and for each index i such that the multiplicity function $m_i(x)$ is not constant on X, the inequality

$$m_i(o) \geq \min(c,\ [d/2]) \tag{6.2.1}$$

holds, where the integers $m_i(o)$, c, and $[d/2]$ were defined in Section 5.3(iv)–(v), 5.3(iii), and 3.1, respectively.

Then for each stratum S_r, there exists an algebraic k-morphism $u_r\colon S_r \to \mathbf{GL}(n)$ which diagonalizes $g(x)$ on S_r; i.e., equality (6.1.1) is valid for this u_r and any $x \in S_r$. □

Proof. Inequalities (iii) and (v) of Lemma 5.3, and inequality (6.2.1) above, imply the inequality

$$m_i(r) \geq \min(c,\ [d/2]), \quad \text{for all } r \in R. \tag{6.2.2}$$

The rest of the proof of Theorem 6.2 is parallel to that of Theorem 6.1, with a replacement of the analytic category onto the algebraic category, and the reference to the analytic purity theorem (Theorem 3.2) by that to the algebraic purity theorem (Theorem 3.3). ■

6.3. Remarks. (1) The condition of (5.1.2) that the divisor F formed by all the nonempty subvarieties $F_{(i,j)}$ is a divisor with *normal crossings*, excludes, in fact, the cases where, among the subvarieties $F_{(i,j)}$, there occur simultaneously the nonempty subvarieties $F_{(i,j)}$ and $F_{(k,l)}$ with two of their four indices (i, j, k, l) coinciding. This condition is quite restrictive for many potential applications. However, a relaxation of this assumption requires some progress toward our purity conjecture (Conjecture 3.5) which is, probably, very difficult.

(2) If the conjecture of Rao quoted in Remark 3.4(3) is true, then the conditions on the rank of E in Theorems 4.2(ii) and 6.2(ii) can be dropped.

(3) Most of the results of this paper, including those of Sections 2 and 6, can be extended in many directions: to germs of real analytic matrix functions $g\colon X \to \mathbf{gl}(n, \mathbf{R})$; and in the algebraic category, to matrix functions $g\colon X \to G$ with values in an arbitrary semisimple algebraic group G over a perfect field k (under some additional assumptions on X and G), etc. We are planning to return to these questions in our subsequent papers.

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