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Théorie des groupes/*Group Theory*

Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional local regular rings

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Abstract – Let X be a regular scheme, G a reductive group scheme over X. Serre and Grothendieck conjectured that any rationally trivial G-torsor is locally trivial in the Zariski topology of X. We prove this conjecture when dim X = 2 and G is quasi-split over X.

Espaces homogènes principaux rationnellement triviaux, pureté et arithmétique des schémas en groupes réductifs sur les extensions d'anneaux locaux réguliers de dimension 2

Résumé – *Soit* X *un schéma régulier, et soit* G *un schéma en groupes réductif sur* X. *Serre et Grothendieck ont conjecturé que tout* G-*torseur rationnellement trivial est localement trivial pour la topologie de Zariski de* X. *Nous démontrons cette conjecture dans le cas où* X *est de dimension* 2 *et* G *quasi déployé.*

Version française abrégée – Cette Note fait suite à [9], dont on garde la terminologie et les notations. Soient X un schéma noethérien intègre et régulier, K = R (X) le corps des fonctions rationnelles sur X et G un schéma en groupes réductif sur X. On a la suite de cohomologie de G :

$$(1.1)\qquad 1 \to H^1(X_{Zar}, G) \to H^1(X_{et}, G) \overset{i(G)}{\to} H^1(K, G).$$

Conjecture 1.2 (Serre [12], Grothendieck [12], [5]). – *La suite* (1.1) *est exacte.*

Lorsque X est de dimension 1, ou local hensélien, et G est un X-groupe semi-simple arbitraire, cette conjecture a été prouvée dans [9]. Dans cette Note on montre que, lorsque dim X = 2 et G est un X-groupe réductif isotrope, la conjecture 1.2 se déduit des conjectures de pureté 1.3 et 1.4 énoncées plus bas. On prouve aussi les conjectures 1.3 et 1.2 lorsque G est quasi déployé et X de dimension 2.

Par la suite on note R un anneau local régulier noethérien, de dimension $d \geqq 1$, $\mathfrak{m}$ son idéal maximal et K son corps de fractions. On choisit un élément $u \in \mathfrak{m} - \mathfrak{m}^2$. On écrit $Q = R[u^{-1}]$, X = Spec R, Y = Spec Q. Soient G un R-groupe réductif et G_{ss} le R-groupe dérivé de G (la partie semi-simple de G). Lorsque R-rang $(G_{ss}) \geqq 1$, soient $\mathscr{P}(R)$ l'ensemble des sous-groupes paraboliques minimaux de G sur R, $R_u(P)$ le radical unipotent de $P \in \mathscr{P}(R)$ et $G^u(Q)$ le sous-groupe de G(Q) engendré par les sous-groupes $R_u(P)(Q)$, pour $P \in \mathscr{P}(R)$.

Conjecture 1.3 (Pureté). – $H^1(Q_{Zar}, G) = 0$.

Conjecture 1.4 (K_1-pureté). – *Soit* R-rang $(G_{ss}) \geqq 1$ *et soit* P *un sous-groupe parabolique minimal de* G *sur* R. *Alors* $G(Q) = G^u(Q)P(Q)$.

La conjecture 1.3 généralise une conjecture (2) formulée par Quillen pour $G = GL_n$ dans [11] et prouvée par Gabber pour $G = GL_n$ et PGL_n en dimension $\leqq 3$, *cf.* [13].

Note présentée par Jean-Pierre Serre.

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1. INTRODUCTION. — This Note is a continuation of [9] and we shall keep here the terminology and the notations of [9].

Let X be an integral regular noetherian scheme, $K=R(X)$ the field of rational functions on X, and G a reductive group scheme over X. Consider the following sequence of the cohomology of G :

$$(1.1) \qquad 1 \to H^1(X_{Zar}, G) \to H^1(X_{et}, G) \to H^1(K, G).$$

CONJECTURE 1.2 (Serre [12], Grothendieck [12], [5]). — *Sequence* (1.1) *is exact.*

In the cases when X is one-dimensional or a local henselian, and G is an arbitrary semisimple X-group, the Conjecture has been proved in [9]. In this Note we show that when $\dim X=2$ and G is a reductive isotropic X-group Conjecture 1.2 follows from purity Conjectures 1.3 and 1.4 formulated below.

Everywhere below R will be a local regular noetherian ring of dimension $d \geqq 1$, $\mathfrak{m}$ the maximal ideal of R and K the quotient field of R. Choose an element $u \in \mathfrak{m}-\mathfrak{m}^2$ and denote $Q=R[u^{-1}]$, $X=\operatorname{Spec} R$, G_{ss} the derived R-group of G (the semisimple part of G). If $R\text{-rank}(G_{ss}) \geqq 1$, let $\mathscr{P}(R)$ be the set of all minimal parabolic R-subgroups of G over R, $R_u(P)$ the unipotent radical of $P \in \mathscr{P}(R)$, and $G^u(Q)$ the subgroup of $G(Q)$ generated by all subgroups $R_u(P)(Q)$, for $P \in \mathscr{P}(R)$.

CONJECTURE 1.3 (Purity). — $H^1(Q_{Zar}, G)=0$.

CONJECTURE 1.4 (K_1-purity). — *If* $R\text{-rank}(G_{ss}) \geqq 1$ *and* P *a minimal parabolic subgroup of* G *over* R, *then* $G(Q)=G^u(Q)\,P(Q)$.

Conjecture 1.3 generalizes Conjecture (2) formulated by Quillen for $G=\mathbf{GL}_n$ in [11] and proved by Gabber for $G=\mathbf{GL}_n$ and $\mathbf{PGL}_n$ and X of dimension $\leqq 3$, *cf.* [13].

We prove here Conjectures 1.2 and 1.3 and a weak version of Conjecture 1.4, sufficient for our purpose, in the case when $\dim X=2$ and G is quasi-split over X. Moreover, we establish in this case several decompositions of the groups $G(Q)$, $G(K)$ and related groups, in particular, versions of the Iwasawa and the Bruhat-Steinberg decompositions for $G(K)$ and $G(Q)$ respectively.

2. LOCALIZATION MAPS AND "LOCAL CLASS SETS". — Let $\mathfrak{b}=uR$ be the ideal generated by u in R. Denote by $\hat{R}$ (resp. R^h) the completion (resp. henselization) of R with respect to $\mathfrak{b}$. Put $\hat{Q}=\hat{R}[u^{-1}]$, $Q^h=R^h[u^{-1}]$, $X^h=\operatorname{Spec} R^h$ and $Y^h=\operatorname{Spec} Q^h$. Consider the "local class set" $c(G)=G(Q)\backslash G(\hat{Q})/G(\hat{R})$ and "henselian local class set" $c^h(G)=G(Q)\backslash G(Q^h)/G(R^h)$ which have distinguished points corresponding to the classes $G(Q)\,G(\hat{R})$ and $G(Q)\,G(R^h)$ respectively.

PROPOSITION 2.1. — *There exists an exact sequence of pointed sets:*

$$(2.1.1) \qquad 1 \to c(G) \to H^1(R_{et}, G) \to H^1(Q_{et}, G) \times H^1(\hat{R}_{et}, G).$$

Proof. — First, we establish the henselian analogue $(2.1.1)^h$ of (2.1.1) in which $c(G)$ is replaced by $c^h(G)$ and $G(\hat{R})$ by $G(R^h)$. $(2.1.1)^h$ is proved by comparing the local cohomology exact sequences for the pairs (X, Y) and (X^h, Y^h) and using the excision for $H^1_Z(X, G)$ where $Z=\operatorname{Spec} R/\mathfrak{b}$. Then we show using the Artin approximation that $(2.1.1)^h$ implies (2.1.1).

Let $m(G) : H^1(R_{et}, G) \to H^1(Q_{et}, G)$ and $m(G \otimes_R \hat{R}) : H^1(\hat{R}_{et}, G) \to H^1(\hat{Q}_{et}, G)$ be the canonical maps.

COROLLARY 2.2. — *Assume that* $\operatorname{Ker} m(G \otimes_R \hat{R})=0$.

Then the following properties (1), (2) *are equivalent:*

(1) $c(G)=0$;

(2) $\operatorname{Ker} m(G)=0$.

PROPOSITION 2.3. — *Let* M *be an* R*-group of the multiplicative type. Then*

(1) *the localization maps* $l(M)$ *and* $l(M\otimes_R Q)$ *are injective;*

(2) $c(M)=0$, i.e. $M(\hat{Q})=M(Q)M(\hat{R})$, *and* $H^1(Q_{Zar}, M)=0$.

The injectivity of $l(M)$ is known [3], [8], and its proof can be extended to $l(M\otimes_R Q)$ (*see* [1] in the case when $M=G_m$). The vanishing of $c(M)$ [resp. $H^1(Q_{Zar}, M)$] follows from (1) and 2.2 (resp. from Conjecture 1.2 for $X=\operatorname{Spec} Q$ proved in [9], and (1)).

3. AN APPROXIMATION PROPERTY OF $G(\hat{Q})$. — Beginning from this section and up to the end of this Note we shall assume that the residue field $k=R/m$ of R is *infinite.*

Equip $\hat{Q}$ with the $\hat{\mathfrak{b}}$-adic topology as an $\hat{R}$-module, where $\hat{\mathfrak{b}}=u\hat{R}$. This uniquely determines the structure of a topological group on $G(\hat{Q})$. For a subgroup $H\subset G(Q)$ denote by $\bar{H}$ its closure in $G(\hat{Q})$.

PROPOSITION 3.1. — $\overline{G(Q)}$ *contains a subgroup* N *which is open in* $G(\hat{Q})$ *and is normalized by* $G(\hat{R})$.

In the case where $\dim R=1$ and, hence, $Q=K$ is a field, an analogue of Proposition 3.1 for the pair $(\overline{G(K)}, G(K))$ has been proved by Harder [6]. Our proof of 3.1 is based on similar ideas and uses heavily the results of [4] on the local structure of the R-scheme $\mathscr{T}$ of maximal tori of G over local rings.

4. SOME DECOMPOSITIONS OF $G(\hat{Q})$.

LEMMA 4.1. — *Let* T *be an* $\hat{R}$*-torus,* P *a parabolic subgroup of* G *over* $\hat{R}$, *and* $U=R_u(P)$. *Then* $U(\hat{Q})\subset\overline{G(Q)}$, $G^u(\hat{Q})\subset\overline{G(Q)}$ *and* $T(\hat{Q})\subset G(Q)G(\hat{R})$.

Let $R_1=\hat{R}_{\hat{\mathfrak{b}}}$ be the localization of $\hat{R}$ with respect to $\hat{\mathfrak{b}}$, K_1 the field of fractions of R_1, $\mathfrak{b}_1=uR_1$. Notice that $\dim R_1=1$. Denote by $\hat{R}_1$ and $\hat{K}_1$ the $\mathfrak{b}_1$-adic completions of R_1 and K_1 respectively.

LEMMA 4.2. — *Let* P *be a parabolic subgroup of* G *over* $\hat{R}$, *which is minimal over* K_1. *Then* $P(\hat{Q})\subset G(Q)G(\hat{R})$.

Lemmas 4.1, 4.2 are generalizations of one-dimensional results of [9], and their proofs follow the same general pattern and use 3.1, 2.3, the local structure of the R-scheme $\mathscr{T}$ of maximal tori of G [4], and some facts from the Bruhat-Tits theory [2], applied to $G(\hat{K}_1)$, as key ingredients. Combining 4.1, 4.2 and 5.2 (2) below, we obtain:

LEMMA 4.3. — *Assume that* $\dim R=2$ *and that* $G(\hat{Q})=\overline{G(Q)}P(\hat{Q})$ *for a parabolic* $\hat{R}$*-subgroup* P *of* G *minimal over* $\hat{R}$. *Then* $G(\hat{Q})=G(Q)G(\hat{R})$.

Let S be a maximal R-split R-torus of G, $\Phi=\Phi(G)$ [resp. $\Delta=\Delta(G)$] the set of all (resp. all simple) R-roots of G, U_α the unipotent root R-subgroup of G corresponding to $\alpha\in\Phi$, G_α the R-subgroup of G generated by all U_β with $\beta=k\alpha$, $k=\pm1, \pm2$. Let W be the Weyl R-group of G and $w=r_{\alpha(w)_1}r_{\alpha(w)_2}\ldots r_{\alpha(w)_{l(w)}}$ a reduced decomposition of $w\in W(Q)$ into the product of reflections $r_{\alpha(w)_i}$ with respect to simple roots $\alpha(w)_i\in\Delta^+$. Denote by $G^1(Q)$ the subgroup of $G(Q)$ generated by all $G_\alpha(Q)$, for $\alpha\in\Delta^+$.

Theorem 4.4. — *Assume that* $\dim R = 2$ *and that* G *has a Borel subgroup* B *over* R. *Denote* $B_\alpha = B \times_G G_\alpha$. *Then, for all* $\alpha \in \Delta^+$:

(1) $G_\alpha(K) = G_\alpha(Q)\, B_\alpha(K)$, *and* $G(K) = G(Q)\, B(K)$.

(2) *There exists a system of representatives* $Y_\alpha(Q)$ *of* $B_\alpha(K) \backslash B_\alpha(K)\, r_\alpha\, B_\alpha(K)$ *in* $G_\alpha(Q)$, *and the group* $G(Q)$ *has a Bruhat-Steinberg decomposition*

$$G(Q) = \bigcup_{w \in W(Q)} B(Q)\, Y_{\alpha(w)_1}(Q)\, Y_{\alpha(w)_2}(Q) \ldots Y_{\alpha(w)_{l(w)}}(Q).$$

Proof. — General results in [10] reduce the proof of Theorem 4.4 to the vanishing of $H^1(Q_{Zar}, B_\alpha)$ which follows from 2.3 (2).

Corollary 4.5. — *Aussume that* $\dim R = 2$ *and that* G *has a Borel subgroup* B *over* R. *Then* $G(Q) = G^1(Q)\, B(Q)$.

Proposition 4.6. — *Let* R, G *and* B *be such as in* 4.5. *Then* $G(\hat{Q}) = \overline{G(Q)}\, B(\hat{Q})$ *and* $G(\hat{Q}) = G(Q)\, G(\hat{R})$. *If* G_{ss} *splits over* $\hat{R}$ *then* $G(\hat{Q}) = G^*(\hat{Q})\, B(\hat{Q})$.

Proof. — We establish, first, the decompositions of 4.6 for G_α and $B_\alpha = B \times_G G_\alpha$, for $\alpha \in \Delta(G)$ or for $\alpha \in \Delta(G \otimes_R \hat{R})$, using the classification of quasi-split simple R-groups of R-rank 1, and the equality $SL_2(\hat{Q}) = SL_2^*(\hat{Q})$ [7]. We show then that if G_{ss} is simply connected, $B_\alpha(\hat{Q}) \subset \overline{G(Q)}$ and, hence, $G_\alpha(\hat{Q}) \subset \overline{G(Q)}$. These facts together with 4.5 and 4.1 imply 4.6.

5. Some properties of parabolic subgroups of G. — In sections 5, 6 we shall assume that $\dim R = 2$ and (as in sections 3, 4) that the residue field of R is *infinite*.

Proposition 5.1. — (1) *Let* P *be a parabolic subgroup of* G *over* Q. *Then the canonical map* $\operatorname{Ker} l(P) \to \operatorname{Ker} l(G \otimes_R Q)$ *is surjective.*

(2) *If* G *is quasi-split over* R, *then* $\operatorname{Ker} l(G \otimes_R Q) = 0$.

Proof. — Statement (1) is true for an arbitrary Dedekind ring D and a reductive D-group G [8]. Statement (2) follows from (1), applied to a Borel R-subgroup of G, and 2.3 (1).

Proposition 5.2. — (1) *Let* P *be a parabolic subgroup of* G *over* $\hat{K}_1$. *Then there exist* $g \in G(\hat{R}_1)$ *and a parabolic subgroup* P' *of* G *over* $\hat{R}$ *such that* $P' \otimes_{\hat{R}} \hat{K}_1 = g P g^{-1}$. *In particular, if* G *is quasi-split over* $\hat{K}_1$, *it is quasi-split over* $\hat{R}$.

(2) *Let* P *be a minimal parabolic subgroup of* G *over* $\hat{R}$. *Then* $P \otimes_{\hat{R}} \hat{K}_1$ *is a minimal parabolic subgroup of* $G \otimes_R \hat{K}_1$ *over* $\hat{K}_1$.

The proof uses the smoothness and the projectivity of the scheme $\mathcal{P}$ of parabolic subgroups of G [4].

6. Reduction and proof of Conjecture 1.2 ($\dim R = 2$). — Let K be the field of fractions of R. Denote $\bar{R} = R/\mathfrak{b}$, $\bar{K} = R_1/\mathfrak{b}_1$. Since $\dim R = 2$, $\bar{R}$ is a discrete valuation ring and $\bar{K}$ is the field of fractions of $\bar{R}$.

Proposition 6.1. — *Assume that Conjecture* 1.3 *is true for* G, *and that* G *has a minimal proper parabolic subgroup* P *over* $\hat{R}$ *for which* $G(\hat{Q}) = \overline{G(Q)}\, P(\hat{Q})$.

Then Conjecture 1.2 *is true for* G.

Remark 6.2. — If Conjecture 1.4 holds for $G \otimes_R \hat{R}$, then $G(\hat{Q}) = \overline{G(Q)}\, P(\hat{Q})$ by 4.1.

Proof. — Notice, first, that since the rings $\hat{R}$ and $\hat{R}_1$ are complete and G is smooth over R, the natural maps $i : H^1(\hat{R}_{et}, G) \to H^1(\bar{R}_{et}, G)$ and $i_1 : H^1(\hat{R}_{1,et}, G) \to H^1(\bar{K}_{et}, G)$ are bijections [4]. It has been proved in [9] that for the discrete valuation rings $\bar{R}$ and $\hat{R}_1$, $\operatorname{Ker} l(G \otimes_R \bar{R}) = \operatorname{Ker} l(G \otimes_R \hat{R}_1) = 0$. It follows from these facts that the map $\alpha(G): H^1(\hat{R}_{et}, G) \to H^1(\hat{R}_{1,et}, G)$ has trivial kernel. The map $m(G \otimes_R \hat{R})$: $H^1(\hat{R}_{et}, G) \to H^1(\hat{Q}_{et}, G)$ factorizes through the composition $l(G \otimes_R \hat{R}_1) \circ \alpha(G)$ and, hence, also has trivial kernel. Notice, that under the assumptions of 6.1, $c(G)=0$ (prop. 4.3), and $\operatorname{Ker} l(G \otimes_R Q)=0$ (Conjecture 1.3 for G). The triviality of $\operatorname{Ker} m(G \otimes_R \hat{R})$ and $c(G)$ implies that $\operatorname{Ker} m(G)=0$ (prop. 2.2). Since $l(G)=l(G \otimes_R Q) \circ m(G)$, we conclude that $\operatorname{Ker} l(G)=0$.

THEOREM 6.3. — *Let* G *be a reductive* R*-group quasi-split over* R.
Then Conjecture 1.2 *is true for* G.

Proof. — The assumptions of 6.1 for such G are satisfied by 5.1 (2) and 4.6.

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