The structure of a class of finite ramified coverings and canonical forms of analytic matrix-functions in a neighborhood of a ramified turning point

Yevsey A. Nisnevich*

Clay Mathematics Institute, 1770 Massachusetts Avenue, Cambridge, MA 02140; and Institut de Mathématiques de Jussieu, 175 rue du Chevaleret, 75013 Paris, France

Communicated by M. Gromov, Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, France, February 6, 2002 (received for review January 10, 2002)

Let X be the germ of a complex- or real-analytic manifold M at a point $x_o \in M$, or the henselian germ of an algebraic manifold M over a field k of characteristic zero at a point $x_o \in M(k)$, $D \subset X$ a divisor. Under some assumptions on D and its singularities we give a description of the structure, the singularities, and the divisor class group of all finite normal coverings of X ramified over D.

Let $g : X \to \mathrm{gl}(n)$ be an analytic or a k-algebraic family, respectively, of semisimple matrices, the eigenvalues of which are ramified on D as functions of $x \in X$. Put $U = X - D$. Using the above results under some quite general assumptions on g and D we construct an irreducible nonsingular variety U_c, a finite etale morphism $a_c : U_c \to U$, and a morphism $u_c : U_c \to \mathrm{GL}(n)$ (all in the same category as X and g), such that $t_c(x) = u_c(x)g(x)u_c(x)^{-1}$ is a diagonal matrix, for all $x \in U_c$. This construction gives, among other things, an extension in a refined form (on the level of U_c-sections) of the classical one-parameter Perturbation Theory of matrices to the case of many parameters, ramified eigenvalues, not necessarily hermitian matrices, etc. We also prove the stable triviality of the eigenbundles of g on U and vanishing of their Chern classes.

MATHEMATICS

1. Introduction and Notation

Let k be a field of characteristic zero.

1.1. In this paper, unless it is explicitly stated otherwise, A will be a local noetherian k-algebra of one of the following two types:

1.1.1. $A = B^h$ is the henselization of a smooth finitely generated k-algebra B of finite type over k with respect to a maximal ideal m of B.

1.1.2. $A = k\{T_1, T_2, \ldots, T_n\}$ is the algebra of *convergent* power series over a field k, where $k = \mathbf{C}$ the field of complex or $k = \mathbf{R}$ the field of real numbers.

In the algebraic case of 1.1.1, denote $X = \mathit{Spec}\, A$, and in the analytic case of 1.1.2, denote $X = \mathit{Specan}\, A$, the analytic spectrum of A (see ref. 1, Ch. 2, and sections 3.3 and 2.1 below). Denote by x_o the closed point of X in the k-algebraic case and the center of the germ X in the k-analytic case. We shall call such an X *k-henselian* or *k-analytic* germ, respectively, and shall refer to these cases as the local k-algebraic henselian and the local k-analytic cases, respectively.

1.1.3. Let $g : X \to \mathbf{gl}(n)$ be a k-analytic or k-polynomial morphism in the same category. We shall call such a g a k-analytic or k-algebraic matrix valued function.

We say that the matrix valued function g is *pointwise diagonalizable over k* if $g(x)$ is a matrix diagonalizable over the residue field $k(x)$ of x, for all x in $X(\bar{k})$, where $\bar{k}$ is the algebraic closure of k.

1.1.4. We say that x_o is *a turning or a transition point* for a matrix function g, if some eigenvalues $e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ of g change their multiplicities in X. We say that a turning point x_o is *unramified* if all the eigenvalues $e_i(x)$ of $g(x)$ are regular functions in X in the same category, as functions of the parameter x, i.e. they are analytic or polynomial functions of the parameter x in X in the analytic or algebraic cases, respectively. Otherwise, we shall say that the turning point x_o is *ramified.* In this case, some of the eigenvalues e_{i_j} changing their multiplicities in X are the branches of one or several ramified analytic functions.

1.2. We say that a pointwise diagonalizable matrix function g is *locally diagonalizable* or admits a *local diagonalization* in a neighborhood of x_o, if there exists an invertible matrix valued function $u : X \to \mathbf{GL}(n)$ in the same category as X and g, such that the following equality holds:

$$u(x)g(x)u(x)^{-1} = t(x), \text{ for all } x \in X, \qquad [1]$$

where $t(x)$ is a diagonal matrix function.

The existence of a local diagonalization of a matrix-function g when $\dim X \geq 1$ is a classical problem that frequently arises and plays an important role in many areas and contexts in mathematics and physics, including nearly all the main branches of differential equations, perturbation theory, gauge theories, etc. See refs. 2–7 and sections 4 and 5 of this paper and literature quoted there for further discussions of these links.

It is well known (see for example, ref. 4, section 2) that the main difficulties presents a diagonalization in a neighborhood of a turning point x_o. In ref. 4 this problem was studied in the case when $\dim X \geq 1$ and the turning point x_o is unramified. Under this assumption and some additional assumptions on $g(x)$, it was shown in ref. 4 that a diagonalization exists on each stratum of a certain stratification of the germ X. However, the ramification of the eigenvalues e_j is a very common feature when $\dim X \geq 2$. For this reason, the assumption of the regularity of the eigenvalues is too restrictive for many important questions and

*Reprint request address: Institut de Mathématiques de Jussieu, Université Paris VI, 175 rue du Chevaleret, 75013 Paris, France. E-mail: nisnevic@math.jussieu.fr or nisnevic@ihes.fr.

potential applications. The situation here is the opposite to that encountered in the classical one-parameter real-analytic Perturbation theory of Hermitian matrices and operators, where the problem of ramification did not arise, because of a theorem of F. Rellich that established its absence on the real line (2, 3, 5).

Thus, assume now that the eigenvalues of g do ramify on a divisor $D \subset X$, as functions of the (multi-)parameter $x \in X$ and $x_o \in D$ is a ramified turning point. In this case it is known that a regular diagonalizing family u for g satisfying Eq. **1** above may not exist on X, in general. Furthermore, in section 4.6 below we give an example of g for which a diagonalizing morphism u does not exist on $U = X - D$ and even on any finite etale covering of U.

One of the main purposes of this paper is to prove the existence of a diagonalizing morphism $u_c : U_c \to \mathbf{GL}(n)$ for g over a suitable finite etale cover $a_c : U_c \to U$ under some restrictions on g (*Theorem 3.5*). This is the first positive result on the local diagonalization in a neighborhood of a ramified turning point when $dim\, X \geq 2$, to our knowledge. In the analytic case, this gives, among other things, an extension in a refined form of the classical Perturbation Theory onto the multi-parameter ramified case, not-necessarily-Hermitian matrices, and in some other directions. Indeed, these results give the triviality of the eigenbundles of g over U_c, rather than just the existence of analytic projections onto them, as the classical analytic constructions of the Perturbation Theory do, and the first property is by far stronger and more delicate than the second in the multi-parameter case, as *Example 4.6* shows.

1.3. The results outlined in section 1.2 are based on several more general results that have an independent interest in broader algebra-geometric and analytic contexts. The main of them are an explicit construction and a description of the singularities of all finite normal coverings $Y \to X$, ramified along a divisor D, which satisfies the following condition: there exists a closed subgerm $N \subset D$ of codimension ≥ 2 in D such that D is a divisor with normal crossings outside N (and D may have arbitrary singularities inside N) (*Theorem 2.9*). *Theorem 2.9* extends the classical Abhyankar Lemma (ref. 8, XIII, section 5), and it provides a much greater flexibility in applications than this Lemma. In particular, using this extension, we construct under some quite general assumptions on g and its eigenvalues an irreducible reduced factorial germ X_c and a finite ramified cover $a_c : X_c \to X$, such that all the (ramified) eigenvalues e_j of g give rise to functions $e_{c,j}$ on X_c that are well defined everywhere and are regular in the corresponding category (section 3). These nice properties of X_c allow us to define on X_c the eigensheaves $E_{c,j}$ of g corresponding to the "regularized" eigenvalues $e_{c,j}$, for all j, and to show that the restrictions of $E_{c,j}$ onto $U_c = X_c - D_c$ are trivial bundles, for all j, where $D_c = a_c^{-1}(D)$.

The last facts imply the results mentioned in section 1.2 (*Theorem 3.5*) and they have many other applications. Applications to the stable triviality of the eigenbundles (including the kernel bundles) of g on U and vanishing of their Chern classes are given in sections 4 and 5. It seems these are the first general results on properties of the eigenbundles in a neighborhood of a ramified turning point. Many explicit calculations of the local Chern classes and holonomies of the eigenbundles in some special low-dimensional ($dim\, X \leq 3$) cases can be found in the literature on the geometric (Berry) phases, quantum Hall effect, and anomalies (see refs. 6 and 7, and literature quoted there). The algebra-geometric methods of this paper are significantly different from those used in the previous works on these questions and they give more precise and complete results under much broader assumptions (e.g., for $dim\, X \geq 4$), where the differential-geometric methods of the previous works (the Chern-Weil theory, etc.) are inapplicable in principle (see section 4.7).

The results of sections 4 and 5 can be extended onto ramified eigenvalues and the corresponding eigenbundles of analytic families of differential operators. The eigenbundles, their Chern classes, and other properties for families of differential operators arising in various branches of geometry and physics (Laplac, Dirac, and Schroedinger operators, etc.) have geometrical or physical interpretations and carry information important for these fields.

2. Construction and Properties of a Class of Finite Ramified Coverings

2.1. Complex-Analytic Varieties Defined over R. For our purposes it will be convenient to use systematically in the analytic category analogues of the concepts of complex algebraic varieties defined over the real field $\mathbf{R}$ and their morphisms defined over $\mathbf{R}$, which have become standard in algebraic geometry since A. Weil and A. Grothendieck. They are defined as follows.

We say that an analytic function $f : U \to \mathbf{C}$ on an open subset $U \subset \mathbf{C}^n$ *is defined over* $\mathbf{R}$, or simply is $\mathbf{R}$-*analytic* if in suitable systems of coordinates in U and $\mathbf{C}$ its Taylor series expansion has only real coefficients. We say that a complex-analytic variety Y (or a complex-analytic germ Y) *is defined over* $\mathbf{R}$ if there exists a collection of local charts (U_i, $i \in I$) covering Y, where each U_i is analytically isomorphic to an open subset of $\mathbf{C}^n$, such that the set of equations ($f_{i,j}$, $j \in J_i$) defining $Y \cap U_i$ in U_i and the functions $u_{i,j}$ gluing the charts U_i can be chosen $\mathbf{R}$-analytic. For a simplicity we shall call such a Y an $\mathbf{R}$-analytic variety or $\mathbf{R}$-analytic germ, respectively.

For $\mathbf{R}$-analytic variety Y we can consider the set $Y(\mathbf{R})$ of its real points—i.e., real solutions of the local equations $f_{i,j}$, as well as the set $Y(\mathbf{C})$ of its complex points. $\mathbf{R}$-analytic morphisms $f : Y \to Z$ between $\mathbf{R}$-analytic varieties Y and Z are defined using $\mathbf{R}$-analytic functions in the obvious way.

For each real-analytic local $\mathbf{R}$-algebra B we can consider $\mathbf{R}$-analytic germ Y, the set of complex points of which is $Y(\mathbf{C}) = Specan(B \otimes_{\mathbf{R}} \mathbf{C}) \subset \mathbf{C}^n$ (see ref. 1, Ch. 2, section 3). We shall denote this $\mathbf{R}$-analytic germ Y by the symbol *Specan B* and shall refer to it as an $\mathbf{R}$-analytic germ corresponding to B.

As in algebraic geometry, all the standard geometric notions and properties concerning $\mathbf{R}$-analytic varieties $Y, Z \ldots$ and their morphisms $f : Y \to Z$ (for example, the dimension, the codimension, the finiteness, or the surjectivity of morphisms, etc.) will be defined as they usually are defined for the complex varieties of their $\mathbf{C}$-points $Y(\mathbf{C})$, $Z(\mathbf{C}) \ldots$ and their analytic morphisms, rather than for their sets of real points $Y(\mathbf{R})$, $Z(\mathbf{R})$ only.

2.2. Let Y be a normal irreducible k-germ in the same category as X and $a : Y \to X$ a finite Galois (ramified) k-covering of X. Then $Y = Spec\, A_Y$ for a suitable local henselian k-algebra A_Y in the henselian algebraic case, or $Y = Specan\, A_Y$ for a local analytic k-algebra A_Y in the k-analytic case.

Denote by $K = Fract(A)$ the field of fractions of A—i.e. K is the field of rational functions on X in the algebraic case and the field of meromorphic functions on X in the analytic case—and by $K_Y = Fract(A_Y)$.

Denote by $D = Ram\ (a) \subset X$ the ramification divisor in X of the morphism a. Let $D = \cup_{i=1}^{s} D_i$ be the decomposition of D into a union of its irreducible components D_i, r_i the ramification index of Y with respect to D_i (defined, for example, in ref. 9, section 2) and $r_Y := \Pi_{i=1}^{s} r_i$ the total ramification index of a. Because X is a nonsingular germ, the divisor D (respectively each D_i) is defined by a single equation $f = 0$ (resp. $f_i = 0$) in X.

2.3. We say that a divisor $D = \cup_i D_i$ is a *k-divisor* if all its irreducible components D_i are defined over k and are geometrically irreducible. In this case, the equations f_i of D_i also can be chosen with the coefficients in k.

2.4. For a divisor D, denote by $N(D)$ the closed subgerm of D with the reduced structure consisting of all points $x \in D$, at which D is not a divisor with normal crossings. (Notice that the irreducible components D_i of D may have self-intersections.) Denote by $ir(D)$ the number of the irreducible components D_i of D and let $c(D) = codim_D N(D)$.

2.5. We say that a divisor D satisfies *condition* (SD_m) if it is a k-divisor, $ir(D) \leq dim X$ and $c(D) \geq m$.

2.6. Denote by $\mathcal{H}_o$ the homotopy category of pointed simplicial sets, and by $Pro - \mathcal{H}_o$ the category of pro-objects from $\mathcal{H}_o$, which was defined and studied by Artin and Mazur in ref. 10. For a noetherian scheme X or an analytic space X or an analytic germ X over $\mathbf{C}$ denote by $ht(X)$ its homotopy type—i.e., the corresponding class in $Pro - \mathcal{H}_o$; see ref. 10 for its definition and properties. For a scheme X, its homotopy groups $\pi_m(X)$ are defined as $\pi_m(ht(X))$ and $\pi_1(ht(X))$ coincides with the etale (profinite) fundamental group $\pi_1^{et}(X)$ defined in ref. 8. For an analytic space or germ X over $\mathbf{C}$, $\pi_m(ht(X))$ coincides with the topological homotopy groups $\pi_m^t(X)$ of this space or germ, for all m. For a scheme X over $\mathbf{C}$, denote by $X(\mathbf{C})^{an}$ the associated analytic space, and by X_{cl} the "classical" topos of X defined in refs. 10 and 11.

PROPOSITION 2.7. *Let $D = \cup_{i=1}^{s} D_i$ be a divisor in X with the irreducible components D_i, $U = X - D$, $U_i = X - D_i$. Assume that D satisfies SD_2.*

(1) If X = Specan A is a nonsingular complex-analytic germ as in 1.1.2, then

$$\pi_1^t(U_i(\mathbf{C})^{an}) = \mathbf{Z} \text{ and } \pi_1^t(U(\mathbf{C})^{an}) = \Pi_{i=1}^{s} \pi_1^t(U_i(\mathbf{C})^{an})$$

(2) If X = Spec A is local henselian algebraic over $\mathbf{C}$ *as in 1.1.1, then*

$$\pi_1^{et}(U_i) = \hat{\mathbf{Z}} \text{ and } \pi_1^{et}(U)) = \Pi_{i=1}^{s} \pi_1^{et}(U_i).$$

Proof: The second statement of (1) was proved in ref. 12, Corollary 2.4.1. Let D_i be an irreducible component of D given by an equation $f_i = 0$ for some analytic function $f_i : X \to \mathbf{C}$. It is proved in ref. 12 that the Milnor fiber of f_i is simply connected. The first statement of (1) follows then from the exact sequence of homotopy groups of the Milnor fibration of f_i, which was constructed in ref. 13. Both statements of (2) follows from those of (1) and PROPOSITION 2.8 below applied to $Y = U$ and $Y = U_i$.

The following proposition is a local henselian version of the "generalized Riemann existence theorem," which was proved by Artin and Mazur for algebraic varieties of finite type over $\mathbf{C}$ (ref. 10, Theorem 12.1):

PROPOSITION 2.8. *Let V be a normal complex algebraic variety, $x_o \in V(\mathbf{C})$ a* $\mathbf{C}$*-point on a V,* and *Z a closed subvariety of V containing x_o, $Y = V - Z$. Let V^h and Z^h be the henselian germs of V* and *Z, respectively, at x_o, V^{an}* and *Z^{an} the analytic germs of V and Z, respectively, at x_o. Put $Y^h = V^h - Z^h$* and *$Y^{an} = V^{an} - Z^{an}$. Then there exists a map of the homotopy types $\varepsilon : h(Y(\mathbf{C})^{an}) \to h(Y_{et})$ in the category Pro − Ho, and the natural homomorphism of the homotopy groups: $\pi_m^t(Y(\mathbf{C})^{an}) \to \pi_m^{et}(Y^h)$, induced by ε, becomes an isomorphism after profinite completion of π_m^t, for all $m \geq 0$.*

Proof: The construction of the map ε is a local version of that given in ref. 10, Theorem 12.1. As in 10, it is sufficient to prove that it induces an isomorphism between the analytic and etale cohomology with the twisted finite coefficients. For this we use the strong comparison (or base change) theorem between the analytic and etale cohomology (ref. 11, XVI, 4.1), applied to the natural embedding $i : U \to X$.

The following theorem is an extension of the classical (absolute) Abhyankar Lemma (ref. 8, XIII, 5.1–5.3), which assumes that D is a divisor with normal crossings.

THEOREM 2.9. *Let X = Spec A or X = Specan A be a nonsingular k-germ in the k-henselian or k-analytic category as in section 1.1.1 or 1.1.2, respectively. Let Y be a normal irreducible k-germ, $a : Y \to X$ a finite Galois ramified k-covering of X in the same category as X, and let the objects A_Y, D, D_i, etc. be defined for this covering as in section 2.2. Assume, in addition, that condition (SD_2) for the ramification divisor D is satisfied and that the extension Y/X induces an isomorphism $k \xrightarrow{\sim} k_Y$ of the residue fields.*

(1) If $d_Y = deg(Y/X) = r_Y$, then

(1i) Y coincides with Spec A'_Y in the henselian k-algebraic case and with Specan A'_Y in the k-analytic case, where

$$A'_Y = A[f_1^{1/r_1}, \ldots, f_s^{1/r_s}]$$

and it is a complete intersection, germ, finite flat over X in the both k-algebraic and k-analytic cases;

(1ii) the singular locus Sin(Y) of Y is contained in the inverse image $N(D)_Y = a^{-1}(N(D))$ in Y under a of the non-normal crossing subgerm N(D) of D.

(2) If $d_Y = deg(Y/X) < r_Y$, then there exists a normal irreducible reduced k-germ Y_c and a finite Galois abelian purely ramified k-covering $b : Y_c \to Y$ of degree r_Y/d_Y (both in the corresponding category), such that the following properties (2i) and (2ii) are satisfied:

(2i) The ramification divisor $Ram(a_c)$ of the composite projection $a_c = a \circ b : Y_c \to X$ in X coincides with D as a set.

(2ii) The germ Y_c is a complete intersection and its singular locus $Sin(Y_c)$ is contained in the inverse image germ $N(D)_c := a_c^{-1}(N(D)) \subset X_c$.

Remark 2.9.1: If the field k is algebraically closed, then automatically $k = k_Y$, and by PROPOSITION 2.7, any finite covering $Y \to X$, etale over $U = X - D$, is Galois.

An outline of the Proof: Assume first that $k = \mathbf{C}$. In the both cases (algebraic henselian over $\mathbf{C}$ case and the complex-analytic case) it follows from PROPOSITION 2.7 that

$$G = Gal(Y/X) = \Pi_{i=1}^{s} \mathbf{Z}/r_i,$$

MATHEMATICS

where the ith factor $R_i = \mathbf{Z}/r_i$ of G is the ramification subgroup of G with respect to the component D_i of D and r_i is the ramification index of K_Y/K with respect to D_i. This implies that $K_Y = K(f_1^{1/r_1}, \ldots, f_s^{1/r_s})$. This K_Y is, clearly, the field of fractions of the ring A'_Y, which was defined in claim (1i) above.

Using the results of ref. 14 and an induction on s one can prove that the ring A'_Y is a complete intersection ring. Denote $Y' = Spec\ A'_Y$ in the k-algebraic case and $Y' = Specan\ A'_Y$ in the k-analytic case. Let $a' : Y' \to X$ be the natural projection and put $N(D)'_Y := (a')^{-1}(N(D))$.

The classical Abhyankar Lemma (ref. 8, expose XIII, 5.1) implies that all the points $x \in Y' - N(D)'_Y$ are smooth. This establishes claim (1ii). Furthermore, (1ii) and the condition $c(D) \geq 2$ implies that $codim_{Y'}Sin(Y') \geq 3$ and by the criterion of normality of Krull–Serre (15) Y' is a normal germ. Therefore, it coincides with the normalization Y of X in K_Y.

(2) Assume now that $d_Y = deg(a) < r_Y$. Define $Y' = Spec\ A'_Y$ in the algebraic case and $Y' = Specan\ A'_Y$ in the analytic case, as in 1(i) above, and put $Y_c = Y'$. Let K_c be the field of rational functions on Y_c in the k-algebraic case or meromorphic functions in the k-analytic case. Then by (1i) Y_c is the normalization of X in K_c and Y_c satisfies property (2i) by its construction and property (2ii) by (1ii).

The case of an arbitrary field k of characteristic zero can be reduced to that of $k = \mathbf{C}$.

COROLLARY 2.10. *Under the assumptions of THEOREM 2.9, the germ Y is the factor Y_c/H of a complete intersection germ Y_c by a finite abelian group $H := Ker(G_c \to G)$, where $G = Gal(Y/X)$ and $G_c = Gal(Y_c/X)$.*

2.11. For a normal Noetherian scheme Z or a normal k-analytic space Z denote by $Cl\ (Z)$ and $Pic\ (Z)$ the (Weil) divisor class group of Z and the Picard group of Z respectively.

2.12. Denote $U = X - D$ and let $D_c = a_c^{-1}(D)$ and $U_c = a_c^{-1}(U)$ be the inverse images of D and U in Y_c, respectively.

COROLLARY 2.13. *Under the notation and assumptions of THEOREM 2.9 assume in addition that condition (SD_3) is satisfied. Then $Cl(Y_c) = 0$—i.e. this scheme is factorial, and*

(i) in the k-algebraic henselian case, $Cl(U_c) = Pic(U_c) = 0$;

(ii) in the k-analytic case, let V be a line bundle on U_c that has an extension $\hat{V}$ onto the whole Y_c as a coherent sheaf. Then its class in $Pic(U_c)$ vanishes.

Proof: By THEOREMS 2.9(1i) and (2ii) Y_c is a complete intersection germ and condition (SD_3) and 2.9(1ii) imply that $codim_{Y_c}(Sin(Y_c)) \geq 4$. The vanishing of $Cl(Y_c)$ follows now from the factoriality theorem of Grothendieck, which proved the Samuel Conjecture (ref. 16, XI, 3.14). In the algebraic case, the natural homomorphism $Cl(Y_c) \to Cl(U_c)$ is surjective. Therefore, the triviality of $Cl(Y_c)$ implies (i).

In the analytic case, denote by W the double dual $\hat{V}^{**}$ of $\mathcal{O}_{Y_c}$-module $\hat{V}$. It is a coherent reflexive $\mathcal{O}_{Y_c}$-module of rank 1 on Y_c that extends $\hat{V}$. The description of the (Weil) divisor class groups given in ref. 17 shows that V is in the image of the natural homomorphism $Cl(Y_c) \to Cl(U_c)$. This implies (ii).

3. A Construction of a Local Diagonal Form and of a Perturbation Theory in the Case of Ramified Eigenvalues

Let A and X be as in section 1.1.1 or section 1.1.2, K the field of fractions of A, $g : X \to \mathbf{gl}(n)$ a k-morphism in the k-algebraic henselian or the k-analytic category, respectively.

3.1. Consider the characteristic equation

$$p(\lambda) = det(\lambda I_n - g(x)) = 0$$

of the matrix $g(x)$ as a polynomial equation with respect to λ over the field K. Here I_n is the unit matrix of order n. Assume that $p(\lambda)$ is irreducible over K. Let L/K be a decomposition field of $p(\lambda)$ over K and $e_1, \ldots, e_n$ the collection of all the roots of p in L.

Denote by K_p the (minimal) Galois field extension of K generated by all the roots e_j of p over K and by $A' = A[e_1, \ldots, e_n]$ the A-subalgebra of K_p generated by all the e_j. Let A_p be the integral closure of A in K. It is a henselian local k-algebra in the algebraic case and an analytic k-algebra in the analytic case. For each root e_j, the characteristic equation is an equation of an integral dependence of e_j over A. Hence, $A' \subset A_p$ and A_p is the integral closure of A' in K.

3.2. Denote $X_p = Spec\ A_p$ in the k-algebraic henselian case and $X_p = Specan\ A_p$ in the k-analytic case, and let $a_p : X_p \to X$ be the natural projection. Let the germs $D = Ram(a)$, D_j, $1 \leq j \leq s$, $N(D)$ and $U = X - D$ be defined as in sections 2.2–2.4 for $Y = X_p$ and $a = a_p$.

Assume that condition (SD_2) and the following condition (RF) are satisfied:

(RF) The extension A_p/A induces an isomorphism $k \xrightarrow{\sim} k_p$ of the residue fields of these rings.

Let X_c be the complete intersection germ and $b : X_c \to X_p$ the ramified covering constructed for $Y = X_p$ in THEOREM 2.9. Put $a_c = a_p \circ b : X_c \to X$. By their constructions, the maps a_p, b, and a_c are defined over k. Denote by $g_c = a_c^*(g)$ the pullback of g onto X_c and by $e_{c,j}(x)$, $j = 1, \ldots, n$, the eigenvalues of the matrix $g_c(x)$, for $x \in X_c$. Notice that the element $e_j \in A_p$ gives rise to a regular function $a_c^*(e_j)$ on X_c in the corresponding category, for all j. (However, these functions may not be defined over k, in general.)

LEMMA 3.3. *Assume that the matrix function g satisfies the following condition (PWD/U):*

(PWD/U) the matrix function g is pointwise diagonalizable over k on $U = X - D$ (cf. 1.1.3).

Then: (i) If $k = \mathbf{R}$, the restriction of the eigenvalue e_j on $U(\mathbf{R})$ is a single-valued real-analytic function on $U(\mathbf{R})$, for all j.

(ii) If in addition conditions (RF) and (SD_2) are satisfied, the eigenvalue $e_{c,j}$ of g_c coincides with some of the elements $a_c^*(e_1)$, …, $a_c^*(e_n)$, say $a_c^*(e_j)$, for each j. Furthermore, $e_{c,j}$ is a k-analytic function on X_c in the k-analytic case and it is a k-polynomial function on X_c in the k-algebraic henselian case.

3.4. Preserve the assumptions of LEMMA 3.3. For each j and for each $x \in X_c$ denote by $E_{c,j}(x)$ the space of all the eigenvectors of $g_c(x)$ in the vector space $k(x)^n$ belonging to the eigenvalue $e_{c,j}(x)$. The collection of $k(x)$-vector spaces $E_{c,j}(x))$ for all x in X_c forms a coherent $\mathcal{O}_{X_c}$-submodule of the free $\mathcal{O}_{X_c}$-module $\mathcal{O}_{X_c}^n$ of rank n on X_c (ref. 4, section 2). It follows from the constructions of X_c and D given above that the eigenvalues $e_{c,j}$, $j = 1, \ldots, n$ may coincide only inside D_c. Hence, they have constant multiplicities $m_{c,j}(x) = dim_{k(x)}(E_{c,j}(x))$ over U_c. By ref. 4, Proposition 2.4, this implies that the restriction $E_{c,j}|U_c$ of $E_{c,j}$ onto U_c is a vector bundle, for all j.

THEOREM 3.5. *In the notation above, assume that the characteristic polynomial $p(\lambda)$ is irreducible over K and that conditions (SD_3) of section 2.5, (RF) of section 3.2, and (PWD/U) of section 3.3, as well as condition (Mu1/U) below, are satisfied:*

(Mu1/U) The multiplicities $m_j(x)$ of all the eigenvalues $e_j(x)$ are equal to one, for all $x \in U$.

Then the k-germ X_c is factorial and the following properties hold:

(i) The restriction $E_{c,j}|U_c$ of the eigensheaf $E_{c,j}$ on U_c is a trivial vector bundle in the corresponding category, for all j.

(ii) There exists a k-morphism $u_c : U_c \to \mathbf{GL}(n)$ in the same category as g, which diagonalizes the matrix-function $g_c = g \circ a_c$ on U_c—i.e., the equality

$$u_c(x)g_c(x)u_c(x)^{-1} = t_c(x), \text{ for all } x \text{ in } U_c$$

is valid, where $t_c(x) = diag(e_{c,1}(x), \ldots, e_{c,n}(x))$ is the diagonal matrix with the eigenvalues $e_{c,j}(x)$ of $g_c(x)$ on the main diagonal.

Proof: Under the assumptions of THEOREM 3.5, the covering $a_c : X_c \to X$ constructed in sections 3.1 and 3.2 satisfies the conditions of THEOREM 2.9 with $Y = X_c$ and $a = a_c$. Then all the eigenbundles $E_{c,j}$ are trivial on U_c by COROLLARY 2.13(i) in the k-algebraic henselian case and by COROLLARY 2.13(ii) in the k-analytic case. This proves (i). The existence of a diagonalizing k-morphism $u_c : U_c \to \mathbf{GL}(n)$ for g_c on U_c follows from the triviality of $E_{c,j}$ and Proposition 2.4 of ref. 4.

4. Rational Stable Triviality of the Eigenbundles of *g*(*x*) on *U*(R)

In this section $k = \mathbf{C}$ or $\mathbf{R}$, the fields or complex or real numbers. We shall preserve in this section the general notation of sections 1–3 and assume that g satisfies condition (PWD/U) of section 3.3.

4.1. By LEMMA 3.3(i), the eigenvalues e_j are single-valued $\mathbf{R}$-analytic functions on $U(\mathbf{R})$. Therefore, we can consider the eigenbundle E_j over $U(\mathbf{R})$, corresponding to the eigenvalue e_j, for each j. It is an $\mathbf{R}$-analytic bundle. Denote by $E_{j,\mathbf{C}} = E_j \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of E_j.

4.2. For a differentiable (C^∞) (respectively real-analytic) manifold M, denote by $Bun^\tau(M, \mathbf{C})$ the category of topological ($\tau = t$) (respectively real-analytic ($\tau = ran$)) vector bundles on M with the complex coefficients, and by $K_o^\tau(M, \mathbf{C})$ the Grothendieck group of this category.

THEOREM 4.3. *Under the notation above, assume that conditions (SD_3), (RF), (PWD/U) and (Mu1/U) of sections 2.5, 3.2, 3.3, and 3.5, respectively, are satisfied. Let M be a closed differentiable (C^∞) submanifold of $U(\mathbf{R})$, $m = dim_{\mathbf{R}}M$, $E_{j,\mathbf{C}}|M$ the restriction of the complexified eigenbundle $E_{j,\mathbf{C}}$ onto M. Then*

(i) The complex vector bundle $d_c(E_{j,\mathbf{C}}|M)$ is topologically stably trivial on M, where d_c is the degree of the extension X_c/X. Furthermore, if $d_c > m/2$, then this bundle is topologically trivial.

(ii) If M is a closed reduced coherent (in the sense of ref. 18, Ch. 2) real-analytic subvariety of $U(\mathbf{R})$, then the vector bundle $d_c(E_{j,\mathbf{C}}|M)$ is real-analytically *stably trivial on M. Furthermore, if $d_c > m/2$, then it is* real-analytically *trivial.*

An outline of the Proof: Denote by $M_c := a_c^{-1}(M) \subset X_c(\mathbf{C})$ the inverse image of M under the projection $a_c : X_c \to X$. Then the restriction of a_c onto M_c gives a continuous etale surjective map $\mu : M_c \to M$ of topological spaces of degree d_c. This map induces the direct image (or transfer) homomorphism of the K_o-groups $\mu_*^t : K_o^t(M_c, \mathbf{C}) \to K_o^t(M, \mathbf{C})$, and the inverse image homomorphism $\mu_t^* : K_o(M, \mathbf{C}) \to K_o(M_c, \mathbf{C})$, such that $\mu_*^t \circ \mu_t^* = d_c$.

The topological stable triviality of $d_cE_{j,\mathbf{C}}|M$ follows from THEOREM 3.1(i) and these properties of the transfer. The analytic stable trivality in (ii) follows from that in (i) and a real analytic version of the Grauert–Oka principle (ref. 18, Ch. 8, Theorem 2.2). The last triviality statement in (i) follows from the stabilization theorems for topological bundles (ref. 19, Ch. 8, Theorem 1.5). The triviality statement in (ii) follows from that in (i) and the same version of the Grauert–Oka principle.

4.4. Let $H^*(M, R)$ be a cohomology theory with the coefficients in a commutative ring R that admits a theory of Chern classes $c_r^\tau : K_o^\tau(M, \mathbf{C}) \to H^{2r}(M, R)$ with the standard properties on the category $Bun^\tau(M, \mathbf{C})$, for $\tau = t$ or $\tau = ran$, (see ref. 19, Ch. 16). If R is a $\mathbf{Q}$-algebra, then for this cohomology theory the Chern character $ch^\tau : K_o^\tau(M) \to H^*(M, R)$ is also defined and it is a ring homomorphism.

As examples of such cohomology theories we can take the following: (*i*) the singular cohomology $H^*_{sin}(M, R)$ of the underlying topological space M with $R = \mathbf{Z}$ or $\mathbf{Q}$; (*ii*) the de Rham cohomology theory $H^*_{dR}(M, \mathbf{R})$ of smooth (C^∞) differential forms on the C^∞-manifold M with $R = \mathbf{R}$.

COROLLARY 4.5. *Under the notation and the assumptions of THEOREM 4.3(i) (respectively 4.3(ii)) we have:*

(i) The first (integral) Chern class $c_1^\tau(d_cE_{j,\mathbf{C}}|M)$ of the restriction of the eigenbundle $d_cE_{j,\mathbf{C}}$ onto M is zero, for all j and $\tau = t$ (respectively $\tau = ran$).

(ii) If R is a $\mathbf{Q}$-algebra, then $ch^\tau(E_{j,\mathbf{C}}|M) = 0$, for $\tau = t$ (respectively for $\tau = ran$).

Example 4.6: Let $X = (\mathbf{C}^3, \mathbf{0})$ the germ of $\mathbf{C}^3$ at zero, $g(x) = \Sigma_{i=1}^3 x_i\sigma_i \in \mathbf{gl}(2)$, where σ_i are the Pauli matrices, $x = (x_1, x_2, x_3) \in \mathbf{C}^3$ [see Example 3.7 of Avron's paper (6)]. Then $D = (\Sigma_i x_i^2 = 0)$ is a cone in the germ X. It has an isolated singularity at its vertex $\mathbf{0} = D(\mathbf{R})$ of codimension 2 in D. The matrix $g(x)$ has two eigenvalues $e_+ = |x|$ and $e_- = -|x|$, where $|x| = (\Sigma_i x_i^2)^{1/2}$. According the Chern–Weil theory, the Chern classes of the eigenbundles E_+ and E_- of g can be calculated by integrating over a small 2-sphere $\mathbf{S}^2 \subset U(\mathbf{R})$ with the center at $\mathbf{0}$ the curvature forms of the *Berry connections* on E_+ and E_-—i.e., the connections induced by the exterior differential d on $\mathbf{C}^3$ (refs. 6 and 7). This gives $c_1(E_+) = -1 \in \mathbf{Z}$, $c_1(E_-) = 1 \in \mathbf{Z}$ (ref. 6, p. 50). COROLLARY 4.5 above and its proof imply then that this $g(x)$ cannot be analytically (and even continuously) diagonalized on any finite cover of U. This shows that the condition $codim_D N(D) \geq 3$ of (SD_3) in THEOREMS 3.5 and 4.3, and all other statements depending on them, is sharp.

Remark 4.7: In the general cituation of the beginning of section 4, the method of the Chern–Weil theory indicated in section 4.6 is applicable only when $U(\mathbf{R})$ is homotopy equivalent to a compact closed (without a boundary) orientable submanifold $M \subset U(\mathbf{R})$. For a germ X it is possible only when $U(\mathbf{R}) = X(\mathbf{R}) - x_o$, or equivalently $D(\mathbf{R}) = x_o$. The last equality is a very strong restriction on the pair (X, g). The methods and results here, including those of sections 4 and 5, do not require this restriction.

5. The Local Structure and Characteristic Classes of the Kernel Bundle

We return in this section to the general notation and assumptions of section 1.1.

5.1. Let *Ker g* be the kernel sheaf of g considered as a homomorphism of free sheaves of $\mathbb{O}_X$-modules $g : \mathbb{O}_X^n \to \mathbb{O}_X^n$. We say that x_o is *a turning or a transition point* for *Ker g* at x_o if the dimension of the fibers of *Ker g* jumps up at x_o. It means that some

MATHEMATICS

eigenvalues of g, which are not zero identically on X, vanish at x_o. We say that x_o is *a ramified turning or transition point* for *Ker g* if some of the nonzero eigenvalues vanishing at x_o ramify at x_o as the functions of x.

In this paper we consider only the case when x_o is a ramified transition point for *Ker g*. In the unramifed case a description of the local structure of *Ker g* under some natural assumptions on g follows easily from the results of ref. 4.

5.2. Assume from now on that x_o is a ramified turning point for *Ker g*. Let $e_o = 0$ be the eigenvalue of g that is identically zero on X, and m_o its multiplicity in g. Let e_j, $m_o + 1 \leq j \leq n$ be the collection of all nonzero eigenvalues of g that vanish at x_o. Then the characteristic polynomial of g over K has a form

$$p(\lambda) = det(\lambda\, I_n - g(x)) = \lambda^{m_o} q(\lambda).$$

Assume also that the polynomial $q(\lambda)$ is irreducible over K. Denote by K_q, A_q, X_q, $a_q : X_q \to X$, D, U, etc. all the objects defined or constructed for the irreducible polynomal q in the same way as in sections 3.1 and 3.2 above for p. Assume that conditions (SD_2), (RF), and (PWD/U) are satisfied. Let X_c and $b : X_c \to X_q$ be the k-germ and the finite covering constructed for $Y = X_q$ in THEOREM 2.9, $a_c = a_q \circ b : X_c \to X$, D_c, and U_c the inverse images of D and U in X_c. Denote $d_c = deg(X_c/X)$.

Let g_c be a pullback of g onto X_c, $e_{c,j}$ the k-regular eigenvalue of g_c on X_c corresponding to e_j, $m_o + 1 \leq j \leq n$, by sections 3.2 and 3.3. Put $e_c = \Pi_{j=mo+1}^{n} e_{c,j}$. Denote by Z_c the divisor of zeros of e_c in X_c, $Z = a_c(Z_c) \subset X$, and $V = X - (D \cup Z)$. Because a_c is a finite k-morphism, Z is a closed subgerm of X of codimension one, defined over k. Therefore, V is open in X and it is defined over k. Notice that by the constructions the restriction *Ker* $g|V$ of the coherent sheaf *Ker g* onto V is a locally free $\mathbb{O}_V$-module.

5.3. For a noetherian k-scheme (respectively for a k-analytic space) Z denote by $K_o(Z)$ the Grothendieck group of the category of coherent, locally free $\mathbb{O}_Z$-modules, or equivalently k-algebraic (respectively k-analytic) vector bundles on Z.

Let $Z \to H^*(Z)$ be any cohomology theory on the category of k-schemes or k-analytic spaces for which there exists the theory of Chern classes $c_r : K_o(Z) \to H^{2r}(Z)$ with the standard properties (ref. 19, Ch. 16). If $H^*(Z)$ is a vector space over a field F of characteristic zero, then the Chern character $ch(?)$ is defined with the values in $H^*(Z)$.

THEOREM 5.4. *Assume that q is an irreducible polynomial over K and that conditions* (SD_3), *(RF), (PWD/U), and (Mu1/U) for the ramified eigenvalues e_j of g are satisfied. Then*

(1) In the k-algebraic category, the restriction Ker $g|V$ is a stably trivial $\mathbb{O}_V$-module on V. In particular, if $rank_{\mathbb{O}_V}(Ker\ g|V) > dim\ X$, then Ker $g|V$ is a free $\mathbb{O}_V$-module on V.

(2) in the k-analytic category, the d_c-multiple $d_c(Ker\ g|V)$ of the restriction Ker $g|V$ is a stably trivial $\mathbb{O}_V$-module on V. In particular, if rank $d_c(Ker\ g|V) > dim\ X$, then $d_c(Ker\ g|V)$ is a free $\mathbb{O}_V$-module on V.

An outline of Proof: Let $E_{c,j}$ be the eigenbundle of the eigenvalue $e_{c,j}$ on X_c, $m_o + 1 \leq j \leq n$. Denote $V_c = a_c^{-1}(V)$. Then we have the following relation on V_c:

$$(Ker\ g_c|V_c) \ \oplus \ (\oplus_{j=m_o+1}^{n} E_{c,j}) = \mathbb{O}_{V_c}^{n}.$$

Because all the eigenbundles $E_{c,j}$, are trivial on U_c, the equality above shows the stable triviality of *Ker* $g_c|V_c$. Using the transfer arguments as in the proof of THEOREM 4.3, we can derive from this the stable triviality of $d_c(Ker\ g|V)$ on V in the both k-algebraic and k-analytic cases. But in the k-algebraic case the group $K_o(V) = \mathbf{Z}$ has no torsion, so *Ker* $g|V$ is stably trivial on V itself. The stable rank of $\mathbb{O}_V$ is $\leq dim\ V$ (20). Combining these facts we see that if $rank_{\mathbb{O}_V}(Ker\ g|V) > dim\ V$, it is free on V.

COROLLARY 5.5. *Under the conditions of 5.4 we have:*

(1) In the k-algebraic category the (integral) Chern classes $c_r(Ker\ g|V)$ of the bundle Ker $g|V$ vanish, for all $r > 0$.

(2) In the k-analytic category all the Chern classes $c_r(d_c(Ker\ g|V))$ of the bundle $d_c(Ker\ g|V)$ in $H^{2r}(V)$ vanish, for all $r > 0$.

(3) If $H^*(V)$ is a vector space over a field F of characteristic zero, then the (rational) Chern character $ch(Ker\ g|V)$ of Ker $g|V$ in $H^*(V)$ also vanishes.

I am grateful to Professors S. Bloch, O. Gabber, I. M. Gelfand, M. Gromov, D. Kazhdan, and B. Mazur for their interest in this work and for very useful conversations and correspondence. This work was done while I was employed by the Clay Mathematics Institute (CMI) and was visiting the Institut de Mathematique de Jussieu (IMJ). I express my deep gratitude to CMI for the financial support of this work and to the IMJ for providing an opportunity to use their research facilities, which made this work physically possible.

1. Grauert, H., Peternell, T. & Remmert, R. (1994) *Encyclopaedia of Mathematical Sciences, Several Complex Variables VII* (Springer, Berlin), Vol. 74.
2. Friedrichs, K.-O. (1965) *Perturbation of Spectra in Hilbert Space*, Lectures in Applied Mathematics (AMS, Providence), Vol. III.
3. Kato, T. (1995) *Perturbation Theory of Linear Operators*, Classics in Mathematics (Springer, Berlin), 4th Ed.
4. Nisnevich, Ye. A. (1998) *Internatl. Math. Res. Notices* **10,** 511–526.
5. Rellich, F. (1969) *Perturbation Theory of Eigenvalue Problems* (Gordon and Breach, New York).
6. Bregola, U., Marmo, G. & Morandi, G., eds. (1990) *Anomalies, Phases, Defects*, Proceedings of the Ferrara Summer School on Theoretical Physics, (Bibliopolis, Napoli).
7. Shapere, A. & Wilczek, F., eds. (1989) *Geometric Phases in Physics*, Advanced Series in Mathematical Physics (World Scientific, Teaneck, NJ), Vol. 5.
8. Grothendieck, A. (1971) *Revétements Etales et Groupes Fundamental,* Lecture Notes in Mathematics (Springer, Berlin), Vol. 224.
9. Grothendieck, A. & Murre, J. (1971) *The Tame Fundamental Group of a Formal Neighbourhood of a Divisor with Normal Crossings on a Scheme*, Lecture Notes in Mathematics (Springer, Berlin), Vol. 208.
10. Artin, M. & Mazur, B. (1965) *Etale Homotopy*, Lecture Notes in Mathematics (Springer, Berlin), Vol. 100.
11. Artin, M., Grothendieck, A. & Verdier, J.-L. (1972) *Théories des Topos et Cohomologie Etale de Schémas*, Lecture Notes in Mathematics (Springer, Berlin), Vol. 305.
12. Le, D. T. & Saito, K. (1984) *Arkiv foer Mathematik* **22,** 1–24.
13. Milnor, J. (1966) *Singularities of Complex Hypersurfaces*, Annals of Mathematics Studies (Princeton Univ. Press, Princeton), Vol. 61.
14. Avramov, L. (1975) *Doklady Acad. Nauk SSSR* **225,** 11–14.
15. Serre, J.-P. (2000) *Local Algebra*, Springer Monographs in Mathematics (Springer, Berlin).
16. Grothendieck, A. (1968) *Cohomologie Locale des Faisceaux Cohérentes et Théorèmes de Lefschetz Locaux et Globaux*, Séminaire de Géométrie Algébriques du Bois Marie, 1962 (SGA 2, Amsterdam).
17. Lipman, J. (1979) *Am. J. Math.* **101,** 203–211.
18. Guaraldo, F., Macri, P. & Tancredi, A. (1986) *Topics on Real Analytic Spaces*, Advanced Lectures in Mathematics (Vieweg and Sons, Braunsweig, Germany).
19. Husemoller, D. (1994) *Fibre Bundles*, Graduate Texts in Mathematics (Springer, New York), Vol. 20, 3rd Ed.
20. Bass, H. (1968) *Algebraic K-theory* (Benjamin, New York).