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THE COMPLETELY DECOMPOSED TOPOLOGY ON SCHEMES AND ASSOCIATED DESCENT SPECTRAL SEQUENCES IN ALGEBRAIC K-THEORY

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To Alexander Grothendieck on his 60th birthday.

ABSTRACT. Let X be a noetherian scheme of finite Krull dimension. A new Grothendieck topology on X, called the completely decomposed topology, is introduced, and the formalism of the corresponding cohomology and homotopy theories is developed. This formalism is applied to construct certain descent (or local-to-global) spectral sequences convergent to various algebraic K-groups of X, or to the homotopy groups of more general spectra. They refine the well-known Brown-Gersten spectral sequences.

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0. INTRODUCTION

This paper has two goals. The first of them is to give a definition and a systematic treatment of some basic properties of the completely decomposed topology on schemes. Our second goal is to use this topology to construct a descent (or local-to-global) spectral sequence for the K-theory of coherent sheaves which refines the well known Brown-Gersten spectral sequence [8].

The cd-topology has been introduced by the author in [28] - [30] as a tool for a study of the arithmetic and (non-abelian) étale cohomology of affine group schemes over Dedekind rings. The definition of this topology has been motivated by its intimate connections with several classical adelic constructions associated with such group schemes, in particular, with their adele groups and adele class groups. Later the cd-topology and the associated cohomology theory have been used by K. Kato and S. Saito for the study of high-dimensional arithmetic schemes, i.e. schemes proper and of finite type over $\mathbb{Z}$ or over a finite field. In particular, these tools have been used for a study of arithmetic of such schemes (a generalization of the idele class group [25], conjectures on special values of L-functions [47]), their Class Field Theory (generalizations of the Artin reciprocity [25] and of the Moore uniqueness theorem [50]), and various cohomological questions (the theory of Brauer groups [51], relationships of the class groups and the Class Field Theory with the motivic cohomology theory [50], existence of which have been conjectured by Beilinson [40] and Lichtenbaum [48]).

However, a systematic treatment of the basic properties of the cd-topology and the associated cohomology theory are still lacking in the literature, and we shall try to fill this gap partially in §1 of this paper. The content of 1.1-1.22 is essentially extracted from our Harvard thesis [30] (1982, unpublished). More recent in this presentation of the theory is a new notion of a point in a topos which is more general than that used in [SGA 4] and is more convenient for our purposes. The theory of the local cd-cohomology and their excision properties have been used in the proofs of [28], [29], [25], [50] but are developed systematically here for the first time.

The rest of this paper is devoted to applications of the technique developed in §1 to a study of the descent problem in Algebraic K-theory.

0.2. Let X be a noetherian scheme of finite Krull dimension, $K_n(X)$ (resp. $G_n(X)$) the Quillen K_n-groups of the category $LF(X)$ of coherent locally free (resp. $Coh(X)$ of coherent) sheaves of $\mathcal{O}_X$-modules on X. $K_n(X,\mathbb{Z}/\ell\mathbb{Z})$ (resp. $G_n(X,\mathbb{Z}/\ell\mathbb{Z})$) the K_n-groups with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients of the category $LF(X)$ (resp. $Coh(X)$), where ℓ is an integer. Let $\tilde{K}_n^{et}$ (resp. $\tilde{G}_n^{et}$) be the sheaf on the etale site X_{et} obtained by sheafifying the presheaf $K_n : Y \longrightarrow K_n(Y)$ (resp. $G_n : Y \longrightarrow G_n(Y)$) on etale topology on X; let $\tilde{K}_n^{et}(\mathbb{Z}/\ell\mathbb{Z})$ and $\tilde{G}_n^{et}(\mathbb{Z}/\ell\mathbb{Z})$ be the etale sheaves on X corresponding to $K_n(X, \mathbb{Z}/\ell\mathbb{Z})$ and $G_n(X,\mathbb{Z}/\ell\mathbb{Z})$ respectively.

0.3. One of the most promising directions in the Algebraic K-theory is a study of its relationships with etale cohomology. Based on an analogy with the approach used in the Algebraic Topology for a study of generalized homology theories [1], [37], especially with the Atiyah-Hirzebruch spectral sequence relating topological K-theory and singular homology [3]. Quillen [34] and Lichtenbaum [13], [26] formulated the following conjectures (we state the first of them in a later, corrected form):

0.4. Conjecture ([13], [26], [34]): Assume that X is regular. Then there exists a descent spectral sequence with the E_2-term

0.4.1. $E_2^{p,q} = H^p(X_{et}, \tilde{K}_q^{et}(\mathbb{Z}/\ell\mathbb{Z}))$, $p \geq 0$, $q \geq 0$,

which converges to $K_{q-p}(X,\mathbb{Z}/\ell\mathbb{Z})$ for $q-p > 2\cdot\mathrm{coh.dim}_\ell\,(X_{et})$, where $\mathrm{coh.dim}_\ell\,(X_{et})$ is the etale cohomological ℓ-dimension of X.

(We assume here and everywhere below that $K_n(X,\mathbb{Z}/\ell\mathbb{Z}) = G_n(X,\mathbb{Z}/\ell\mathbb{Z}) = 0$ if $n < 0$. The indexation of the terms of this and other spectral sequences in this paper follows to that of Bousfield-Kan [5], [38] (see §2.21 for details)).

Notice, that as it is well known, the spectral sequence does not converge to $K_n(X,\mathbb{Z}(\ell\mathbb{Z}))$ for small n already when $X = \mathrm{Spec}\,k$, where k is a field.

The second conjecture of Quillen-Lichtenbaum, proved recently by joint efforts of Suslin, Gabber, Gillet and Thomason [35], [36], [12], [20], asserts that the sheaf $\tilde{K}_n^{et}(\mathbb{Z}/\ell\mathbb{Z})$ is constant, and equal to $\mathbb{Z}/\ell\mathbb{Z}(i)$, if $n = 2i$, and to zero, if $n = 2i+1$.

Thus, combining this result with conjecture 0.4, if it is true, one would have effective tools for a study and computations of

$K_n(X,\mathbb{Z}/\ell\mathbb{Z})$ for sufficiently big n. Notice, however, that the situation for $n \leq 2\cdot\mathrm{coh.dim}_\ell\,(X_{et})$ and also for a singular X is left open by Conjecture 0.4 in its current form.

The best known and frequently used approximation to Conjecture 0.4 is the Brown-Gersten spectral sequence on the Zariski topology:

0.4.2. $E_2^{p,q} = H^p(X_{Zar}, \tilde{G}_q^{Zar}) \Rightarrow G_{q-p}(X)$, $p \geq 0$, $q \geq 0$, $q-p \geq 0$.

and a similar spectral sequence for $G_n(\mathbb{Z}/\ell\mathbb{Z})$ [8], [38][1]. If X is regular, we can replace $G_n(X)$ on $K_n(X)$ and $G_n(\mathbb{Z}/\ell\mathbb{Z})$ on $K_n(\mathbb{Z}/\ell\mathbb{Z})$ in these spectral sequences respectively.

Unfortunately, the sheaves $\tilde{K}_n^{Zar}$, $\tilde{K}_n^{Zar}(\mathbb{Z}/\ell\mathbb{Z})$, $\tilde{G}^{Zar}$ and $\tilde{G}_n^{Zar}(\mathbb{Z}/\ell\mathbb{Z})$ on X_{Zar} are very complicated, and this makes direct computations with the Brown-Gersten spectral sequence usually impossible. However, some information on the cohomology of the sheaves $\tilde{K}_n^{Zar}$, $\tilde{K}_n^{Zar}(\mathbb{Z}/\ell\mathbb{Z})$ can be obtained from the Gersten (or Cousin in the terminology of Grothendieck [24]) resolutions of these sheaves if X

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1 We are not discussing here works of Thomason [38] and Friedlander [10] in which descent spectral sequences on etale topology have been constructed for different although related K-theories $K_n(X,\mathbb{Z}/\ell\mathbb{Z})[\beta^{-1}]$ (the Bott periodized algebraic K-theory [74]) and $K_n^{top}(X,\mathbb{Z}/\ell\mathbb{Z})$ (the etale topological K-theory [10]). Relationships of these theories with the Quillen K-theory are the subject of many current works and conjectures [11], [39], [40], [48], [73] - [78]. In particular, Conjecture 0.4 can be formulated in terms of such relationships.

satisfies the assumptions under which the Gersten conjecture is currently proved, i.e. if X is smooth and of finite type over a field or (in the case of finite coefficients) over a discrete valuation ring.

0.5. In this paper we shall construct a local-to-global (or descent) spectral sequence on the cd-topology X_{cd} of X defined in §1:

0.5.1. $E_2^{p,q} = H^p(X_{cd}, \tilde{G}_q^{cd}) \Rightarrow G_{q-p}(X)$, $p \geq 0$, $q \geq 0$, $q-p \geq 0$,

and its analogues $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ for $G_n(X,\mathbb{Z}/\ell\mathbb{Z})$. Actually, our construction gives similar spectral sequences for the homotopy groups $\pi_2(F(X))$ of a more general class of presheaves $F: X_{cd} \longrightarrow FSp$ with values in the category FSp of fibrant spectra (in the sense of Homotopy Theory) which are additive and satisfy certain cd-excision property (see §2 - §4 for precise definitions and results). By a result recently announced by Thomason and Trobaugh [62], there exist Bass type extensions $\underline{K}^B(X)$ and $\underline{K}^B(\mathbb{Z}/\ell\mathbb{Z})$ of the connective K-theory spectra $\underline{K}(X)$ and $\underline{K}(\mathbb{Z}/\ell\mathbb{Z})$ of the category LF(X) of locally free sheaves onto negative degrees which are additive and satisfy this cd-excision condition. Therefore, there exist variants of spectral sequences 0.5.1 and $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ for their homotopy groups $K_n^B(X)$ and $K_n^B(\mathbb{Z}/\ell\mathbb{Z})$ (see 3.8, 4.5 for details). The version of $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ for $K_n^B(\mathbb{Z}/\ell\mathbb{Z})$ combined with the computation of the fibres of the sheaf $\tilde{K}_n^{cd}(\mathbb{Z}/\ell\mathbb{Z})$ described below (and in Lemma 4.6), and the comparison theorem of [38] for fields imply an extension of the global comparison theorem $K_n(X,\mathbb{Z}/\ell\mathbb{Z})[\beta^{-1}] \xrightarrow{\sim} K_n^{top}(X,\mathbb{Z}/\ell\mathbb{Z})$ of [38] to singular schemes X (see [62]).

If X is regular the spectral sequences for G_n and K_n^B (respectively for $G_n(\mathbb{Z}/\ell\mathbb{Z})$ and $K_n^B(\mathbb{Z}/\ell\mathbb{Z})$) coincide.

The cd-topology is stronger than the Zariski topology, but weaker than etale topology in X. Hence, spectral sequence 0.5.1 and its variants refine the Brown-Gersten spectral sequence and can be considered as a step toward problem 0.4 of the etale localization of K-groups in which we restrict our attention only to the geometric etale extensions, i.e., extensions with the fixed residue fields; but they include G_n and K_n^B, for all $n \geq 0$. Moreover, the existence of spectral sequence 0.5.1 and its variants show that the only obstructions to the existence of the etale descent for the G_n- and K_n^B-sheaves are coming from the residue fields. (The last fact for a regular X and from a different point of view has been obtained in [38]).

Although the sheaf $\tilde{K}_n^{cd}(\mathbb{Z}/\ell\mathbb{Z})$ is not constant as $\tilde{K}_n^{et}(\mathbb{Z}/\ell\mathbb{Z})$, its fibres can be easily computed in terms of groups $K_n(k',\mathbb{Z}/\ell\mathbb{Z})$ of all finite etale extensions k' of the residue fields k(x) of X (see Lemma 4.6). Thus, our approach gives an opportunity to reduce directly various questions concerning $K_*(X,\mathbb{Z}/\ell\mathbb{Z})$ for possibly singular X to the corresponding questions for $K_*(k',\mathbb{Z}/\ell\mathbb{Z})$ for all finite etale extensions k' of the residue fields k(x), for all $x \in X$, avoiding any use of the Gersten conjecture and the Gersten resolution and the restrictions which the current status of the Gersten conjecture imposes.

Notice also, that the Gersten-Cousin complex for a singular X is not exact, and, thus, the methods based on it and on the Brown-Gersten spectral sequence are not applicable to singular X in principle.

Unfortunately, the current knowledge of K-theory of fields [27], [77], [78] gives only a quite restricted opportunity to use our spectral sequence for direct computations of $K_i(X,\mathbb{Z}/\ell\mathbb{Z})$, $i \leq 2$. But any progress in the understanding of K-theory of fields will increase its applicability.

0.6. The construction of spectral sequence 0.5.1 given in this paper is based on a suitably generalized method of the construction of the Brown-Gersten spectral sequence outlined by Thomason in ([38], §2). Thomason's construction combines elements of the original Brown-Gersten construction [8] and the ideas of Grothendieck ([23], II; [24]) and Quillen [33] on a use of the filtration by the codimension of points of X. The theory of local presheaves of spectra on the Zariski topology, and a notion of the hypercohomological spectrum of a presheaf of spectra are the main new tools used in [38] to combine the two approaches mentioned above (see also §§4.7, 4.8 for further comments on this method).

The first mentioned theory is a spectrum level version of the theory of local homotopy and homology developed by Grothendieck in [SGA 2] and [SGA 4]. The second notion is a spectrum level version of the hypercohomological complex of a complex of sheaves in the derived category of complexes. In this paper we shall develop both of these tools in the context of presheaves of spectra on the cd-topology.

In §2 we give the definition and study some properties of the

hypercohomological spectrum $\mathbb{H}(X_{cd},F)$ corresponding to a presheaf of spectra F on X_{cd} in a form used in this paper. In §3 we develop the theory of presheaves of local spectra $\underline{\Gamma}_x(F)$ on the cd-topology for a point $x \in X$ and presheaves of spectra F which have a cd-excision property. The theory is more complicated than the corresponding theory on the Zariski topology outlined in [38], because the cd-presheaves $\underline{\Gamma}_x(F)$ are not constant on the closure $\overline{x}$ of x as in the case of the Zariski topology, and the proof of the acyclicity of the associated sheaves $\tilde{\underline{\Gamma}}_x(F)$ on X_{cd} given in §4 required some extra efforts. This proof is close in its spirit with the proof of the acyclicity of the adelic resolutions given in ([29], [30], Ch. I). The weak homotopy eqivalence $F(X) \xrightarrow{\sim} \mathbb{H}(X_{cd},F)$ underlining the descent spectral sequence for F is proved by induction based on the acyclicity of the sheaves $\tilde{\underline{\Gamma}}_x(F)$ for all $x \in X$ in §4. §4 is concluded by some further comments and conjectures.

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The author thanks H. Gillet and R. Thomason for discussions of their papers [16], [38] which triggered his interest in the descent problem in K-theory and which were the starting points of the K-theoretical part of this paper. He also thanks J.F. Jardine for a careful reading of an early version of this paper and valuable critical remarks and suggestions.

This paper is dedicated to Alexander Grothendieck who created the brave new world in the realm of which this work became possible.

1. THE COMPLETELY DECOMPOSED TOPOLOGY

1.0. In this section we shall develop the formalism of the cd-topology and the cd-cohomology following the general scheme of [SGA 4]. Due to limitations of space and time we tried to include here only most basic facts which are used in this paper or in other works and/or which are specific features of the cd-topology. Between them are a more general than that in [SGA 4] theory of points in a topos, the local cohomology theory and its excision properties, the behaviour of the cd-cohomology and the local cd-cohomology under limits, and an estimate on the cohomological dimension of the topos X_{cd} (Theorem 1.32). The local cohomology and the excision properties for the cd-topology have been used frequently in [28] - [32] and later in [25], [50] but have not been developed systematically with the necessary completeness and/or details in any of these papers. This theory, developed in 1.23-1.31, can also be considered as an introduction into its homotopy-theoretical version developed in §3. The estimate on the cohomological dimension of X_{cd} is due to Kato and Saito [25] but we included it here with a detailed proof because the proof of this important fact is only

indicated in [25].

1.1. Let X be a schema, Et/X the category of all schemes etale over X.

For an etale morphism $\varphi: X' \longrightarrow X$ consider the following subset $cd(X'/X)$ of X:

1.1.1. $cd(X'/X) = \{x \in X | \exists\, x' \in \varphi^{-1}(x) \mid \varphi^{a}(k(x)) \xrightarrow{\sim} k(x')\}$,

where φ^{a} is the canonical map of the residue fields induced by φ. If X'/X is a Galois extension, the condition above means that x is completely decomposed (or split) in X' in the classical terminology of the Number Theory. This explains our notations and terminology.

For each $X' \in Ob(Et/X)$ consider the category $Cov_{cd}(X')$ of coverings on X' which consists of all families $\{\varphi_i: X'_i \longrightarrow X',\ i \in I\}$ of etale morphisms φ_i such that

1.1.2 $\bigcup_{i \in I} cd(X'_i/X') = X'$

The correspondence $X' \longrightarrow Cov_{cd}(X')$ satisfies all of the axioms for a pretopology ([SGA 4], II, 1.3; [SGA 3], IV, §6) and, hence, it defines a Grothendieck topology on Et/X which we shall call the *completely decomposed topology* or, more shortly, the *cd-topology*.

Denote by X_{cd} the corresponding site, i.e. the category Et/X equipped with the cd-topology, and by X_{cd} the topos of sheaves on X_{cd}.

1.2. Examples. (1) $\dim X = 0$. Let k be a field and set $X = \operatorname{Spec} k$. Then the category X_{cd} consists of the spectra $\operatorname{Spec} A$ of finite etale k-algebras A. Any such k-algebra is a finite direct sum $A = \bigoplus_{i=1}^{k} L_i$ of finite separable field extensions L_i of k. This fact and condition 1.1.2 imply that for any finite separable extension k'/k, and cd-covering $Y \longrightarrow X' = \mathrm{Spec}(k')$ has the form

1.2.1. $Y = \mathrm{Spec}\, L_0 \amalg \mathrm{Spec}\, L_1 \amalg \ldots \amalg \mathrm{Spec}\, L_k$

(the disjoint sum), where $L_0 = k'$ and L_i/k' is a finite separable extension of k', $1 \leq i \leq k$. Any such covering can be refined by the trivial covering $\mathrm{Id}_{X'}: X' \xrightarrow{\sim} X'$.

Let $F: X_{cd} \longrightarrow (\mathrm{Sets})$ be a presheaf of sets on X_{cd}. It is known ([SGA 4], II, 2.4) that F is a sheaf if and only if for any $X' \in \mathrm{Ob}(X_{cd})$, and any covering $\{X_i' \longrightarrow X', i \in I\}$ the sequence of sets

1.2.2. $F(X') \xrightarrow{\alpha} \prod_{i \in I} F(X_i') \rightrightarrows \prod_{(i,j) \in I^2} F(X_i' \times_X X_j)$

is exact in the first and the second terms (in particular, α is injective).

On the coverings in the form 1.2.1 the exactness of sequence 1.2.2 is equivalent to the bijectivity of the canonical map

1.2.3. $F(X_0' \amalg X_1' \amalg \cdots \amalg X_k') \xrightarrow{\sim} \prod_{i=0}^{k} F(X_i')$

induced by the canonical inclusions $X_i \longrightarrow \coprod_{i=0}^{k} X_i$, $0 \leq i \leq k$, where $X_i' = \mathrm{Spec}\, L_i$, $1 \leq i \leq k$ and $X_0' = X'$. Since they are the only coverings in $\mathrm{Cov}_{cd}(X')$ we see that the presheaf F is a sheaf if it is *additive* in the sense of the following definition (see for example, [38], 1.52):

1.2.4. <u>Definition</u>. Let C be a site with finite coproducts $\amalg$. A presheaf $F: C \longrightarrow (\mathrm{Sets})$ is called *additive* if for any $X_1, X_2 \in \mathrm{Ob}(C)$ the canonical map

1.2.5. $F(X_1 \amalg X_2) \longrightarrow F(X_1) \times F(X_2)$

induced by the canonical embeddings $X_i \longrightarrow X_1 \amalg X_2$, $i = 1,2$, is bijective.

Notice that condition 1.2.2 shows that any sheaf of sets on C is additive.

Return now to our example. The characterization of sheaves on X_{cd} given above (or the construction of the sheaf associated with a presheaf given in [SGA 4], II, §3) shows that the canonical map $F \longrightarrow \tilde{F}$ of a presheaf $F: X_{cd} \longrightarrow (\mathrm{Sets})$ into its sheafification $\tilde{F}$ on X_{cd} induces a bijectivity on global sections:

1.2.6. $\Gamma(X',F) \xrightarrow{\sim} \Gamma(X'\tilde{F})$

for any irreducible $X' \in \mathrm{Ob}(X_{cd})$, i.e. the spectrum $X' = \mathrm{Spec}\, k' \in \mathrm{Ob}(X_{cd})$ of an etale field extension k'/k. Hence, the functor of global sections $F \longrightarrow \Gamma(X',F)$ is exact for any such X'. Since any $Y \in \mathrm{Ob}(X_{cd})$ is a finite disjoint union of irreducibles $Y_i = \mathrm{Spec}\, L_i$ with L_i as in 1.2.1, the functor of sections

1.2.7. $\Gamma(Y,-): F \longrightarrow \Gamma(Y,F)$, $F \in \mathrm{Ob}(X_{cd})$

is exact for any $Y \in \mathrm{Ob}(X_{cd})$.

(2) $\dim X = 1$. Let $X = \mathrm{Spec}\, R$ be the spectrum of an integral noetherian one-dimensional ring R. It follows from condition 1.1.2 on

the generic point η of X that any cd-covering $\{\varphi_i: X_i \longrightarrow X,\ i \in I\}$ must contain an open immersion $X_{i_0} \hookrightarrow X$. Let $X - X_{i_0} = \{x_1 \cup x_2 \cup \cdots \cup x_r\}$, where x_α, $1 \le \alpha \le r$ are closed points of X. Then the covering also contains a family of etale extensions

$$\varphi_\alpha: X_{i_\alpha} \longrightarrow X, \quad 1 \le \alpha \le r$$

such that $x_{i_\alpha} \in cd(X_{i_\alpha}/X)$, $1 \le \alpha \le r$. (Some of φ_{i_α}'s also might be open immersions and some of them might coincide).

In a similar way one can construct inductively cd-coverings of a noetherian scheme X of a finite Krull dimension > 1. Notice, that the coverings constructed in this way are finite.

1.3. Assume that X is a noetherian scheme. Then any etale X-scheme* $f: Y \longrightarrow X$ is also noetherian, f is an open map ([SGA 1], IV, 6.6) and any subscheme $Y' \hookrightarrow Y$ is noetherian and therefore has a finite number of irreducible components. It is easy to see using the inductive method of construction of cd-coverings of Y indicated in 1.2(2) that under the noetherianess assumption any cd-covering $\{f_i: Y_i \longrightarrow Y,\ i \in I\}$ contains a finite cd-subcovering.

<u>Functorial properties of the cd-topology</u>

1.4. Let $f: Y \longrightarrow X$ be a morphism of schemes. It is easy to see from the definitions of 1.1 that the functor "inverse image"

1.4.1. $f^*: X' \longrightarrow Y \times_X X' = Y'$

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* of finite type

induces the functor

1.4.2. $f^*: X_{cd} \longrightarrow Y_{cd}$

which commutes with finite inverse limits and transforms a cd-covering $\{\varphi_i: V_i \longrightarrow X',\ i \in I\}$ of $X' \in Ob(X_{cd})$ into the cd-covering $\{\varphi_{i,Y'}: V_i \times_{X'} Y' \longrightarrow Y',\ i \in I\}$ of Y'. Hence, the functor f^* is continuous ([SGA 4], III, 1.6) and it defines the morphism of the sites

1.4.3. $f_{cd}: Y_{cd} \longrightarrow X_{cd}$

([SGA 4], IV, 4.9.2). Therefore, f_{cd} defines the morphism of topoi of sheaves

1.4.4. $\tilde{f}_{cd} = (f^*_{cd}, f^{cd}_*): \tilde{Y}_{cd} \longrightarrow \tilde{X}_{cd}$,

where f^{cd}_* is the functor "direct image" of sheaves

1.4.5. $f^{cd}_*: \tilde{Y}_{cd} \longrightarrow \tilde{X}_{cd}$, $f^{cd}_*(F) = F \circ f^*$, for all $F \in Ob(\tilde{Y}_{cd})$

and the functor "inverse image"

1.4.6. $f^*_{cd}: \tilde{X}_{cd} \longrightarrow \tilde{Y}_{cd}$

is defined as the left adjoint functor to f^{cd}_*. (The existence of the left adjoint functor to f^{cd}_* follows from the general results of the Category Theory (see, for example, [49], Ch. II, prop. 2.2)).

We often shall consider also the direct and inverse image functors in the categories of *presheaves*

1.4.7. $f^{cd}_\#: \hat{Y}_{cd} \longrightarrow \hat{X}_{cd}$, $f^{cd}_\#(F) = F \circ f^*$, $\forall F \in Ob(\hat{Y}_{cd})$

and

1.4.8. $f^{\#}_{cd}: X^{\wedge}_{cd} \longrightarrow Y^{\wedge}_{cd}$

respectively associated with f, where $Y^{\wedge}_{cd}$ and $X^{\wedge}_{cd}$ are the categories of presheaves on Y_{cd} and X_{cd} respectively.

The functor $f^{\#}$ is again defined as the left adjoint to $f_{\#}$. Recall that for a *sheaf* F on X_{cd}

1.4.9. $f_{\#}(F) = f_{*}(F)$, $f^{*}(F) = (f^{\#}(F))^{\sim}$,

where $(f^{\#}(F))^{\sim}$ is the sheafification of $f^{\#}(F)$ on Y_{cd} (see ([49], ch. II, 2.7 and p. 68) for these facts in a more general situation).

We often will drop the lower and upper indices "cd" and denote f^{cd}_{*}, $f^{cd}_{\#}$, f^{*}_{cd} and $f^{\#}_{cd}$ simply as f_{*}, $f_{\#}$, f^{*} and $f^{\#}$ respectively where it does not cause a confusion.

<u>Generalized points in the cd-topos</u>

Working with the cd-topology it is convenient to use a more general notion of a point in a topos than that used in ([SGA 4], IV).

1.5. <u>Definition</u>: Let S be a site, $S^{\sim}$ the topos of the sheaves on S.

(a) We say that the site S and the topos $S^{\sim}$ are *acyclic* if for any $X \in Ob(S)$ the functor of X-sections

$$\Gamma(X,-): F \longrightarrow \Gamma(X,F)$$

is exact on $S^{\sim}$.

(b) Assume that the category S has a terminal object X_0. We say that the site S and the topos $S^{\sim}$ are *connected* if $\Gamma(X_0, M_S) = M$ for any set M and the constant sheaf M_S on S associated with M.

1.5.1. <u>Remark</u>. For a sheaf of groups (resp. abelian groups) F on S condition (a) implies the vanishing of its cohomology $H^1(X,F) = 0$ (resp. $H^i(X,F) = 0$, $i > 0$, for all $X \in Ob(S)$. This explains the term "acyclic" in Definition 1.5 (a).

1.6. <u>Definition</u>. Let $\mathcal{T}$ be a topos. A pair (ρ, α_ρ), consisting of an acyclic connected site ρ with a terminal object and a morphism of topoi $\alpha_\rho: \rho^{\sim} \longrightarrow \mathcal{T}$ is called a point of $\mathcal{T}$.

The usual notions of a conservative family of points, the stalk of a sheaf at a point, the Godement resolution, etc. can be extended to points in the sense of Definition 1.6.

We shall discuss below these notions in the case of the cd-topology.

1.7. Let X be a scheme. We shall identify a point $x \in X$ with the spectrum $\mathrm{Spec}\, k(x)$ of its residue field $k(x)$. Let $i_x: x = \mathrm{Spec}\, k(x) \hookrightarrow X$ be the canonical embedding. Then i_x induces morphisms of the corresponding cd-sites and of the topoi of sheaves:

1.7.1. $i_{x,cd}: x_{cd} \longrightarrow X_{cd}$,

1.7.2. $i^{\sim}_{x,cd}: x^{\sim}_{cd} \longrightarrow X^{\sim}_{cd}$,

(see 1.4).

By 1.2(1) the functor $\Gamma: F \longrightarrow \Gamma(x',F)$ of global sections is exact on $x^{\sim}_{cd}$, for all $x' \in Ob(x_{cd})$, i.e. x_{cd} and $x^{\sim}_{cd}$ are acyclic in the sense of Definition 1.5(a). On the other hand x_{cd} and

$\tilde{x}_{cd}$ are obviously connected. Hence, the pair $(x_{cd}, \tilde{i}_{cd})$, with $\tilde{i}_x: \tilde{x}_{cd} \to \tilde{X}_{cd}$ induced by i_x, is a "point" of the topos $\tilde{X}_{cd}$ in the sense of Definition 1.6.

<u>Neighborhoods of points</u> x_{cd}, $x \in X$

1.8. <u>Definition</u>: Let $x \in X$ be a point of X. An etale X-scheme $\varphi: U \to X$ is called a *neighborhood of the point* x_{cd} of the site X_{cd} (or of the point x in the cd-topology) if $x \in cd(U/X)$ *.

Recall, that this condition means that there exists a point $i_y: y \hookrightarrow U$ such that $\varphi(y) = x$, φ induces an isomorphism of the residue fields $\varphi_y^a: k(x) \xrightarrow{\sim} k(y)$, and the following diagram is commutative:

$$\begin{array}{ccc} y = \operatorname{Spec} k(y) & \overset{i_y}{\hookrightarrow} & U \\ \varphi_y \downarrow \wr & & \downarrow \varphi \\ x = \operatorname{Spec} k(x) & \underset{i_x}{\hookrightarrow} & X \end{array}$$

(Compare with the notion of ξ-punctured etale neighborhoods in ([SGA 4], VIII, 4), where ξ is a "geometric" point of X).

More generally, let $Z \hookrightarrow X$ be a subscheme of X, $Z_U = \varphi^{-1}(Z)$. We shall call an etale scheme $\varphi: U \to X$, a *cd-neighborhood* of Z if there exists a subscheme $Z' \subset Z_U$ such that φ induces an isomorphism $\varphi|_{Z'}: Z' \xrightarrow{\sim} Z$.

Denote by $N_{cd}(Z,X)$ the category of all cd-neighborhoods of Z in X. When no confusion can arise we shall write simply $N_{cd}(Z)$ instead of $N_{cd}(Z,X)$.

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* See the Note Added in Proof on page 342.

1.9. Let $U \hookrightarrow X$ be an affine open subscheme of X containing x, $N_{cd}^U(x)$ the subcategory of $N_{cd}(x)$ consisting of all cd-neighborhoods of x, affine over U. As in ([SGA 4], VII, 4.5), we can see that the categories $N_{cd}(x)$ and $N_{cd}^U(x)$ are pseudo-filtered. $N_{cd}^U(x)$ is confinal in $N_{cd}(x)$, and consists of affine schemes. Recall also that

1.9.1. $$\varprojlim_{X' \in Ob(N_{cd}^U(x)^o)} X' = \operatorname{Spec} \mathcal{O}_{x,X}^h$$

where $\mathcal{O}_{x,X}^h$ is the henselization of the local ring $\mathcal{O}_{x,X}$ of x on X with respect to its maximal ideal $m_x \subset \mathcal{O}_{x,X}$ ([EGA], IV, 18.6.5), and $N_{cd}^U(x)^o$ is the category dual to $N_{cd}^U(x)$.

<u>Stalks (or fibres) of a sheaf on X_{cd}</u>

1.10. Let $x \in X$ be a point of X, $(x_{cd}, \tilde{i_x})$ the corresponding point of the topos X_{cd}, F a presheaf on X_{cd}. Define the *presheaf-stalk* F_x^p of F at x_{cd} as its *presheaf* inverse image on x_{cd}:

1.10.1. $F_x^p = i_x^{\#}(F)$.

Define the *sheaf-theoretical inverse image* $i_x^{*}(F)$ of the *presheaf* F *on* x_{cd} and the *sheaf-stalk* (or simply the *stalk*) F_x of F at x_{cd} as the sheafification of the presheaf F_x^p:

1.10.2. $F_x \overset{=}{\underset{def}{=}} i_x^{*}(F) \overset{=}{\underset{def}{=}} (F_x^p)^{\sim}$.

We shall see in Proposition 1.11 (4) below that if $\tilde{F}$ is the sheafification of a presheaf F on X_{cd}, and $u: F \to \tilde{F}$ is the canonical map, then the natural map

1.10.3. $u_x: F_x \xrightarrow{\sim} (\tilde{F})_x$

is an isomorphism, for all $x \in X$.

Let $x' \in Ob(x_{cd})$ and $M(x', X)$ be the category of $V \in Ob(X_{cd})$ such that there exists a morphism $g_V: x' \longrightarrow V$ which makes the diagram

1.10.4
$$\begin{array}{ccc} x' & \xrightarrow{g_V} & V' \\ \downarrow & & \downarrow \\ x & \xhookrightarrow{i_x} & X \end{array}$$

commutative.

1.11. Proposition. (compare [SGA 4], VIII, 3.9). Let $x' \in Ob(x_{cd})$ and $M(x',X)^o$, $N_{cd}(x)^o$ and $N^U_{cd}(x)^o$ be the categories dual to $M(x',X)$, $N_{cd}(x)$ and $N^U_{cd}(x)$ respectively. Then

(1) The categories $M(x',X)^o$, $N_{cd}(x)^o$ and $N^U_{cd}(x)^o$ are filtered.

(2) For any sheaf F on X_{cd} and $x' \in Ob(x_{cd})$

1.11.1. $$F^p_x(x') = \varinjlim_{V \in Ob(N_{cd}(x',X')^o)} F(V) = \varinjlim_{V \in Ob(N^{U'}_{cd}(x',X')^o)} F(V)$$

where X' is any etale X-scheme such that $x' = X' \times_X x$, and $U' \hookrightarrow X'$ a fixed open affine subscheme of X' containing x'. The $\varinjlim$ does not depend on a choice of X' and U' with these properties.

(3) If F is an additive presheaf on X_{cd} then F^p_x is also additive, $F^p_x = F_x$, and we can replace F^p_x by F_x in 1.11.1 for such

F.

(4) Let $\tilde{F}$ be the sheaflfcation of a presheaf F on X_{cd}. Then

1.11.2.

$$(\tilde{F})^p_x(x') = (\tilde{F})_x(x') = \varinjlim_{V \in Ob(N_{cd}(x',X')^o)} \tilde{F}(V) = \varinjlim_{V \in Ob(N^{U'}_{cd}(x',X')^o)} \tilde{F}(V),$$

for all $x' \in Ob(x_{cd})$,

1.11.3. $F_x(x') = F^p_x(x') \xrightarrow{\sim} (\tilde{F})^p_x(x') = (\tilde{F})_x(x')$

for any irreducible $x' \in Ob(x_{cd})$, and the natural map of the stalks $u_x: F_x \longrightarrow (\tilde{F})_x$, induced by the canonical morphism $u: F \longrightarrow \tilde{F}$, is a bijection.

(5) If F is additive then

1.11.4. $F_x = F^p_x \xrightarrow{\sim} (\tilde{F})^p_x = \tilde{F}_x$, for all $x \in X$.

Proof: (1) Let $X' \in Ob(X_{cd})$ be such that $x' = X' \times_X x$. Then the categories $M(x',X)$, $N_{cd}(x',X') = M(x',X')$ and $N^{U'}_{cd}(x',X')$ obviously have fibre products induced from Sch/X. Hence, the dual categories $M(x',X)^o$, $N_{cd}(x',X')^o$ and $N^{U'}_{cd}(x',X')$ are filtered ([49], Ch. II, Prop. 2.3). This proves (1).

(2) By ([49], Ch. II, p.57)

1.11.5. $$F^p_x(x') = \varinjlim_{V \in Ob(M(x',X)^o)} F(V)$$

Since the categories $N_{cd}(x',X')$ and $N^{U'}_{cd}(x',X')$ are cofinal in $M(x',X)$ 1.11.5 implies 1.11.1.

Let $X'' \in Ob(X_{cd})$ be another etale X-scheme such that $X'' \times_X x = x'$. Replacing X'' by $X'' \times_X X'$, we can assume that there exists an etale morphism $X'' \longrightarrow X'$. Then, clearly, the category $N_{cd}(x',X'')$ is cofinal in $N_{cd}(x',X')$. Hence, the $\varinjlim$ does not depend on a choice of X' with these properties. This proves (2).

(3) For $x_1, x_2 \in Ob(x_{cd})$ consider a commutative diagram

1.11.6.

$$\begin{array}{ccc} F^p_x(x_1 \sqcup x_2) & \xrightarrow{\;u\;} & F^p_x(x_1) \times F^p_x(x_2) \\ \wr\downarrow & & \wr\downarrow \\ \varinjlim\limits_{V \in Ob(M(x_1 \sqcup x_2, X)^o)} F(V) & \xrightarrow{\;v\;} & \varinjlim\limits_{V_1 \in Ob(M(x_1,X)^o)} F(V_1) \times \varinjlim\limits_{V_2 \in Ob(M(x_2,X)^o)} F(V_2) \end{array}$$

where u is the natural map and the vertical maps are bijections 1.11.5. To define the lower horizontal map v observe that the subcategory $M_1(x_1 \sqcup x_2, X)$ of $M(x_1 \sqcup x_2, X)$ consisting of schemes $V = V_1 \sqcup V_2$, where $V_i \in Ob(M(x_i,X))$, $i = 1,2$, is cofinal in $M(x_1 \sqcup x_2, X)$ and, hence, we have a canonical bijection γ

1.11.7.

$$\gamma: \varinjlim_{V \in Ob(M(x_1 \sqcup x_2,X)^o)} F(V) \xrightarrow{\sim} \varinjlim_{V_1 \in Ob(M(x_1,X)^o)} \varinjlim_{V_2 \in Ob(M(x_2,X)^o)} F(V_1 \sqcup V_2)$$

Let now for all $V_i \in Ob(M(x_i,X))$, $i = 1,2$

$$v_{V_1,V_2}: F(V_1 \sqcup V_2) \longrightarrow F(V_1) \times F(V_2)$$

be a canonical map. Define

1.11.8. $$v_1 = \varinjlim_{V_1 \in Ob(M(x_1,X)^o)} \varinjlim_{V_2 \in Ob(M(x_2,X)^o)} v_{V_1,V_2}$$

and put $v = v_1 \circ \gamma$.

For an additive presheaf F, v_{V_1,V_2} is a bijection for all $V_i \in Ob(M(x_i,X))$, $i = 1,2$ and, hence, v_1 and v are bijections. Then diagram 1.11.6 shows that u is a bijection, i.e. F^p_x is additive. By 1.2(1) an additive presheaf on x_{cd} is a sheaf, hence, $F^p_x = F_x$.

(4) Since any sheaf is additive, $\tilde{F}^p_x$ is additive by (3). Then (2) and (3) imply all equalities of 1.11.2. On the other hand the construction of the sheaf associated with a presheaf given in ([SGA 4], II, 3.19) shows that

1.11.9. $$\varinjlim_{V \in Ob(N_{cd}(x',X')^o)} F(V) = \varinjlim_{V \in Ob(N_{cd}(x',X')^o)} \tilde{F}(V)$$

for any irreducible $x' \in Ob(x_{cd})$ and any $X' \in Ob(X_{cd})$ such that $x' = X' \times_X x$ (this fact remains true for any topology on X', see arguments in the proof of [SGA 3], VIII, 3.9). This together with 1.11.1 and 1.11.2 implies 1.11.3.

(5) If F is additive the middle bijection and the first equality of 1.11.3 can be extended to any reducible $x' = \coprod_{i=1}^{k} x'_i$ using 1.2.3. Q.E.D.

1.12. Let I be a small filtered category (see [60], ch. IX), $i \longrightarrow X_i \in Ob(X_{cd})$ a filtered projective system in X_{cd} indexed by

$i \in I$, i.e. a contravariant functor $I \longrightarrow X_{cd}$. We often will consider projective systems satisfying the following condition:

1.12.1. All schemes X_i are quasi-compact and quasi-separated and all the transition maps $u_{ij}: X_i \longrightarrow X_j$ are affine and flat.

1.12.2. <u>Example</u>: Let X be a locally noetherian scheme, $U \hookrightarrow X$ an affine open noetherian neighborhood of $x \in X$, $N^U_{cd}(x,X)$ the category of cd-neighborhoods of x affine over U. Then the dual category $I = N^U_{cd}(x,X)^o$ is filtered by 1.11(1) and the projective system $N^U_{cd}(x,X) \hookrightarrow X_{cd}$, canonically indexed by I, satisfies condition 1.12.1.

1.12.3. For $x \in X$, a finite separable $k(x)$-algebra A' and $x' = \operatorname{Spec} A' \in Ob(x_{cd})$ denote by $\mathcal{O}^h_{x'}$ the unique henselian $\mathcal{O}^h_{x,X}$-algebra such that $\mathcal{O}^h_{x'} \otimes_{\mathcal{O}^h_{x,X}} k(x) = A'$. (Notice that $\mathcal{O}^h_{x'}$ exists and is unique by ([EGA], IV, 18. 5.15); it coincides with $\sum\limits_{j=1}^{k} \mathcal{O}^h_{x'_j,X'}$, where $x' = \coprod\limits_{j=1}^{k} x'_j$ is the decomposition of x' into the disjoint union of irreducible components x'_j, and X' is an etale X-scheme such that $X' \times_X x = x'$).

1.13. <u>Corollary</u>. Let X be a locally noetherian scheme, $\mathcal{C}$ be a faithful subcategory*) and all schemes $\operatorname{Spec} \mathcal{O}^h_{x',X'}$ for all $X' \in Ob(X_{cd})$ and for all $x' \in X'$. Let $F: \mathcal{C} \longrightarrow (Sets)$ be a contravariant functor. Assume that F commutes with filtered projective limits in $\mathcal{C}$ which satisfy condition 1.12.1 above:

*) of Sch (X) which contains the category X_{cd} as a faithful subcategory *

1.13.1. $F(\varprojlim_{i\in I} X_i) = \varinjlim_{i\in I} F(X_i)$.

Let $F: X_{cd} \longrightarrow (Sets)$ be the restriction of F on X_{cd} and $\tilde{F}$ be the sheafification of the presheaf F on X_{cd}. Let $x \in X$ be a point.

(1) The presheaf stalk F^p_x of F at the point x_{cd} of X_{cd} can be described as

1.13.2. $F^p_x(x') = F(\mathcal{O}^h_{x'}) = F(\sum\limits_{j=1}^{k} \mathcal{O}^h_{x'_j})$

where $x' = \coprod\limits_{j=1}^{k} x'_j \in Ob(x_{cd})$ and other notations as in 1.12.3.

Moreover, if F is additive

1.13.3. $F^p_x(x') = \prod\limits_{j=1}^{k} F(\mathcal{O}^h_{x'_j})$

(2) The (sheaf-theoretic) stalk $\tilde{F}_x$ of the sheaf $\tilde{F}$ at x_{cd} is

1.13.4. $\tilde{F}_x(x') = \prod\limits_{j=1}^{k} F(\mathcal{O}^h_{x'_j})$,

for all $x' = \coprod\limits_{j=1}^{k} x'_j \in Ob(x_{cd})$.

In particular,

1.13.5. $F^p_x(x') = \tilde{F}_x(x') = F(\mathcal{O}^h_{x'}) = \tilde{F}(\mathcal{O}^h_{x'})$

if x' is irreducible.

<u>Proof</u>: Let $X' \in Ob(X_{cd})$ be such that $x' = X' \times_X x$ and $U \hookrightarrow X'$ be a Zariski open noetherian affine neighborhood of x', $N^U_{cd}(x',X')^o$

the category dual to the category $N^U_{cd}(x',X')$ of neighborhoods of x' on X', affine over U (see §1.9).

Using 1.11.1, 1.11.2, 1.12.2 and 1.9.1 we obtain:

$$F^p_x(x') = \varinjlim_{V \in Ob(N^U_{cd}(x',X')^o)} F(V) = F(\varprojlim_{V \in Ob(N^U_{cd}(x',X')^o)} V) = F(\mathcal{O}^h_{x',X'})$$

and similarly for $\tilde{F}_x(x')$. This proves 1.13.2. If F is additive presheaf then F^p_x is also additive by 1.11(3), and any sheaf is additive. This together with 1.13.2 proves 1.13.3. The equalities of 1.13.4 and 1.13.5 follow from 1.11(3), 1.13.2 and 1.2.3.

1.14. <u>Example</u>: Let F be one of the functors $G_n: X' \longrightarrow G_n(X')$ or $G_n(\mathbb{Z}/\ell\mathbb{Z}): X' \longrightarrow G_n(X',\mathbb{Z}/\ell\mathbb{Z})$ of the G-theory (see 0.2), which are defined and contravariant on the category $(Sch/X)_{fl}$ of all X-schemes and flat morphisms, or one of the functors $K_n: X' \longrightarrow K_n(X')$ or $K_n(\mathbb{Z}/\ell\mathbb{Z}): X' \longrightarrow K_n(X;,\mathbb{Z}/\ell\mathbb{Z})$ of the K-theory which are defined and contravariant on the category Sch/X of all X-schemes and all X-morphisms. Then these functors are additive and satisfy the conditions of Corollary 1.13 by ([33], 7.2.2). Therefore, for all $x \in X$ and all $x' \in Ob(x_{cd})$

1.14.1. $(\tilde{G}^{cd}_n)_x(x') = G(\mathcal{O}^h_{x'})$, $\tilde{G}^{cd}_n(\mathbb{Z}/\ell\mathbb{Z})_x(x') = G(\mathcal{O}^h_{x'},\mathbb{Z}/\ell\mathbb{Z})$

1.14.2. $(\tilde{K}^{cd}_n)_x(x') = K_n(\mathcal{O}^h_{x'})$, $\tilde{K}^{cd}_n(\mathbb{Z}/\ell\mathbb{Z})_x(x') = K_n(\mathcal{O}^h_{x'},\mathbb{Z}/\ell\mathbb{Z})$.

1.15. <u>Proposition</u>: Let F be a sheaf on X_{cd}. Then the family of stalks F_x, $x \in X$ is conservative, i.e. any homomorphism of sheaves $u: F \longrightarrow F'$ on X_{cd} is an isomorphism if and only if the induced homomorphisms $u_x: F_x \longrightarrow F'_x$ on the stalks are isomorphisms of sheaves, for all $x \in X$.

This proposition can be proved by a modification of the proof of the similar property of etale topology ([SGA 4], VIII, 3.5b) using 1.9.1 and 1.11.2. We omit details.

As a formal consequence of Proposition 1.15 we obtain:

1.16. <u>Corollary</u>: A homomorphism of sheaves $u: F \longrightarrow F'$ on X_{cd} is a monomorphism (resp. an epimorphism) if and only if $u_x: F_x \longrightarrow F'_x$ is a monomorphism (resp. an epimorphism) in the category of sheaves on x_{cd} for all $x \in X$.

1.17. <u>Corollary</u>: Let $F \longrightarrow G \longrightarrow H$ be a sequence of homomorphisms of sheaves on X_{cd}. Then this sequence is exact if and only if the sequences $F_x \longrightarrow G_x \longrightarrow H_x$ are exact, for all $x \in X$.

1.17.1. <u>Remark</u>: It is enough to check all conditions on stalks in 1.15–1.17 for irreducible $x' \in Ob(x_{cd})$, i.e. the spectra of finite separable field extensions k'/k. In particular a homomorphism of sheaves on X_{cd} $u: F \longrightarrow G$ is an isomorphism (resp. monomorphism, resp. epimorphism) if and only if for any $x \in X$ and any <u>irreducible</u> $x' \in Ob(x_{cd})$ the induced homomorphism $u_x(x'): F_x(x') \longrightarrow G_x(x')$ is an isomorphism (resp. monomorphism, resp. epimorphism).

The cd-cohomology and higher direct images

In 1.18–1.44 we shall assume that X is a *locally noetherian* scheme.

1.18. Let $\mathcal{A}b$ be the category of abelian groups, $\mathcal{P}(X_{cd})$ (resp. $\mathcal{S}(X_{cd})$) be the category of presheaves (resp. sheaves) of abelian groups on X_{cd},

$$\Gamma: \mathcal{S}(X_{cd}) \longrightarrow \mathcal{A}b, \quad F \longrightarrow \Gamma(X_{cd},F), \quad F \in Ob(\mathcal{S}(X_{cd}))$$

the functor of global sections. The functor Γ is left exact. Its right derived functors $\mathbb{R}^q\Gamma(F) = H^q(X_{cd},F)$, $q \geq 0$ are called the q-th cohomology groups of X_{cd} with coefficients in F.

1.18.1 Lemma: Let $X = \operatorname{Spec} R$ be the spectrum of a local henselian ring R. Then for any sheaf of abelian groups F on X_{cd}, $H^i(X_{cd},F) = 0$, for all $i > 0$.

Proof: Let $x \in X$ be the closed point and

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

be an exact sequence of sheaves on X_{cd}. Then by Corollary 1.17 the sequence of stalks evaluated on x must be exact:

$$0 \longrightarrow G'_x(x) \longrightarrow G_x(x) \longrightarrow G''_x(x) \longrightarrow 0$$

Since X is a local henselian scheme, each etale morphism $X' \longrightarrow X$ with $x \in cd(X'/X)$ admits an X-section $X \longrightarrow X'$. Hence, for any sheaf G on X_{cd}

$$G_x(x) = \varinjlim_{X' \in Ob(N_{cd}(x,X)^o)} G(X') = G(X)$$

Therefore, the functor of global sections Γ is exact on $\mathcal{S}(X_{cd})$, and $H^i(X_{cd}, F) = 0$, for all $i > 0$.

1.19. Let I be a filtered category, $\{i \longrightarrow X_i,\ i \in I\}$ a filtered projective system of X-schemes. Assume that all schemes X_i, for all $i \in I$ are quasi-compact and quasi-separated and that all transition morphisms $u_{ij}: X_j \longrightarrow X_i$ are affine. Then the limit $X_\infty = \varprojlim_{i\in I} X_i$ exists in Sch/X ([EGA], IV, 8). Let F be a sheaf on X_{cd}, F_i (resp. F_∞) the inverse image of F on X_i (resp. X_∞). Then the canonical projections $u_i: X_\infty \longrightarrow X_i$, $i \in Ob(I)$, induce the canonical maps $H^q(X_{i,cd},F_i) \longrightarrow H^q(X_{\infty,cd},F_\infty)$ and therefore the map

1.19.1. $$\varinjlim_{i\in Ob(I)} H^q(X_{i,cd},F_i) \longrightarrow H^q(X_{\infty,cd},F_\infty).$$

1.20. Theorem: Under the assumptions and notations of 1.19 canonical map 1.19.1 is an isomorphism.

Proof of 1.20 follows the general scheme of the proof of a similar fact for etale topology ([SGA 3], VII, §§3,5) and is too long and technical to give here.

1.21. Let $f: X \longrightarrow Y$ be a morphism of schemes, $f_*^{cd}: \mathcal{S}(X_{cd}) \longrightarrow \mathcal{S}(Y_{cd})$ be the direct image functor on the categories of abelian sheaves induced by f. This functor is left exact and its right derived functors $\mathbb{R}^q f_*^{cd}$, $q \geq 0$ are called the higher direct images of f.

1.22. Theorem: Let $f: X \longrightarrow Y$ be a morphism of schemes, $F \in Ob(\mathcal{S}(X_{cd}))$ a sheaf of abelian groups on X_{cd}. Then

(1) $\mathbb{R}^q f_*^{cd}$ is the sheaf associated with the presheaf $Y' \longrightarrow H^q((f^{-1}(Y')_{cd}, F')$ on Y_{cd}, where $F' = f^*(F)$ is the inverse image of F on the X-scheme $X' = f^{-1}(Y')$. The formation of $\mathbb{R}^q f_*^{cd}$ commutes with the localization on Y_{cd}.

(2) There exists the Cartan-Leray spectral sequence

1.22.1. $E_2^{p,q} = H^p(Y_{cd}, \mathbb{R}^q f_* F) \Rightarrow H^{p+q}(X_{cd}, F)$, $p \geq 0$, $q \geq 0$.

If Y is noetherian and $\dim Y < \infty$ then this spectral sequence is strongly convergent.

(3) The fibres of the sheaf $\mathbb{R}^q f_*(F)$ at $x \in X$ can be described as

1.22.2. $\mathbb{R}^q f_*(F)_x(k') = H^q((\mathcal{O}^h_{x'})_{cd}, F^h_{x'})$,

for any etale k-algebra k', where $x' = \operatorname{Spec} k'$, the henselian $\mathcal{O}^h_{x,X}$-algebra $\mathcal{O}^h_{x'}$ is defined in 1.12.3, 1.13, and $F^h_{x'}$ is the inverse image of F on $(\operatorname{Spec} \mathcal{O}^h_{x'})_{cd}$.

<u>Proof</u>: Statement (1) and the existence of spectral sequence 1.22.1 are special cases of the results proved in ([SGA 4], V, 5.1, 5.3) for any continuous morphism of topoi $f: T_1 \longrightarrow T_2$. The strong convergence of this spectral sequence follows from vanishing of $E_2^{p,q}$ for $p > \dim X$ (Theorem 1.32 below). (3) follows from (1), Theorem 1.20 and Corollary 1.13.

<u>Local cohomology theory for the cd-topology</u>

1.23. Let $i: Y \longrightarrow X$ be a closed subscheme of X, $U = X-Y$, $j: U \hookrightarrow X$ the natural open immersion. U and Y determine the open and the closed subsites of X_{cd} respectively. It follows then from the general results of ([SGA 4], §§IV, 13, 14.5) that the functor $i_*^{cd}: \tilde{Y}_{cd} \longrightarrow \tilde{X}_{cd}$ has the right adjoint $i^!_{cd}: \tilde{X}_{cd} \longrightarrow \tilde{Y}_{cd}$ and the adjunction morphism $i_*^{cd} \circ i^!_{cd} \longrightarrow \mathrm{Id}$ is injective. Moreover, for any sheaf of groups F on X_{cd} (not necessary abelian) the sequence

1.23.1 $1 \longrightarrow i_*^{cd} i^!_{cd} F \longrightarrow F \longrightarrow j_*^{cd} j^*_{cd} F$

is exact ([SGA 4], IV, 1.4.6). In particular, for any etale morphism $X' \longrightarrow X$ we have

1.23.2 $\Gamma(X', i^!_{cd}(F)) \overline{\overline{\text{def}}}\ \Gamma(X', i_*^{cd} i^!_{cd}(F)) = \operatorname{Ker}(F(X') \longrightarrow F(U \times_X X'))$,

i.e. $i_*^{cd}(F)$ can be characterized as the maximal subsheaf of F, sections of which have their supports in Y ([SGA 4], IV, 14.8).

Beginning from this point in this section and in §§1.24-1.30 below we shall drop indices "cd" in the notations of all these functors and write simply i_*, i^*, $i^!$ ... instead of i_*^{cd}, i^*_{cd}, $i^!_{cd}$ .... It will not cause a confusion.

The functor $\phi: F \longrightarrow i_* i^!(F)$ (resp. $\psi: F \longrightarrow \Gamma(X, i_* i^! F))$, $F \in \operatorname{Ob}(\mathcal{S}(X_{cd}))$ is left exact. Its right derived functors $\mathbb{R}^n\phi = \mathcal{H}^n_Y(X_{cd}, F)$, $n \geq 0$ (resp. $\mathbb{R}^n\psi = H^n_Y(X_{cd}, F)$, $n \geq 0$) are called the *n-th local cd-cohomology sheaf (resp. group) of* X *modulo* Y *with coefficients in* F. There exists a long exact cohomology sequence

1.23.3. $\longrightarrow H^{n-1}(U_{cd},F) \longrightarrow H^n_Y(X_{cd},F) \longrightarrow H^n(X_{cd},F) \longrightarrow H^n(U_{cd},F) \longrightarrow$

relating the ordinary and the local cd-cohomology ([SGA 4], V, 6.5.3). We shall call it the *cohomological sequence of the pair* (X_{cd},U_{cd}).

1.24. <u>Theorem</u>: Let I be a filtered category, $\{i \longrightarrow X_i,\ i \in Ob(I)\}$ be a filtered projective system in Sch/X. Assume that all schemes X_i, $i \in Ob(I)$ are quasi-compact and quasi-separated and that all the transition morphisms $u_{ij}\colon X_i \longrightarrow X_j$ are affine. Denote $X_\infty = \varprojlim_{i \in Ob(I)} X_i$. Let $Y \hookrightarrow X$ be a closed subscheme, $Y_i = X_i \times_X Y$ and $Y_\infty = X_\infty \times_X Y$ are inverse image of Y on X_i, $i \in Ob(I)$, and X_∞ respectively; F a sheaf of abelian groups on X_{cd}, F_i and F_∞ are the inverse images of F on $X_{i,cd}$ and $X_{\infty,cd}$ respectively. Then the canonical map

1.24.1. $\varinjlim_{i \in Ob(I)} H_{Y_i}(X_{i,cd},F) \longrightarrow H^n_{Y_\infty}(X_{\infty,cd},F_\infty)$

is an isomorphism.

<u>Proof</u>: Let $U_i = X_i - Y_i$, $U_\infty = X_\infty - Y_\infty$. For all $i \in Ob(I)$ we have a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^{n-1}(X_{i,cd},F_i) & \longrightarrow & H^{n-1}(U_{i,cd},F_i) & \longrightarrow & H^n_{Y_i}(X_{i,cd},F_i) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^{n-1}(X_{\infty,cd},F_\infty) & \longrightarrow & H^{n-1}(U_{\infty,cd},F_\infty) & \longrightarrow & H^n_{Y_\infty}(X_{\infty,cd},F_\infty) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^n(X_{i,cd},F_i) & \longrightarrow & H^n(U_{i,cd},F_i) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
\longrightarrow & H^n(X_{\infty,cd},F_\infty) & \longrightarrow & H^n(U_{\infty,cd},F_\infty) & \longrightarrow \cdots
\end{array}$$

rows of which are exact sequences 1.23.3. for the pairs (X_i,U_i) and (X_∞,U_∞). Since exact sequences in the category of abelian groups $\mathcal{A}b$ are preserved by filtered direct limits ([55], ch. I, th. 5), applying $\varinjlim_{i \in Ob(I)}$ to the upper row of this diagram we obtain again a commutative diagram with the exact rows. By Theorem 1.20 four external vertical arrows of the new diagram are isomorphisms. By 5-lemma ([60], ch. VIII, §4, lemma 4) the middle arrow is also an isomorphism.

1.24.2. <u>Remark</u>: The same arguments can be used to prove an analogue of 1.24 for etale cohomology. This analogue seems to be lacking in the literature on etale cohomology although is quite useful (see the proof of Corollary 1.28 below).

1.25. <u>Theorem</u>: Let $Y \hookrightarrow X$ be a closed subscheme, F a sheaf of abelian groups on X_{cd}.

(1) $\mathcal{H}^q_Y(X_{cd},F)$ is the sheaf associated with the presheaf $X' \longrightarrow H^q_{Y'}(X'_{cd},F')$ on X_{cd} where $Y' = X' \times_X Y$, F' is the inverse image of F on $(X')_{cd}$. The formation of this sheaf commutes with the a localization on X_{cd}.

(2) For $x \in X$ and $x' \in Ob(x_{cd})$, let $X^h_{x'} = \operatorname{Spec} \mathcal{O}^h_{x'}$, $Y^h_{x'} = X^h_{x'} \times_X Y$, and $F^h_{x'}$ be the inverse image of F on $(X^h_{x'})_{cd}$. Then

1.25.1. $\mathcal{H}^q_Y(X_{cd},F)_x(x') \xrightarrow{\sim} H^q_{Y^h_{x'}}((X^h_{x'})_{cd},F^h_{x'})$

(3) There exists strongly convergent spectral sequence

1.25.2. $E_2^{p,q} = H^p(X_{cd}, \mathcal{H}^q_Y(X_{cd},F)) \Rightarrow H^{p+q}_Y(X_{cd},F)$

<u>Proof</u>: Statement (1) is a special case for the topos X_{cd} of the general results of ([SGA 4], V, 6.4(1),(2)). Statement (2) follows from (1) and Corollary 1.13 applied to the additive functor $X' \longrightarrow H^q_{Y\times_X X'}(X'_{cd},F')$. Condition 1.13.1 of Corollary 1.13 is satisfied by Theorem 1.24. The existence of spectral sequence 1.25.2 follows from the general results of ([SGA 4], V, 6.4(3)). Its strong convergence comes from the vanishing of $E_2^{p,q}$ for $q > \dim X$ which is proved below (Theorem 1.32).

1.26. <u>Lemma</u>: Let $\varphi\colon X_1 \longrightarrow X$ be an etale morphism, $i\colon Y \hookrightarrow X$ a closed subscheme of X, $Y_1 = Y \times_X X_1$, $i_1\colon Y_1 \hookrightarrow X_1$ the natural closed immersion. Assume that φ induces an isomorphism $\varphi_{Y_1}\colon Y_1 \xrightarrow{\sim} Y$. Let F be a sheaf of groups on X_{cd} (not necessarily abelian), $F_1 = \varphi^*(F)$. Then the canonical homomorphism

1.26.1. $\tau\colon \Gamma(X, i_* i^!(F)) \longrightarrow \Gamma(X_1, i_{1*} i_1^!(F_1))$

induced by φ is an isomorphism.

<u>Proof</u>: Consider the commutative diagram

$$\begin{array}{ccccccc}
1 \longrightarrow & \Gamma(X, i_* i^!(F)) & \longrightarrow & \Gamma(X,F) & \longrightarrow & \Gamma(U,F) \\
& \downarrow \tau & & \downarrow \varphi_X & & \downarrow \varphi_U \\
1 \longrightarrow & \Gamma(X_1, i_{1*} i_1^!(F)) & \longrightarrow & \Gamma(X_1,F_1) & \longrightarrow & \Gamma(U_1,F_1)
\end{array}$$

rows of which are exact by 1.23.1.

(1) <u>Injectivity of τ</u>: Let $s \in \operatorname{Ker} \tau$. Then we can consider s

as an element of $\Gamma(X,F)$ such that $s|_U = 1$ and $s|_{X_1} = 1$. But the couple of etale morphisms $\{U \hookrightarrow X, X_1 \longrightarrow X\}$ is a covering of X in the cd-topology (because $Y \in cd(X_1/X)$). Therefore, $s = 1$ in $\Gamma(X,F)$ by a characteristic property of sheaves ([SGA 4], II, 2.4), see also 1.2.2 above.

(2) <u>Surjectivity of τ</u>: Let $s \in \Gamma(X_1, i_{1*} i_1^!(F))$. Then the sections $1 \in \Gamma(U,F)$ and $s \in \Gamma(X_1,F_1)$ agree on $U_1 = X_1 \times_X U$. Since the couple of etale morphisms $\{U \hookrightarrow X, X_1 \longrightarrow X\}$ is a cd-covering, by another property which characterizes sheaves in the category of presheaves ([SGA 4], II, 2.4) there exists a section $t \in \Gamma(X,F)$ such that $\varphi_X(t) = s$ and $t|_U = 1$. Hence, $t \in \Gamma(X, i_* i^!(F))$ and $\tau(t) = s$.

1.27. <u>Theorem</u>: (excision for $H^n_Y(X_{cd},F)$) Let $\varphi\colon X_1 \longrightarrow X$ be an etale map, $i\colon Y \hookrightarrow X$ a closed subscheme, and $Z \hookrightarrow X_1$ a closed subscheme of X_1 such that the restriction of φ on Z induces an isomorphism $\varphi_Z\colon Z \xrightarrow{\sim} Y$, F a sheaf of abelian groups on X_{cd}. Then there exists a canonical isomorphism

1.27.1. $H^n_Y(X_{cd},F) \xrightarrow{\sim} H^n_Z(X_{1,cd},\varphi^*(F))$, for all $n \geq 0$.

<u>Proof</u>: (1) Denote $Y_1 = \varphi^{-1}(Y) = Y \times_X X_1$. Clearly, $Z \hookrightarrow Y_1$. Assume first that $Z = Y_1$. Since $X_1 \in Ob(X_{cd})$, the functor inverse image $\varphi^*\colon \mathcal{S}(X_{cd}) \longrightarrow \mathcal{S}(X_{1,cd})$ is exact and has exact left adjoint functor $\varphi_!\colon \mathcal{S}(X_{1,cd}) \longrightarrow \mathcal{S}(X_{cd})$, called the "extension by zero" ([SGA 4], IV, §11). Therefore φ^* transforms injective sheaves to injectives ([SGA 4], V, 4.11). It follows from a well known general result of

homological algebra ([58], 2.4.1) that it is sufficient to prove the theorem for $n = 0$. But since by Definition 1.23 $H^0_Y(X_{cd},F) = \Gamma(X, i_* i^!(F))$, isomorphism 1.27.1 for $n = 0$ is just 1.26.1.

(2) Consider now the general case of the Theorem. Since $\varphi_Z: Z \xrightarrow{\sim} Y$ is an isomorphism, its inverse $\psi: Y \xrightarrow{\sim} Z$ can be considered as a section of the etale morphism $\varphi_{Y_1}: Y_1 \longrightarrow Y$, and by ([EGA], IV, 17.9.3) its image $Z = \psi(Y)$ is open in Y_1. (If Y is connected, Z coincides as a space with a connected component of Y_1.) Hence, $Z_1 = Y_1 - Z$ is closed in Y_1. Since Y_1 is closed in X_1, Z_1 is closed also in X_1 and therefore $X_2 = X_1 - Z_1$ is open in X_1. Notice that $Z = Y \times_X X_2$ by the construction of X_2 and, hence, the pair (X_2, Z) satisfies the assumption of case (1) proven above. Applying the result of case (1) to the etale morphisms $\pi = \varphi_{X_2}: X_2 \longrightarrow X$, and $i: X_2 \hookrightarrow X_1$, one obtains canonical isomorphisms

1.27.2. $$H^n_Y(X_{cd},F) \xrightarrow{\sim} H^n_Z(X_{2,cd},\pi^*(F)) \xleftarrow{\sim} H^n_Z(X_{1,cd},\varphi^*(F))$$

Notice, that $\varphi \circ i = \pi$ and $i^* \circ \varphi^* = \pi^*$. Therefore, the composition of this isomorphisms of 1.27.2 gives 1.27.1.

1.28. Corollary: Let $x \in X$ be a closed point, F a sheaf of abelian groups on X_{cd}, F^h_x the inverse image of F under the canonical map i^h_x: Spec $\mathcal{O}^h_x \hookrightarrow X$. Then the canonical map

1.28.1. $$H^n_x(X_{cd},F) \xrightarrow{\sim} H^n_x((\mathcal{O}^h_{x,X})_{cd},F^h_x)$$

induced by i^h_x is an isomorphism.

Proof: The map 1.28.1 can be factored into the composition

$$H^n_x(X_{cd},F) \xrightarrow{\alpha} \varinjlim_{U \in Ob(N_{cd}(x)^o)} H^n_x(U_{cd},F) \xrightarrow{\beta} H^n_x((\mathcal{O}^h_{x,X})_{cd},F^h_x)$$

The map α (resp. β) is an isomorphism by Theorem 1.27 (resp. by Theorem 1.24).

1.28.2. Remarks: (1) Analogues of Theorem 1.27 and Corollary 1.28 are true for $H^i(X_{cd},G)$, $i = 0,1$ and a sheaf of non-abelian groups G with essentially the same proofs. These non-abelian versions have been used in the proofs of the results of our works [28]-[32].

(2) Virtually the same arguments give the following excision theorem for etale cohomology:

1.28.3. Theorem: Let $\varphi: X_1 \longrightarrow X$, Y, Z be such as in Theorem 1.27, F a sheaf of abelian groups on X_{et}. Then there exists a canonical isomorphism

$$H^n_Y(X_{et},F) \xrightarrow{\sim} H^n_Z(X_{1,et},\varphi^*(F)), \quad \text{for all } n \geq 0$$

This theorem is more general than the excision theorem of Milne ([49], ch. II, Theor. 1.27) which corresponds to the case when $Z = \varphi^{-1}(Y)$.

1.29. For an arbitrary point $x \in X$ and an abelian sheaf $F \in Ob(\mathcal{S}(X_{cd}))$ define the *local cohomology* $H^q_x(X_{cd},F)$ *of* X_{cd} *modulo* x *with values in* F by the formula

1.29.1. $$H^q_x(X_{cd},F) = \varinjlim_{V\in N_{Zar}(x,X)^o} H^q_{\bar{x}\times_X V}(V_{cd},F)$$

where $\bar{x}$ is the closure of x in X with the reduced scheme structure. Since $\bar{x}\times_X V$ is closed in V the local cohomology under the $\varinjlim$ are defined.

If U is an open affine noetherian subscheme of X and $N^U_{Zar}(x,X)$ the subcategory of affine open subschemes of U, then its dual category $N^U_{Zar}(x,X)^o$ is cofinal in $N_{Zar}(x,X)^o$ and we can replace the limit with respect to $N_{Zar}(x,X)^o$ in 1.29.1 by the limit with respect to $N^U{}_{Zar}(x,X)^o$. The projective system $N^U_{Zar}(x,X)$ satisfies the conditions of Theorem 1.24 and we obtain by this theorem:

1.29.2. $$H^q_x(X_{cd},F) \xrightarrow{\sim} H^q_x(\mathcal{O}_{x,cd},F_{\mathcal{O}_x})$$

where $F_{\mathcal{O}_x}$ is the inverse image of F on $(\mathrm{Spec}\ \mathcal{O}_x)_{cd}$.

Since x is closed in $\mathrm{Spec}\ \mathcal{O}_x$, Corollary 1.28 is applicable to $H^q_x(\mathcal{O}_{x,cd},F)$ and it gives a canonical isomorphism

1.29.3. $$H^q_x(X_{cd},F) \xrightarrow{\sim} H^q_x(\mathcal{O}^h_{x,cd},F^h_x).$$

1.30. Let X_p be the set of points of codimension p on X, i.e.

1.30.1. $$X_p = \{x\in X \mid \dim \mathcal{O}_{x,X} = p\}.$$

Let $Z_p(X)$ be the set of all closed subschemes of X of codimension $\geq p$. Following the construction of Grothendieck for etale cohomology ([23], III, §10) consider the coniveau filtration $F^pH^i(X_{cd},F)$ on $H^i(X_{cd},F)$:

1.30.2.

$$F^pH^i(X,F) = \bigcup_{Z\subset Z_p(X)} \mathrm{Ker}\ (H^i(X_{cd},F) \longrightarrow H^i((X-Z)_{cd},F)),\quad 0\leq p\leq \dim X$$

1.31. <u>Proposition</u>: (1) There exists a spectral sequence

1.31.1. $$E_1^{p,q} = \sum_{x\in X_p} H^{p+q}_x(X_{cd},F) \quad (E_1^{p,q} = 0 \text{ if } p > \dim(X), \text{ for all } q\geq 0),$$

whose E_∞-term is the graded group $Gr_F H^{p+q}(X_{cd},F)$ associated to the coniveau filtration.

(2) $$E_1^{p,q} = \sum_{x\in X_p} H^{p+q}_x((\mathcal{O}^h_x)_{cd},F^h_x).$$

<u>Proof</u>: (1) The construction of Grothendieck ([23], III, §10) actually gives spectral sequence 1.31.1 for any topology which is stronger than the Zariski topology and has a reasonable theory of "cohomology with supports" (see also [56], §3; [24], ch. VII).

The presentation of $E_1^{p,q}$ in the form (2) follows immediately from 1.29.3.

We have now all necessary tools to give a complete proof of the following important theorem of Kato-Saito, which is stated with some indications on its proof in ([25], §1.2). The theory of local cohomology developed above allows us to supply the details lacking in [25].

1.32. <u>Theorem</u>: (Kato-Saito [25]). Let X be a noetherian scheme of finite Krull dimension d, and F a sheaf of abelian groups on X. Then

1.32.1. $H^n(X_{cd},F) = 0$, for all $n > d$.

Therefore, $\mathrm{coh.dim}(X_{cd}) \leq n$ (here $\mathrm{coh.dim}(X_{cd})$ is the cohomological dimension of X_{cd}).

Proof: If $d = 0$, X is a sum of spectra of fields, and $H^n(X_{cd},F) = 0$ for all $n > 0$ by 1.2(1) and 1.5.1. Assume by induction that the theorem is true for all schemes of dimension $< d$. Let $X_x^h = \mathrm{Spec}\, \mathcal{O}_x^h$, $Y_x^h = X_x^h - x$. Exact sequence 1.23.3 for the pair (X_x^h, Y_x^h) implies that

1.32.2. $H^{n-1}((Y_x^h)_{cd},F) \xrightarrow{\sim} H_x^n((X_x^h)_{cd},F)$, $n \geq 2$.

If $x \in X_p$, with $p \leq d$, $\dim X_x^h = p$, and $\dim Y_x^h = p-1 < d$. By our inductive assumption $H^{n-1}((Y_x^h)_{cd},F) = 0$ for $n > p$ and, therefore, $E_1^{p,q} = \sum_{x \in X_p} H_x^{p+q}((X_x^h)_{cd},F) = 0$ for $q > 0$, for all $p \geq 0$. But then the spectral sequence degenerates, and, hence ([21], ch. I, 4.4.1) $H^p(X_{cd},F) = E_1^{p,0}$ for all $p \geq 0$. In particular, $H^p(X_{cd},F) = E_1^{p,0} = 0$ for $p > \dim(X)$.

1.33. Corollary: Under the assumptions and notations of 1.32

1.33.1. $H_x^n(X_{cd},F) = 0$, $n > \mathrm{codim}_X(x)$.

Proof: The vanishing in 1.33.1 follows from Theorem 1.32, 1.32.2 and 1.29.3.

The comparison with other topologies

1.34. Denote by X_{Zar}, X_{et} and X_{fl} the Zariski, small etale and small flat (= ffpf) sites of X respectively, and let $\tilde{X}_{Zar}$, $\tilde{X}_{et}$ and $\tilde{X}_{fl}$ be the topoi of sheaves on X_{Zar}, X_{et} and X_{fl} respectively. For all $X' \in \mathrm{Ob}(X_\tau)$ let $\mathrm{Cov}_\tau(X')$ be the category of coverings of X_τ where τ is one of the symbols Zar, et or fl.

1.35. The natural embedding of the categories of coverings

$$v_\tau^*: \mathrm{Cov}_{cd}(X') \longrightarrow \mathrm{Cov}_\tau(X'), \quad \text{for all } X' \in \mathrm{Ob}(Et/X),$$

where τ is one of the symbols et or fl, induces a morphism of sites $v: X_\tau \longrightarrow X_{cd}$, τ = et, fl. It induces also a morphism of the corresponding topoi of sheaves:

1.35.1. $u^\tau = (u_*^\tau, u_\tau^*): \tilde{X}_\tau \longrightarrow \tilde{X}_{cd}$, τ = et, fl.

where the functor "direct image" $u_*^\tau: \tilde{X}_\tau \longrightarrow \tilde{X}_{cd}$ is defined by the formula

1.35.2. $u_*^\tau(F) = F \circ v^*$, for all $F \in \mathrm{Ob}(\tilde{X}_\tau)$,

and the functor "inverse image" $u_\tau^*: \tilde{X}_{cd} \longrightarrow \tilde{X}_\tau$ is defined as the left adjoint to u_*^τ which exists by ([59], ch. IX, Theor. 5.1).

1.36. For a sheaf of abelian groups F on X_τ consider the Cartan-Leray spectral sequence corresponding to the morphism of topoi u^τ ([SGA 4], V, 5.3):

1.36.1. $E_2^{p,q} = H^p(X_{cd}, \mathbb{R}^q u_*^\tau F) \Rightarrow H^{p+q}(X_\tau,F)$, τ = et, fl.

If X is a noetherian scheme of finite Krull dimension then this spectral sequence converges strongly, because by Theorem 1.32 the cohomological dimension of x_{cd} is $\leq n$ and, therefore, $E_2^{p,q} = 0$ for $p > n$. As usual, it implies the existence of the canonical homomorphisms

1.36.2. $H^i(X_{cd}, u_*^\tau(F)) \longrightarrow H^i(X_\tau, F)$, for all $i \geq 0$

and the exactness of the sequence of lower terms of 1.36.1

1.36.3. $0 \longrightarrow H^1(X_{cd}, u_*^\tau(F)) \longrightarrow H^1(X_{et}, F) \longrightarrow H^0(X_{cd}, R^1u_*^\tau(F)) \longrightarrow$

$$\longrightarrow H^2(X_{cd}, u_*(F)) \longrightarrow H^2(X_{et}, F)$$

(see [21], ch. 1, §4.5).

This exact sequence is still defined and is exact (at least in its first three terms) for a sheaf of nonabelian groups F on X_τ ([82], ch. III, IV). A detailed study of this exact sequence in the case when X is the spectrum of a Dedekind ring and F is a reductive group scheme over X has been undertaken in [28]-[31].

1.37. Proposition: For any point $x \in X$, any separable $k(x)$-algebra A' and $x' = \operatorname{Spec} A' \in \operatorname{Ob}(x_{cd})$ the stalk $R^q u_*^\tau(F)_x$ of the sheaf $R^q u_*^\tau(F)$ on X_{cd} at x can be described as

1.37.1. $R^q u_*^\tau(F)_x(x') \xrightarrow{\sim} H^q((\mathcal{O}_{x'}^h)_\tau, F_x^h)$, for all $q \geq 0$, $\tau = et, fl$,

where $\mathcal{O}_{x'}^h$ is the unique henselain $\mathcal{O}_{x,X}^h$-algebra such that $\mathcal{O}_{x'}^h \otimes k(x) = A'$ ([EGA], IV, 18.5.15), and $F_{x'}^h$ is the inverse image of F on $(\operatorname{Spec} \mathcal{O}_{x'}^h)_\tau$.

Proof: Description 1.36.1 follows from Corollary 1.13 applied to the functor $X' \longrightarrow H^q(X'_\tau, F')$ on X_{cd}, where F' is the inverse image of F on X'_τ. This functor satisfies Condition 1.13.1 of Corollary 1.13 by ([SGA 4], VII, theor. 5.7 for $\tau = et$; [23], III, p. 172 for $\tau = fl$).

1.38. Lemma: Let X be a noetherian scheme of finite Krull dimension, F a sheaf of abelian groups on X_τ, $\tau = et$ or fl. Then the following properties are equivalent:

(1) $R^i u_*^\tau(F) = 0$, $1 \leq i \leq n$.

(2) For any etale X-scheme X' the canonical homomorphism

$$H^i(X'_{cd}, u_*^\tau(F)) \to H^i(X'_\tau, F)$$

is an isomorphism for $0 \leq i \leq n$ and a monomorphism for $i = n+1$.

(3) For any point $x \in X$ and $x' \in \operatorname{Ob}(x_{cd})$

$$H^i((\mathcal{O}_{x'}^h)_\tau, F_{x'}^h) = 0, \quad 1 \leq i \leq n,$$

where $\mathcal{O}_{x'}^h$ and $F_{x'}^h$ are defined as in Proposition 1.36.

Proof: The implication (1) ⇒ (2) follows from spectral sequence 1.35.3 which converges (even strongly) under the noetherianness assumption, and a general property of convergent spectral sequences ([57], ch. XV, Theor. 5.12). The implication (2) ⇒ (3) follows from Lemma 1.18.1, Theorem 1.20 and 1.9.1. The equivalence (3) ⇔ (1) follows from Proposition 1.37.

1.39. Example: Let X be an irreducible curve over an algebraically closed field k, G a smooth abelian group scheme of finite type over X with the connected affine generic fibre G_η. Since k is algebraically closed, for any closed point $x \in X$, $k(x) \xrightarrow{\sim} k$ and the category x_{cd} is trivial. Therefore,

1.39.1. $H^i((\mathcal{O}_x^h)_{fl}, G_x^h) \xrightarrow{\sim} H^i((\mathcal{O}_x^h)_{et}, G_x^h) \xrightarrow{\sim} H^i(k(x)_{et}, G) = 0$

by ([23], III, Theor. 11.7). On thg other hand, if η is the generic point of X, $k(\eta)$ is a field of cohomological dimension ≤ 1 in the sense of Serre ([54], ch. II, §3). Then $H^i(k(\eta)_{et}, G_\eta)$ coincides with the Galois cohomology groups $H^i(k(\eta), G_\eta)$, for all $i \geq 0$, τ = et, fl, and the Galois cohomology vanish for $i = 1$ and 2 by the theorems of Steinberg and Grothendieck respectively ([54], ch. III). Therefore,

1.39.2. $H^i(X_{cd}, G) \xrightarrow{\sim} H^i(X_{et}, G) \xrightarrow{\sim} H^i(X_{fl}, G) \quad i = 1,2$

It is clear, that the conditions of Lemma 1.38 are satisfied only in very special cases. As rule, $H^i(X_{cd}, F)$ is very different from $H^i(X_{et}, F)$ or $H^i(X_{fl}, F)$.

1.40. Consider now relationships with the Zariski topology. For any Zariski open subscheme $U \hookrightarrow X$ the natural embedding of the categories of coverings

1.40.1. $t^*: Cov_{Zar}(U) \hookrightarrow Cov_{cd}(U)$

induces a morphism of sites

1.40.2. $t: X_{cd} \longrightarrow X_{Zar}$.

As in §1.35 t induces also a morphism of topoi of sheaves

1.40.3. $r = (r_*, r^*): \tilde{X}_{cd} \longrightarrow \tilde{X}_{Zar}$

with $r_*(F) = F \circ t^*$, for any sheaf F on X_{cd}, and r^* defined as

the left adjoint functor to r_*. For an abelian sheaf F on X_{cd} consider the Cartan-Leray spectral sequence

1.40.4. $E^{p,q} = H^p(X_{Zar}, \mathbb{R}^q f_*(F)) \Rightarrow H^{p+q}(X_{cd}, F)$

If X is a noetherian scheme of finite Krull dimension n, then $E_2^{p,q} = 0$ for $p > n$ by the vanishing theorem of Grothendieck ([58], Theor. 3.6.5) and therefore the spectral sequence converges strongly. Again, it implies the existence of the canonical homomorphisms

1.40.5. $H^i(X_{Zar}, r_*(F)) \longrightarrow H^i(X_{cd}, F), \quad i \geq 0$

and the existence and the exactness of the sequence of lower terms of 1.40.4:

1.40.6. $0 \longrightarrow H^1(X_{Zar}, r_*(F)) \longrightarrow H^1(X_{cd}, F) \longrightarrow H^0(X_{Zar}, \mathbb{R}^1 r_*(F)) \longrightarrow$
$\longrightarrow H^2(X_{Zar}, r_*(F)) \longrightarrow H^2(X_{cd}, F)$.

1.41. <u>Lemma</u>: Let $y \in X$, $X_y = \mathrm{Spec}\ \mathcal{O}_{y,X}$. Then the stalks of the sheaf $\mathbb{R}^i r_*(F)$ at y can be described by the formula

1.41.1. $(\mathbb{R}^i r_*(F))_y \xrightarrow{\sim} H^i(X_{y,cd}, F_{\mathcal{O}_y})$,

where $F_{\mathcal{O}_y}$ is the inverse image of F on $(X_y)_{cd}$.

The proof proceeds as in Proposition 1.37 with Theorem 1.20 replacing the results on limits for etale and flat cohomology cited there.

1.42. <u>Lemma</u>: Let X be a noetherian scheme of finite Krull dimension, F a sheaf of abelian groups on X_{cd}. Then the following properties are equivalent:

(1) $R^i r_*(F) = 0, \quad 1 \leq i \leq n.$

(2) For any open subscheme $U \hookrightarrow X$ the canonical homomorphism

$$H^i(U_{Zar}, r_*(F)) \longrightarrow H^i(U_{cd}, F)$$

is an isomorphism for $0 \leq i \leq n$, and a monomorphism for $i = n+1$.

(3) For any point $x \in X$

$$H^i((\mathcal{O}_{x,X})_{cd}, F_{\mathcal{O}_x}) = 0, \quad 1 \leq i \leq n,$$

where $F_{\mathcal{O}_x}$ is the inverse image F on $(\mathrm{Spec}\ \mathcal{O}_{x,X})_{cd}$.

The proof is analogous to that of Lemma 1.38.

1.43. Remark: Lemmas 1.38, 1.42 and their proofs are analogous to a lemma of Grothendieck on the relationships of etale and flat cohomology ([23], III, 11.1).

1.44. Examples: (1) Let X be a regular irreducible noetherian scheme of finite Krull dimension, K the field of rational functions on X, and G a reductive group scheme over X. It is shown in our papers [30], [31] that the canonical maps

1.44.1. $\lambda_x : H^1(\mathcal{O}^h_{x,et}, G) \longrightarrow H^1(K^h_{x,et}, G)$

are injective for all $x \in X$, where $\mathcal{O}^h_x$ is the henselization of the local ring $\mathcal{O}_x$ of x on X with respect to its maximal ideal m_x, and K^h_x is the quotient field of $\mathcal{O}^h_x$. It follows from this and exact sequence 1.36.3 that the sequence

1.44.2. $1 \longrightarrow H^1(X_{cd}, u^{et}_*(G)) \longrightarrow H^1(X_{et}, G) \xrightarrow{\lambda} H^1(K, G)$

is exact. On the other hand it has been conjectured by Serre ([53], exp. 1) and Grothendieck ([53], exp. 5; [23], II, 2.10) that $\mathrm{Ker}\ \lambda = H^1(X_{Zar}, w_*(G))$, where $w = u^{et} \circ r : \tilde{X}_{et} \longrightarrow \tilde{X}_{Zar}$ is the canonical morphism of these topoi. Therefore, the exactness of 1.44.2 implies that the remaining part of the conjecture is equivalent to the bijectivity of the canonical map

1.44.3. $H^1(X_{Zar}, w_*(G)) \longrightarrow H^1(X_{cd}, u^{et}_*(G))$

The conjecture of Serre and Grothendieck and, therefore, the bijectivity of 1.44.3 has been proved in the cases when G is an X-torus ([28], [63]); $\dim X = 1$ and G an arbitrary reductive X-group [30], [31]; and $\dim X = 2$ and G is a quasi-split X-group [30], [32].

(2) Let D be a Dedekind ring with the quotient field K, $X = \mathrm{Spec}\ K$, η the generic point of X, D_x the Zariski local ring of $x \in X$, $v(x)$ the valuation of K corresponding to x. Denote by $\hat{D}_{v(x)}$ and $\hat{K}_{v(x)}$ the $v(x)$-adic completions of D and K respectively.

Let G be a flat affine group D-scheme of a finite type over D with a smooth generic fibre $G_\eta = G \otimes_D K$.

Consider the set of double classes $c_x(G) = G(K)\backslash G(\hat{K}_{v(x)})/G(\hat{D}_{v(x)})$ which we shall call the local class set of G. It has been proved in [28]-[30] that there exists a canonical bijection

1.44. $H^1(D_{x,cd}, G) \xrightarrow{\sim} c_x(G)$.

A global, adelic analogue of 1.44.4 is also established there.

We say that G_η (or G) has the *weak approximation property* with respect to $v(x)$ if the group $G(K)$ is dense in $G(\hat{K}_{v(x)})$ in the $v(x)$-adic topology.

If G_η has the weak approximation property with respect to $v(x)$, $G(\hat{K}_{v(x)}) = G(K)G(\hat{D}_{v(x)})$ and, hence, by 1.44.4.

1.44.5. $H^1(D_{x,cd}, G) \xrightarrow{\sim} c_x(G) = 0$

If G has the weak approximation property with respect to any $v(x)$, $x \in X - \eta$ then 1.44.5 and Lemma 1.42 imply that the canonical map

1.44.6. $H^1(X_{Zar}, w_*(G)) \longrightarrow H^1(X_{cd}, u_*^{et}(G))$

is a bijection. This assumption is satisfied in the following cases:

(i) K is a number field and G_η is a simply connected semisimple K-group [68], [69].

(ii) K is a number field, and G_η is a K-torus which splits over a cyclic extension of K [79], [71].

Notice, that if G_η is semisimple but is not simply connected, or if G_η is a K-torus which splits over a non-cyclic extension, the weak approximation may fail even over number fields (see examples in [68], [79]). For other (non-arithmetic) fields the weak approximation property occurs (or, at least, is known) only in very few and special cases [68], [44], [31]. The following example shows that the loss of the weak approximation implies non-surjectivity of the canonical map 1.44.6 for certain affine and flat models of G_η over D.

(3) Let $D, X, G \ldots$ as in (2) above. Assume that G_η is semisimple but does not have the weak approximation property with respect to $v(x)$ for some $x \in X$. Let $m_{v(x)}$ be the maximal ideal of $\hat{D}_{v(x)}$. Consider the family of congruence subgroups
$\Gamma_n = \mathrm{Ker}(G(\hat{D}_{v(x)}) \longrightarrow G(\hat{D}_{v(x)}/m^n_{v(x)}))$, $n > 0$.

Γ_n is open in $G(\hat{D}_{v(x)})$, and it follows from the definition that if G_η does not have the weak approximation $G(\hat{K}_{v(x)}) \neq G(K)\Gamma_n$ for sufficiently big $n > 0$. On the other hand, it is known that for $n >> 0$ there exists an affine and flat model G_n of G_η over D such that $G_n(\hat{D}_{v(x)}) = \Gamma_n$. Hence, $H^1(D_{x,cd}, G) \xrightarrow{\sim} c_x(G_n) \neq 0$, and by Lemma 1.42 the canonical map

1.44.7. $H^1(U_{Zar}, w_*(G)) \longrightarrow H^1(U_{cd}, u_*^{et}(G))$

is not surjective for some open subscheme U of X (it is always injective).

Example (i) shows, however, that map 1.44.6 is still bijective if the reduction modulo m_x is the best possible, i.e. semisimple for all $x \in X$. Therefore, the bijectivity of 1.44.6 occurs, as rule, only in very regular situations: a regular base X and regular fibres of the sheaf G, no degenerations!

2. THE SIMPLICIAL GODEMENT COMPLEX AND THE HYPERCOHOMOLOGICAL SPECTRA ON THE CD-TOPOLOGY

2.1. In this section we shall review and specify for the cd-topology some general homotopy-theoretical and sheaf-theoretical constructions of [5], [41], [38] which will be used through the rest of this paper.

The cosimplicial resolution of a sheaf of topological spaces was introduced by Godement ([21], Appendix) and extended to sheaves of spaces on a topos by Deligne ([SGA 4], XVII, 4.2) and Illusie ([45], I, 1.5.3; II, 6.1), see also Johnstone [83]. Thomason ([38], §1), combined the Godement resolution and the inverse homotopy limit consturction (see [5]) defined the hypercohomological spectrum $\mathbb{H}(X,F)$ of a presheaf of spectra on a site $\mathscr{C}$ which has sufficiently many points, for all $X \in Ob(\mathscr{C})$. This construction allows one to define on the spectrum level a sort of a sheaf "hyperhomotopy", which "interpolates" the sheaf cohomology and the homotopy groups of a spectrum. It complements the Čech simplicial "hyperhomotopy" theory of a simplicial spectra which emerged in the works of Grothendieck (see Segal [52]), Deligne ([SGA 4], V^{bis}; [9]), Beilinson [4] and Gillet [16] on cohomological descent for simplicial schemes in various cohomology and homotopy theories. See also works of Illusie [45], Brown [7], Brown-Gersten [8], Breen [6], Jardine [46], [66], [67] for related or intermedian homotopy-theoretical constructions.

Our notion of points of the topos X_{cd} requires a careful reexamination of all steps of the general schemes used in [21], [45] and [38] for the constructions of the cosimplicial Godement resolution of F, the hypercohomological spectrum $\mathbb{H}(X_{cd},F)$ and a related hypercohomological spectral sequence convergent to $\pi_*(\mathbb{H}(X_{cd},F))$, as these schemes applied to X_{cd}. This reexamination is carried out in this section. Most complications are due to a difference between the presheaf-theoretic stalk (or the presheaf inverse image) $F_x^p = i_x^{\#}(F)$, and the sheaf-theoretic stalk (or the sheaf inverse image) $F_x = i_x^*(F)$ which arises with our definition of points (see also §§1.10, 1.11). However, it does not affect the final result of this section - Theorem 2.22.

By the sign " $\xrightarrow{\sim}$ " in §§2-4 we shall denote weak homotopy equivalences of simplicial sets or spectra.

2.2. For a scheme X and a point $x \in X$ let $i_x\colon x \hookrightarrow X$ be the natural embedding, and $i_x\colon x_{cd}^{\sim} \longrightarrow X_{cd}^{\sim}$ the corresponding morphism of topoi of sheaves on the cd-topology of the corresponding schemes (see, §1.4). Consider the product

2.2.1. $P^{\sim} = \prod_{x \in X} x_{cd}^{\sim}$

For a sheaf F on X_{cd} put

2.2.2. $p^*(F) = \prod_{x\in X} i_x^*(F) \in Ob(P^{\sim})$;

and for $G = \prod_{x\in X} G_x \in Ob(P^{\sim})$ put

2.2.3. $p_*(G) = \prod_{x\in X} i_{x,*}(G_x) \in Ob(X_{cd}^{\sim})$.

The pair (p^*, p_*) of adjoint functors defines the morphism of topoi

2.2.4. $p: P^{\sim} \longrightarrow X^{\sim}_{cd}$.

Let $T = p_* p^*: X^{\sim}_{cd} \longrightarrow X^{\sim}_{cd}$. The adjunction morphisms $\eta: Id \longrightarrow p_* p^*$ and $\epsilon: p^* p_* \longrightarrow Id$ induce the natural transformations of functors $\eta: Id \longrightarrow T$ and $\mu: TT \longrightarrow T$ which satisfy to the relationship $\mu = p_* \epsilon p^*$.

The cosimplicial sheaf

2.2.5. $$T^{\cdot}F \underset{def}{=} \{TF \rightrightarrows TTF \rightrightarrows TTTF \rightrightarrows \cdots\}$$

is called the *cosimplicial Godement resolution* of F. The term in the codimension n of this sheaf is $F^n = T^{n+1}F$. The coface maps are

2.2.6.

$$d^i_n \underset{def}{=} T^i \eta T^{n+1-i}: F^n = T^{n+1}F \longrightarrow F^{n+1} = T^{n+2}F, \quad 0 \leq i \leq n+1,$$

and the codegeneracies are

2.2.7. $$s^i_n \underset{def}{=} T^i \mu T^{n-i}: F^{n+1} = T^{n+2}F \longrightarrow F^n = T^{n+1}F, \quad 0 \leq i \leq n.$$

One can check that d^i_n's and s^i_n's satisfy the standard cosimplicial identities ([21], App., §2) or ([5], ch. X, §2.1) using the method of Godement ([21], App., §§2,3).

The map $\eta: Id \longrightarrow T$ induces a canonical augmentation $\eta(F): F \longrightarrow T^{\cdot}F$.

Let $u: F \longrightarrow F'$ be a morphism of sheaves on X_{cd}. Then it is easy to see that u induces a morphism of cosimplicial presheaves $T^{\cdot}(u): T^{\cdot}F \longrightarrow T^{\cdot}F'$ which is compatible with the augmentations, i.e.

such that

2.2.8. $$T^{\cdot}(u) \circ \eta(F) = \eta(F') \circ u$$

2.2.9. Remark. Let $\mathcal{C}$ be a faithfull subcategory of Sch/X which satisfies all conditions of 1.13 and has fibre products. Let $F: \mathcal{C} \longrightarrow (Sets)$ be a contravariant functor satisfying all conditions of 1.13, and let $\tilde{F}$ be the sheafification of the restriction $F: X_{cd} \longrightarrow (Sets)$ of F on X_{cd}. Then on an affine X-scheme X' the sheaf $T\tilde{F}$ is given by a formula:

2.2.10. $$T\tilde{F}(X') = \prod_{x \in X} (i_{x*} i^*_x \tilde{F})(X') = \prod_{x \in X} F(\mathcal{O}^h_x \otimes_R R').$$

where $R = \Gamma(X, \mathcal{O}_X)$ (resp. $R' = \Gamma(X, \mathcal{O}_{X'})$) are the rings of global sections of the structure sheaf $\mathcal{O}_X$ of X (resp. $\mathcal{O}_{X'}$ of X'), and $\mathcal{O}^h_x$ is the henselization of the local ring $\mathcal{O}_x$ of x on X with respect to its maximal ideal $m_x \subset \mathcal{O}_x$. This formula underlines the geometric adelic constructions of §2 of Ch. I of [31].

2.3. For a simplicial pointed set Y let $\Omega Y = Map_*(S^1, Y)$ be the *loop space* of Y, where S^1 is the simplicial 1-sphere and Map_* is the function complex of based maps of the pointed set S^1 into Y ([5], VIII, §1).

Recall, that a *prespectrum of simplicial sets* $E = \{E_n, n \geq 0\}$ is a collection of pointed simplicial sets E_n together with the structural maps $\omega_n: E_n \longrightarrow \Omega E_{n+1}$.

A *fibrant spectrum* $E = \{E_n, n \geq 0\}$ is a prespectrum E such that all simplicial sets E_n are fibrant (i.e. satisfy the Kan condition ([5], VIII, 3.3), and all the structural maps ω_n are weak homotopy equivalences).

The homotopy groups $\pi_k(E)$ of a prespectrum $E = \{E_n\}$ are given as the direct limit

2.3.1. $\pi_k(E) = \varinjlim_n \pi_{n+k}(E_n)$, for all $k \in \mathbb{Z}$

with respect to the system of the canonical maps on the homotopy groups

2.3.2. $\pi_{k+n}(E_n) \longrightarrow \pi_{k+n}(\Omega E_{n+1}) \xrightarrow{\sim} \pi_{k+n+1}(E_{n+1})$, for all $k \in \mathbb{Z}$,

induced by $\omega_n: E_n \longrightarrow \Omega E_{n+1}$.

If E is a fibrant spectrum, $\pi_k E \xrightarrow{\sim} \pi_k E_0$, for all $k \geq 0$, and $\pi_k E = \pi_{k+n} E_n$ for all $k < 0$, where $n \geq -k$.

A map $f = \{f_n\}: E \longrightarrow E'$ of prespectra is a collection of maps $f_n: E_n \longrightarrow E'_n$, for all $n \geq 0$ such that $f_{n+1} \circ \omega_n = \omega'_n \circ f_n$.

A map of prespectra $f: E \longrightarrow E'$ is a *weak homotopy equivalence* if it induces an isomorphism on their homotopy groups. A map $f = \{f_n\}: E \xrightarrow{\sim} E'$ of *fibrant* spectra is a weak homotopy equivalence if and only if each $f_n: E_n \xrightarrow{\sim} E'_n$ is a weak homotopy equivalence of simplicial sets. A map of fibrant spectra $f = \{f_n\}$ is a *fibration* if each f_n is (Kan) fibration of simplicial sets.

A sequence of maps of fibrant spectra $E' \xrightarrow{g} E \xrightarrow{u} E'$ is a *homotopy fibre sequence* if for all n the corresponding sequences of maps of their n-th components

$$E''_n \xrightarrow{g_n} E_n \xrightarrow{f_n} E'_n, \quad n \geq 0,$$

are homotopy fibre sequences of fibrant simplicial sets, i.e. f_n is a homotopy fibration, and g_n is a weak homotopy equivalence onto the

homotopy fibre of f_n.

The category of prespectra has a structure of a closed model category in the sense of Quillen [54] such that the corresponding homotopy category is the usual stable category ([42], §2).

2.4. The category FSp of fibrant spectra is closed under filtered direct limits. A filtered direct limit of fibrations (resp. of weak homotopy equivalences) is a fibration (resp. weak homotopy equivalence ([38], §5.5.5, [45], I, 2.1.21).

Let $F = \{F_n\}: \mathcal{C} \longrightarrow FSp$ be a functor from a small category $\mathcal{C}$ to FSp. Then for each $X \in Ob(\mathcal{C})$ $F_n(X)$ is a fibrant simplicial set for all n and the homotopy limit $\operatorname{holim}_{\mathcal{C}} F_n(X)$ is defined in the category of fibrant simplicial sets ([5], XI, 3.2). The structural weak homotopy equivalences $\omega_n: F_n(X) \xrightarrow{\sim} \Omega F_{n+1}(X)$ of the spectrum $\{F_n(X), n \geq 0\}$ induce weak homotopy equivalences

2.4.1. $\operatorname{holim}_{\mathcal{C}} F_n(X) \xrightarrow{\sim} \operatorname{holim}_{\mathcal{C}} \Omega F_{n+1}(X) \xrightarrow{\sim} \Omega \operatorname{holim}_{\mathcal{C}} F_{n+1}(X)$.

Thus, $\operatorname{holim}_{\mathcal{C}} F_n(X)$ is a fibrant spectrum, for all $X \in Ob(\mathcal{C})$.

Let $F, G: \mathcal{C} \longrightarrow FSp$ be two contravariant functors with values in FSp. A morphism $u: F \longrightarrow G$ is a family of maps of spectra $u(X): F(X) \longrightarrow G(X)$ for all $X \in Ob(\mathcal{C})$ such that for the map $\varphi: X' \longrightarrow X$ in $\mathcal{C}$ the diagram

2.4.2.
$$\begin{array}{ccc} F(X) & \xrightarrow{u(X)} & G(X) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\varphi)} \\ F(X') & \xrightarrow[u(X')]{} & G(X') \end{array}$$

is commutative in the category of fibrant spectra.

We say that a morphism $u: F \longrightarrow G$ is *a weak homotopy equivalence of the functors* if the maps $u(X): F(X) \longrightarrow G(X)$ are weak homotopy equivalences for all $X \in Ob(\mathscr{C})$ i.e. if the induced maps $\pi_q(u): \pi_q(F) \longrightarrow \pi_q(F')$ are isomorphisms of the presheaves of groups, for all $q \in \mathbb{Z}$.

We say that a map $u: F \longrightarrow F'$ of contravaariant functors $F, F': \mathscr{C} \longrightarrow FSp$ is a *homotopy fibration* if for any $X \in Ob(\mathscr{C})$ the map $u(X): F(X) \longrightarrow F'(X)$ is a homotopy fibration.

We say that a sequence of maps $F_1 \xrightarrow{u} F \xrightarrow{v} F_2$ of contravariant functors $F, F_1, F_2: \mathscr{C} \longrightarrow FSp$ is a *homotopy fibre sequence* if for all $X \in Ob(\mathscr{C})$ the sequence of maps of spectra

$$F_1(X) \xrightarrow{u(X)} F(X) \xrightarrow{v(X)} F_2(X)$$

is a homotopy fibre sequence of spectra.

2.5. <u>Definition</u>: ([38], 1.52) We say that a presheaf $F: X_{cd} \longrightarrow FSp$ is *additive* if for any $X_1, X_2 \in Ob(X_{cd})$ the canonical map of spectra

2.5.1. $\quad F(X_1 \amalg X_2) \longrightarrow F(X_1) \times F(X_2)$,

induced by the natural embeddings $X_i \longrightarrow X_1 \amalg X_2$, $i = 1,2$, is a weak homotopy equivalence, i.e. if for all $q \in \mathbb{Z}$ the induced map on the homotopy groups

2.5.2. $\quad \pi_q(F)(X_1 \amalg X_2) \longrightarrow \pi_q(F)(X_1) \times \pi_q(F)(X_2)$

is an isomorphism.

Hence, F is additive if and only if the presheaves of abelian groups $\pi_q(F)$ are additive for all $q \in \mathbb{Z}$ in the sense of Definition 1.2.4.

For a *sheaf* of spectra $F: X_{cd} \longrightarrow FSp$ map 2.5.1 is actually an equality.

2.6. <u>Example</u>: The presheaves of K-theory spectra $\underline{G}: X' \longrightarrow \underline{G}(X')$ and $\underline{G}(\mathbb{Z}/\ell\mathbb{Z}): X' \longrightarrow \underline{G}(X', \mathbb{Z}/\ell\mathbb{Z})$, (resp. $\underline{K}: X' \longrightarrow \underline{K}(X')$ and $K(\mathbb{Z}/\ell\mathbb{Z}): X' \longrightarrow \underline{K}(X', \mathbb{Z}/\ell\mathbb{Z})$) of the category $Coh(X)$ (resp. $LF(X)$) with the integral and $\mathbb{Z}/\ell\mathbb{Z}$ - coefficients constructed in [80], [70], [72], [42], [38] are additive [33]. Notice, that all these constructions are weakly homotopically equivalent [81].

2.7. Let $x = Spec\ k$ be the spectrum of a field k, and $F: x_{cd} \longrightarrow FSp$ be an additif presheaf of fibrant spectra, $\tilde{F}$ its sheafification. Then condition 2.5.1 shows that the natural map $F \longrightarrow \tilde{F}$ is a weak homotopy equivalence of presheaves.

If $x = Spec\ k(x)$ is a point of a scheme X and $F: X_{cd} \longrightarrow FSp$ is an additive presheaf then the presheaf inverse image $i_x^{\#}(F)$ is an additive presheaf as the arguments of the proof of Proposition 1.11(3) show. Hence, the canonical map into its sheafification

2.7.1. $\quad i_x^{\#}(F) \longrightarrow i_x^{*}(F)$.

is a weak homotopy equivalence of additive presheaves.

2.8. Let $f: X \longrightarrow Y$ be a morphism of schemes, $F = (F_n): X_{cd} \longrightarrow FSp$ (resp. $G = (G_n): Y_{cd} \longrightarrow FSp$) a presheaf of fibrant spectra on X_{cd} (resp. on Y_{cd}). Then applying $f_{\#}$ (resp. $f^{\#}$) to the family of presheaves $F_m: X_{cd} \longrightarrow (FSSets)$ (resp. $G_m: Y_{cd} \longrightarrow (FSSets)$) with values in the category (FSSets) of fibrant simplicial pointed sets, we obtain a family of presheaves $f_{\#}(F) = \{f_{\#}(F_m), m \geq 0\}$ (resp. $f^{\#}(G) = \{f^{\#}(G_m), m \geq 0\}$) which forms a presheaf of fibrant spectra $f_{\#}(F)\colon Y_{cd} \longrightarrow FSp$ (resp. $f^{\#}(G)\colon X_{cd} \longrightarrow FSp$). On the other hand, if $\tilde{F} = \{\tilde{F}_m, m \geq 0\}$ is the sheafification of F on X_{cd}, it is easy to see from the construction of the associated sheaf in ([SGA 4], II) that $\tilde{F}$ is also a sheaf of fibrant spectra.

In particular, for a point $x \in X$ and a presheaf (resp. sheaf) $F\colon X_{cd} \longrightarrow FSp$ the presheaf $i_x^{\#}(F)$ (resp. the sheaves $i_x^{*}(F)$, $p^{*}(F)$, and $T^m F$, for all $m > 0$) are well defined in the category of presheaves (resp. sheaves) of fibrant spectra.

2.9. Lemma: Let x be a point of X, $E\colon X_{cd} \longrightarrow FSp$ a presheaf of fibrant spectra $\tilde{E}$ the associated sheaf on X_{cd}. Then

(1) The functors π_q, and $\tilde{\pi}_q$ commute with $i_{x,\#}$:

2.9.1. $\pi_q(i_{x,\#}(E)) = i_{x,\#}(\pi_q(E))$, for all $q \in \mathbb{Z}$;

2.9.2. $\tilde{\pi}_q(i_{x,\#}(E)) = i_{x,*}(\tilde{\pi}_q(E))$, for all $q \in \mathbb{Z}$.

(2) If E is additive then $\pi_q(E)$ and $\pi_q(i_{x,\#}(E))$ are sheaves and for all $q \in \mathbb{Z}$

2.9.3. $\pi_q(i_{x,\#}(E)) = i_{x,*}(\pi_q(E)) \xrightarrow{\sim} i_{x,*}(\pi_q(\tilde{E})) = \pi_q(i_{x,*}(\tilde{E}))$.

(3) The functors $i_{x,\#}$ and $p_{\#} = \prod_{x \in X} i_{x,\#}$ preserve weak homotopy equivalences, homotopy fibrations and homotopy fibre sequences.

Proof: (1) For all $X' \in Ob(X_{cd})$ let $x' = X' \times_X x$. Using the definitions of the functors involved we obtain:

$$i_{x,\#}(\pi_q(E))(X') = \pi_q(E)(x') = \pi_q(i_{x,\#}(E)(X')) = \pi_q(i_{x,\#}(E))(X').$$

This proves 2.9.1. It is easy to see that $i_{x,*}(\tilde{\pi}_q(E))$ is the sheafification of the presheaf $i_{x,\#}(\pi_q(E))$ on X_{cd}. This fact and 2.9.1 implies 2.9.2.

(2) For an additive presheaf E the presheaf of its homotopy groups $\pi_q(E)$ is additive (see §2.5). By §1.2(1) $\pi_q(E)$ is sheaf on X_{cd}, and, therefore, $i_{x,\#}(\pi_q(E))$ is a sheaf equal to $i_{x,*}(\pi_q(E))$ by 1.4.9. The first equality in 2.9.3 follows now from 2.9.1, and it shows that $\pi_q(i_{x,\#}(E))$ is a sheaf on X_{cd}.

For an additive presheaf E the canonical map $E \longrightarrow \tilde{E}$ induces the canonical isomorphisms

2.9.4. $\pi_q(E) \xrightarrow{\sim} \pi_q(\tilde{E})$, for all $q \in \mathbb{Z}$

(see §2.7). This together with the proven part of (2) gives the middle canonical isomorphism and the last equality of 2.9.3.

(3) Let $u\colon E \xrightarrow{\sim} E'$ be a weak homotopy equivalence of presheaves of spectra on X_{cd}. For all $q \in \mathbb{Z}$ consider a commutative diagram of the presheaves of the homotopy groups

2.9.4.
$$\begin{array}{ccc} \pi_q(i_{x,\#}(E)) & \longrightarrow & \pi_q(i_{x,\#}(E')) \\ \| & & \| \\ i_{x,\#}(\pi_q(E)) & \xrightarrow{\sim} & i_{x,\#}(\pi_q(E')) \end{array}$$

where the vertical maps are the equalities of 2.9.1. Since $\pi_q(u)\colon \pi_q(E) \xrightarrow{\sim} \pi_q(E')$ is an isomorphism of presheaves by our assumption, the bottom horizontal map is an isomorphism of the presheaves. Hence, the top horizontal map is an isomorphism of presheaves on x_{cd}, for all $q \in \mathbb{Z}$, i.e. the presheaves $i_{x,\#}(E)$ and $i_{x,\#}(E')$ are weakly homotically equivalent.

The preservation by $i_{x,\#}$ of homotopy fibrations and homotopy fibre sequences of presheaves on x_{cd} follows directly from the definitions.

2.10. For a *presheaf* of sets $F: X_{cd} \longrightarrow (Sets)$, $i_x^*(F)$ is defined as the sheafification of $i_x^{\#}(F)$ on x_{cd}, see §1.10, and $p^*(F) = \prod_{x \in X} i_x^*(F)$ is a sheaf on P. Hence, $TF = p_*p^*(F)$, T^nF, $n > 0$, and T^*F are defined as before for the *sheaf* p^*F. It is clear from this definition and the bijection $F_x \xrightarrow{\sim} \tilde{F}_x$ of Proposition 1.11(4) that

2.10.1. $T^mF = T^m\tilde{F}$, for all $m > 0$.

Analogously, for a presheaf of spectra $F: X_{cd} \longrightarrow FSp$, T^nF, $n > 0$, and T^*F are defined using the *sheaves* $i_x^*(F)$ and $p^*(F)$ as above. We shall see below (Lemma 2.11(3)) that for any presheaf F, T^nF is weakly homotopically equivalent to $T^n\tilde{F}$, where $\tilde{F}$ is the sheafification of F on X_{cd}.

2.11. <u>Lemma</u>: LEt $F: X_{cd} \longrightarrow FSp$ be a presheaf of spectra on X_{cd}, and x a point of X. Then

(1) the functor π_q, $q \in \mathbb{Z}$, commutes with the functors $i_x^{\#}$, i_x^* and T^m, $m > 0$:

2.11.1. $\pi_q(i_x^{\#}(F)) = i_x^{\#}(\pi_q(F))$

2.11.2. $\tilde{\pi}_q(i_x^*(F)) = \pi_q(i_x^*(F)) \xrightarrow{\sim} i_x^*(\pi_q(F)) \xrightarrow{\sim} i_x^*(\tilde{\pi}(F))$

2.11.3.

$$\tilde{\pi}_q(T^m(F)) = \pi_q(T^mF) \xrightarrow{\sim} T^m(\pi_q(F)) \xrightarrow{\sim} T^m(\pi_q(F)), \text{ for all } m > 0;$$

In particular, $\pi_q(i_x^*(F))$ and $\pi_q(T^m(F))$, for all $m > 0$, are sheaves on x_{cd} and X_{cd} respectively.

2. The functors $i_x^{\#}$, i_x^*, T^m, $m > 0$, and T^* preserve weak homotopy equivalences, homotopy fibrations and homotopy fibre sequences of presheaves of spectra on X_{cd}.

3. Let $\tilde{F}$ be the sheafification of F on X_{cd}, $u: F \longrightarrow \tilde{F}$ the canonical map. Then

2.11.4. $i_x^*(F)(x') \xrightarrow{\sim} i_x^{\#}(F)(x') \xrightarrow{\sim} i_x^{\#}(F)(x') \xrightarrow{\sim} i_x^*(F)(x')$

for any irreducible $x' \in Ob(x_{cd})$, and u induces canonical isomorphisms of spectra and homotopy groups:

2.11.5 $i_x^*(u): F_x = i_x^*(F) \xrightarrow{\sim} i_x^*(\tilde{F}) = \tilde{F}_x$, for all $x \in X$;

2.11.6. $\tilde{\pi}_q(u): \tilde{\pi}_q(F) \xrightarrow{\sim} \tilde{\pi}_q(\tilde{F})$, for all $q \in \mathbb{Z}$;

2.11.7.

$$\pi_q(T^mF) \xrightarrow{\sim} \pi_q(T^m\tilde{F}), \text{ for all } q \in \mathbb{Z}, \text{ for all } m > 0.$$

In particular, the canonical map $T^m(u): T^mF \longrightarrow T^m\tilde{F}$ induced by u is a weak homotopy equivalence, for all $m > 0$.

<u>Proof</u>: (1) Since the homotopy groups of a spectra commute with filtered direct limits ([45], I, 2.1.2.1) we have by 1.11(1) and 1.11.5 for $x' \in Ob(x_{cd})$:

$$\pi_q(i_x^{\#}(F))(x') = \pi_q(\varinjlim_{X' \in Ob(M(x',X)^o)} F(X'))$$
$$= \varinjlim_{X' \in Ob(M(x',X)^o)} \pi_q(F)(X')$$
$$= i_x^{\#}(\pi_q(F))(x')$$

This proves 2.11.1. Furthermore, for an irreducible $x' \in Ob(x_{cd})$, using 1.11.3 for F and $\pi_q(F)$ and 2.11.1 we obtain:

$$(\pi_q(i_x^{*}(F))(x') = \pi_q(i_x^{*}(F)(x')) \xrightarrow{\sim} \pi_q(i_x^{\#}(F)(x'))$$
$$= i_x^{\#}(\pi_q(F))(x') \xrightarrow{\sim} i_x^{*}(\pi_q(F))(x')$$

This gives the middle isomorphism of 2.11.2 for an irreducible $x' \in Ob(x_{cd})$.

If now $x' \in Ob(x_{cd})$ is reducible, and has the decomposition $x' = \coprod_{j=1}^{k} x'_j$ into irreducible components x'_j, then applying 1.11.3 to $i_x^{\#}(F)$ and $i_x^{\#}(\pi_q(F))$ and 1.2.3 to the sheaf $i_x^{*}(\pi_q(F))$ we obtain:

$$(\pi_q(i_x^{*}(F)))(x') \underset{def}{=} \pi_q(i_x^{*}(F)(x')) \xrightarrow{\sim} \pi_q(\prod_{j=1}^{k} i_x^{*}(F)(x'_j)) \xrightarrow{\sim} \prod_{j=1}^{k} \pi_q(i_x^{*}(F)(x'_j))$$
$$= \prod_{j=1}^{k} i_x^{\#}(\pi_q(F)(x'_j)) = \prod_{j=1}^{k} i_x^{*}(\pi_q(F))(x'_j) \xrightarrow{\sim} i_x^{*}(\pi_q(F))(x')$$

This proves the middle isomorphism of 2.11.2. It implies that $\pi_q(i_x^{*}(F))$ is a sheaf, i.e. the first equality of 2.11.2. The third

follows from Proposition 1.11(4). The equalities of 2.11.3 follow from 2.11.1, 2.11.2 and 2.9.1 because by our definitions $T = p_* p^*$ where $p^* = \prod_{x \in X} i_x^*$ and $p_* = \prod_{x \in X} i_{x,*}$.

(2) Let $u: F_1 \xrightarrow{\sim} F_2$ be a weak homotopy equivalence of presheaves of spectra on X_{cd}. Then for all $x' \in Ob(x_{cd})$ passing to the limit over the filtered category $M(x',X)$ (see §1.10) of the weak equivalences $u(X'): F_1(X') \longrightarrow F_2(X')$ and using 1.11.5 and the fact that the filtered direct limits preserve weak homotopy equivalences of fibrant spectra by 2.4 we see that the map

2.11.8. $i_x^{\#}(u)(x') = \varinjlim_{X' \in Ob(M(x',X)^o)} u(X'): i_x^{\#}(F_1)(x') \longrightarrow i_x^{\#}(F_2)(x')$

is a weak homotopy equivalence.

Furthermore, for an irreducible $x' \in Ob(x_{cd})$, $i_x^{\#}(F)(x') \xrightarrow{\sim} i_x^{*}(F)(x')$ by 1.11.3, and for a reducible $x' \in Ob(x_{cd})$ with the decomposition $x' = \coprod_{j=1}^{k} x'_j$ into the irreducible components x'_j we have a commutative diagram

$$\begin{array}{ccc} i_x^{*}(F_1)(x') & \longrightarrow & i_x^{*}(F_2)(x') \\ \wr\downarrow & & \wr\downarrow \\ \prod_{j=1}^{k} i_x^{\#}(F_1)(x'_j) & \xrightarrow{\sim} & \prod_{j=1}^{k} i_x^{\#}(F_2)(x'_j) \end{array}$$

Since the bottom horizontal map is a weak homotopy equivalence by 2.11.8, the top horizontal map is also a weak homotopy equivalence.

Since i_x^* preserves weak homotopy equivalences, the functor

$p^* = \prod_{x\in X} i_x^*$ also preserves them.

The same is true for $p_* = \prod_{x\in X} i_{x,*}$ by Lemma 2.9(3). Hence, the functor T^m, $m > 0$, which is an iterated composition of p^* and p_*, preserves such equivalences.

Similarly, we can prove that these functors preserve homotopy fibrations and homotopy fibre sequences.

(3) Bijections 2.11.4 and 2.11.5 are true for a presheaf of spectra $F = \{F_m, m \geq 0\}$ because they are true for each component F_m, by 1.11(3). To prove that $\tilde{\pi}_q(u)$ in 2.11.6 is an isomorphism of sheaves it is sufficient by Proposition 1.15 to check that it induces isomorphisms of stalks:

2.11.9. $\tilde{\pi}_q(u)_x\colon i_x^*(\tilde{\pi}_q(F)) \xrightarrow{\sim} i_x^*(\tilde{\pi}_q(\tilde{F}))$, for all $x \in X$.

The last isomorphisms follow from 2.11.2 and 2.11.5, for all $x \in X$. Finally, using 2.11.3 and 2.11.6 we obtain for all $q \in \mathbb{Z}$, and for all $m > 0$:

$$\pi_q(T^m(F)) \xrightarrow{\sim} T^m(\tilde{\pi}_q(F)) \xrightarrow{\sim} T^m(\tilde{\pi}_q(\tilde{F})) \xrightarrow{\sim} \pi_q(T^m(\tilde{F}))$$

This proves 2.11.7.

2.12. Lemma: Let $F\colon X_{cd} \to FSp$ be an additive presheaf of fibrant spectra, $u\colon F \to \tilde{F}$ the canonical map of F into its sheafification $\tilde{F}$ on X_{cd}, x a point of X. Then

(1) the presheaf $i_x^{\#}(F)$ is additive;

(2) the canonical maps in the diagram

2.12.1.
$$\begin{array}{ccc} i_x^{\#}(F) & \xrightarrow[\sim]{g_x} & i_x^*(F) \\ {\scriptstyle u_x^{\#}}\ \big\downarrow & & \big\downarrow \\ i_x^{\#}(\tilde{F}) & \xrightarrow[\tilde{g}_x]{\sim} & i_x^*(\tilde{F}) \end{array}$$

are weak homotopy equivalences, where $u_x^* = i_x^*(u)$ and $u_x^{\#} = i_x^{\#}(u)$ are induced by u, and g (resp. $\tilde{g}$) is the inclusion of $i^{\#}(F)$ (resp. $i^{\#}(\tilde{F})$) into its sheafification $i_x^*(F)$ (resp. $i_x^*(\tilde{F})$).

Proof: (1) Let $x_1, x_2 \in Ob(x_{cd})$. Since F is additive, the presheaf $\pi_q(F)$ is additive, for all $q \in \mathbb{Z}$, and by 1.11(3) $i_x^{\#}(\pi_q(F))$ is also additive, for all $q \in \mathbb{Z}$. Using (2.11.1) we obtain:

$$\begin{aligned}\pi_q(i_x^{\#}(F))(x_1 \cup x_2) &= i_x^{\#}(\pi_q(F))(x_1 \cup x_2) = i_x^{\#}(\pi_q(F))(x_1) \times i_x^{\#}(\pi_q(F))(x_2) \\ &= \pi_q(i_x^{\#}(F))(x_1) \times \pi_q(i_x^{\#}(F))(x_2),\end{aligned}$$

as desired in (1).

(2) Let now $x' = \coprod_{j=1}^{k} x'_j \in Ob(x_{cd})$, where x'_j are irreducible summands, $1 \leq x'_j \leq k$. Consider a commutative diagram of the homotopy groups:

2.12.2.
$$\begin{array}{ccc} \pi_q(i_x^{\#}(F))(x') & \xrightarrow[\sim]{\alpha_q} & \prod_{j=1}^{k} \pi_q(i_x^{\#}(F))(x'_j) \\ {\scriptstyle v_q}\ \big\downarrow & & \big\downarrow\ {\scriptstyle w_q} \\ \pi_q(i_x^{\#}(\tilde{F}))(x') & \xrightarrow[\tilde{\alpha}_q]{\sim} & \prod_{j=1}^{k} \pi_q(i_x^{\#}(\tilde{F}))(x'_j) \end{array}$$

where $v_q = \pi_q(u_x^{\#}(x'))$, $v_{q,j} = \pi_q(u_x^{\#}(x'_j))$, $w_q = \prod_{j=1}^{k} v_{q,j}$ and the horizontal maps α_q and $\tilde{\alpha}_q$ are the natural maps, induced by the canonical embeddings $x'_j \longrightarrow \coprod_{j=1}^{k} x'_j = x'$, $1 \leq j \leq k$.

Since presheaf F is additive, $i_x^{\#}(F)$ is also additive by (1) and, hence, α_q and $\tilde{\alpha}_q$ are isomorphisms. All maps $v_{q,j}$, $1 \leq j \leq k$ are isomorphisms by 2.11.4. The commutativity of the diagram implies that v_q is an isomorphism, for all $q \in \mathbb{Z}$, i.e. $u_x^{\#}$ is a weak homotopy equivalence. Similarly, we can prove that the maps u_x^{*}, g_x and $\tilde{g}_x$ in 2.12.1 are weak homotopy equivalences.

2.13. <u>The hypercohomological spectrum</u>. For a presheaf $F: X_{cd} \longrightarrow FSp$ of fibrant spectra define the *hypercohomological spectrum* $\mathbb{H}(X'_{cd},F)$ of F as the homotopy limit of the cosimplicial fibrant spectrum $(T^{*}F)(X)$ (see §§2.2, 2.10):

2.13.1. $\mathbb{H}(X_{cd},F) = \underset{\overleftarrow{\Delta}}{\text{holim}}\,(T^{*}F)(X')$,

where Δ is the category of standard simplices $\Delta_n = \{1, \cdots, n\}$ and nondecreasing maps [5], [9], [84].

Let $u: F \longrightarrow F'$ be a map of presheaves of spectra. Then it follows from the covariant behavior of T^{*} and <u>holim</u> with respect to such maps §2.2 and ([5], Ch. XI, §3.2) that u induces a map of hypercohomological spectra

2.13.2. $\mathbb{H}(u): \mathbb{H}(X_{cd},F) \longrightarrow \mathbb{H}(X_{cd},F')$

compatible with the canonical argumentations of F and F', i.e. such that

2.13.3. $\mathbb{H}(u) \circ \eta(F) = \eta(F') \circ u$.

Since the functors T^m, $m > 0$, and <u>holim</u> preserve weak equivalences, homotopy fibrations and homotopy fibre sequences by Lemma 2.11(3) and ([5], Ch. II, §§5.5-5.6), the functor $\mathbb{H}(X_{cd},\cdot)$ also preserves them. In particular, for a presheaf F the canonical map $u: F \longrightarrow \tilde{F}$ induces the weak homotopy equivalences $T^m(u): T^m(F) \xrightarrow{\sim} T^m\tilde{F}$, for all $m > 0$ by 2.11.7 and, hence, the weak homtopy equivalence

2.13.4. $\mathbb{H}(u): \mathbb{H}(X_{cd},F) \xrightarrow{\sim} \mathbb{H}(X_{cd},\tilde{F})$.

To proceed further we need some acyclicity results.

2.14. <u>Lemma</u>: Let $i_x: x \hookrightarrow X$ be a point of X, $H: x_{cd} \longrightarrow \mathcal{A}b$ be a sheaf of abelian groups on x_{cd}. Then

2.14.1. $H^p(X_{cd}, i_{x,*}(H)) = 0$, for all $p > 0$.

<u>Proof</u>: Write down the Cartan-Leray spectral sequence for H and the morphism of sites $i_x: x_{cd} \longrightarrow X_{cd}$:

$$E_2^{p,q} = H^p(X_{cd}, \mathbb{R}^q i_{x,*}(H)) \Rightarrow H^{p+q}(x_{cd}, H).$$

By definition $\mathbb{R}^q i_{x,*}(H)$ is the sheaf associated with the presheaf

2.14.2. $X' \longrightarrow H^q(i_x^{-1}(X')_{cd}, H)$

on X_{cd}. But $i_x^{-1}(X') = x_{X'}$ is a finite disjoint union $x_{X'} = \coprod_{j=1}^{k} x'_j$ of the spectra $x'_j = \operatorname{Spec} L_j$ of finite separable field extensions L_j of the residue field $k(x) = \mathcal{O}_{x,X}/m_{x,X}$ because $x_{X'}$ is an etale x-scheme when $X' \longrightarrow X$ is etale. Therefore, 2.14.3. $H^q(i^{-1}(X')_{cd},H) = \bigoplus_{j=1}^{k} H^q(x'_{j,cd},H) = 0,$ for all $q > 0$

and $R^q i_{x,*}(H) = 0,$ for all $q > 0$. Hence spectral sequence 2.14.2 degenerates to the canonical isomorphisms

$$H^p(X_{cd},i_{x,*}(H)) \xrightarrow{\sim} H^p(x_{cd},H) = 0, \quad \text{for all } p > 0.$$

2.15. Corollary: Let L be a sheaf of abelian groups on X_{cd}. Then the sheaves $L^m = T^m L$ are acyclic, for all $m > 0$.

Proof: Since $T^m L = p_*(p^*(T^{m-1}(L)))$, Lemma 2.14 applied to $H = p^*(T^{m-1}(L))$ implies the Corollary.

2.16. Corollary: Let $E: x_{cd} \longrightarrow FSp$ be a presheaf of fibrant spectra, $\tilde{\pi}_q(E)$ the sheafification of the presheaf of its homotopy groups $\pi_q(E)$, $q \in \mathbb{Z}$. Then

2.16.1.

$$H^p(X_{cd},\tilde{\pi}_q(i_{x,\#}(E))) = H^p(X_{cd},i_{x,*}(\tilde{\pi}_q(E))) = 0, \quad \text{for all } p > 0.$$

Proof: The equalities of 2.16.1 follow from Lemmas 2.9(1) and 2.14 respectively.

2.17. Let $L: X_{cd} \longrightarrow \mathcal{A}b$ be a sheaf of abelian groups on X_{cd}, $L^\bullet = \{L^m = T^{m+1}L,\ m \geq 0\}$ the Godement cosimplicial resolution of L (see §2.2). Consider the complex of abelian sheaves

2.17.1. $A(L^\bullet) = \{L^m,\ d_m;\ m \geq 0\}$

which corresponds to $L^\bullet$ under the Godement-Dold-Puppe correspondence [21], [84]. The differentials d_m of $A(L^\bullet)$ are

2.17.2. $d_m = \sum_{i=0}^{m+1} (-1)^i d_m^i: L^m \longrightarrow L^{m+1}, \quad m \geq 0.$

where $d_m^i: L^m \longrightarrow L^{m+1}$, $0 \leq i \leq m+1$ are the coface maps defined in §2.2.

2.18. Proposition: The complex $A(L^\bullet)$ of 2.17.1 determines a resolution of L by acyclic sheaves, i.e. the sequence of sheaves

2.18.1. $0 \longrightarrow L \xrightarrow{\eta(L)} L^1 \xrightarrow{d_1} L^2 \xrightarrow{d_2} L^3 \xrightarrow{d_3} L^4 \longrightarrow \ldots$

where $\eta(L)$ is the canonical augmentation $\eta: Id \longrightarrow TL = L^1$, is exact.

Proof: The acyclicity of the sheaves L^i is proved in Corollary 2.15. Since the family of points $\{x_{cd},\ x \in X\}$ is conservative, to prove that $A(L^\bullet)$ is a resolution of L it is enough to prove that the corresponding complex of stalks $A(L_x^\bullet)$ is a resolution of L_x, for all $x \in X$ (see §1.16-1.17), or, by taking the product on $x \in X$ that the complex

2.18.2. $A(p^*(L^\bullet)) = \{p^*(L^m),\ p^*(d_m);\ m \geq 0\}$

is a resolution of $p^*(L)$.

It is shown in ([21], App., §5) that for this it is enough to construct a retraction $h: p^*p_*p^*(L) \longrightarrow p^*(L)$ of the natural augmentation $\eta(p^*(L)): p^*(L) \longrightarrow p^*p_*p^*(L)$, i.e. a map such that $h \circ \eta(p^*(L)) = Id_{p^*(L)}$. In fact, it is enough to show that the map $h = \varepsilon(p^*(L)): p^*p_*(p^*(L)) \longrightarrow p^*(L)$ induced by the adjunction

$\epsilon\colon p^*p_* \longrightarrow \mathrm{Id}$, is such a retraction, i.e.

2.18.3 $\epsilon(p^*(L))\circ\eta(p^*(L)) = \mathrm{Id}_{p^*(L)}$.

Equality 2.18.3 can be checked stalkwise. According to Remark 1.17.1 it is enough to check that the composition

2.18.4. $L_x(x') \xrightarrow{\eta_x(x')} (TL)_x(x') \xrightarrow{\epsilon_x(x')} L_x(x')$

where η_x and ϵ_x are the x-components of $\eta(p^*(L))$ and $\epsilon(p^*L))$ respectively, is the identity on $L_x(x')$, for all $x \in X$, and any irreducible $x' \in \mathrm{Ob}(x_{cd})$. The last fact can be checked by the arguments of ([21], App. §§4,5). We omit details.

2.19. For a presheaf $F\colon X_{cd} \longrightarrow FSp$ consider the sheaves of abelian groups

2.19.1. $L^m_q(F) \overset{=}{\underset{\mathrm{def}}{=}} \pi_q(F^m) = \tilde{\pi}_q(F^m)$, $m \geq 0$, $q \in \mathbb{Z}$,

on X_{cd}, where $F^m = T^{m+1}F$ as in §2.2, and the second equality of 2.19.1 follows from 2.11.3. This equality shows that $\pi_q(F^m)$ are actually sheaves, for all $m \geq 0$ and $q \in \mathbb{Z}$.

For a fixed $q \in \mathbb{Z}$, $L^*_q(F) = \{L^m_q(F),\ m \geq 0\}$ is a cosimplicial sheaf of abelian groups with the coface maps $\pi_q(d^i_{m-1})\colon L^{m-1}_q(F) \longrightarrow L^m_q(F)$, $0 \leq i \leq m$, and the codegeneracies $\pi_q(s^i_m)\colon L^{m+1}_q(F) \longrightarrow L^m_q(F)$, $0 \leq i \leq m$, where d^i_m and s^i_m are defined for F^* in §2.2. To this abelian sheaf corresponds a cochain complex

2.19.2. $A(L^*_q(F)) = \{L^m_q(F),\ d_{q,n};\ m \geq 0\}$

where the differentials $d_{q,m}$ are defined by the formula

2.19.3. $d_{q,m} = \sum_{i=0}^{m+1} (-1)^i \pi_q(d^i_m)$, for all $m \geq 0$.

The complex $A(L^*_q(F))$ has a canonical augmentation $\eta_q = \tilde{\pi}_q(\eta)\colon \tilde{\pi}_q(F) \longrightarrow \tilde{\pi}_q(F^0) = L^0_q(F)$ induced by the augmentation $\eta\colon F \longrightarrow F^0 = TF$.

2.20. <u>Proposition</u>: The complex $A(L^*_q(F))$ with the augmentation η_q defines a resolution of $\tilde{\pi}_q(F)$ by acyclic sheaves, i.e. the sequence of sheaves

2.20.1.

$$0 \longrightarrow \tilde{\pi}_q(F) \xrightarrow{\eta_q} \pi_q(F^0) \xrightarrow{d_{q,0}} \pi_q(F^1) \xrightarrow{d_{q,1}} \pi_q(F^2) \longrightarrow \ldots$$

is exact.

<u>Proof</u>: Since $\pi_q(T^m(F)) = T^m(\pi_q(F))$, for all $m > 0$, and for all $q \in \mathbb{Z}$ by 2.11.3, the acyclicity of the sheaves $L^m_q(F) = \pi_q(T^{m+1}(F))$ follows from Corollary 2.15. Moreover, the same equality shows that $L^*_q(F)$ is the Godement cosimplicial resolution $T^*(\tilde{\pi}_q(F))$ of the sheaf of abelian groups $\tilde{\pi}_q(F)$ on X_{cd}. Hence, the exactness of 2.20.1 follows from Proposition 2.18.

The construction of the hypercohomological spectral sequence given below is based on the following special case of a result of Bousfield and Kan [5], extended to spectra in [38]:

2.21. <u>Theorem</u>: ([5], XI, 7.1, 7.3; [38], 5.13, 5.31). Let $X^* = \{X^m,\ m \geq 0\}$ be a cosimplicial fibrant spectrum with the coface maps $d^i_m\colon X^m \longrightarrow X^{m+1}$, $0 \leq i \leq m+1$.

2.21.1. $A(L_q^*) = \{L_q^m = \pi_q(X^m),\ d_{q,m} = \sum_{i=0}^{m+1} (-1)^i \pi_q(d_m^i);\ m \geq 0\}$

the cochain complex of the homotopy groups of X^*. Then there exists a spectral sequence

2.21.2. $E_1^{p,q} = \pi_q(X^p) \Rightarrow \pi_{q-p}(\underset{\Delta}{\operatorname{holim}}\,(X^*)),\ q \in \mathbb{Z},\ p \geq 0,$

with the E_2-term $E_2^{p,q} = H^p(A(L_q^*))$ and the differentials $d_r\colon E_r^{p,q} \to E_r^{p+r,q+r-1}$ of bidegrees $(r,r-1)$. This spectral sequence converges strongly if there exists $N \geq 0$ such that $H^p(A(L_q^*)) = 0$ for all $p > N$ and for all $q \in \mathbb{Z}$, or $L_q^m = \pi_q(X^m) = 0$ for all $q > N$ and for all $m \geq 0$.

2.22. Theorem: Let $F\colon X_{cd} \to FSp$ be a presheaf of fibrant spectra, $\tilde{\pi}_q(F)$ the sheafification of the presheaf $\pi_q(F)$ of q-th homotopy groups of F, for all $q \in \mathbb{Z}$. Then there exists a hypercohomological type spectral sequence which abuts to the homotopy groups of $\mathbb{H}(X_{cd},F)$:

2.22.1. $E_2^{p,q} = H^p(X_{cd},\tilde{\pi}_q(F)) \Rightarrow \pi_{q-p}(\mathbb{H}(X_{cd},F)),\ q \in \mathbb{Z},\ p \geq 0.$

The differentials $d_r\colon E_2^{p,q} \to E^{p+r,q+r-1}$ of 2.22.1 have the bidegrees $(r,r-1)$. If there exists $N > 0$ such that $H^pX_{cd},\tilde{\pi}_q(F)) = 0$, for all $p > N$ and for all $q \in \mathbb{Z}$, then spectral sequence 2.22.1 converges strongly. In particular, if X is a noetherian scheme of finite Krull dimension, then $E_2^{p,q} = 0$ for $p > \dim X$ and for all $q \in \mathbb{Z}$, and spectral sequence 2.22.1 converges strongly.

Proof: Theorem 2.21 applied to the cosimplicial fibrant spectrum $F^*(X) = \{F^m(X) = (T^{m+1}F)(X),\ m \geq 0\}$ gives the spectral sequence

$$E_1^{p,q} = \pi_q((T^{p+1}F)(X)) \Rightarrow \pi_{q-p}(\mathbb{H}(X_{cd},F)),\quad q \in \mathbb{Z},\ p \geq 0,$$

and $E_2^{p,q}$ is the p-th cohomology group of the complex

$$A(L_q^*(F)(X)) = \{L_q^m(F)(X) = \pi_q(F^m(X)),\ d_{q,m}(X);\ m \geq 0\}.$$

This complex is the complex of global sections of acyclic resolution 2.20.1 of $\tilde{\pi}_q(F)$. Therefore, the p-th cohomology group of $A(L_q^*(F)(X))$ is isomorphic to $H^p(X_{cd},\tilde{\pi}_q(F))$. This shows that $E_2^{p,q} = H^p(X_{cd},\tilde{\pi}_q(F))$.

The statement about the strong convergence follows from Theorem 2.21. The vanishing of $E_2^{p,q}$ for $p > \dim X$ and for all $q \in \mathbb{Z}$ follows from Theorem 1.32.

2.23. Remark: Theorem 2.22 justifies the term "hypercohomological spectrum" for $\mathbb{H}(X_{cd},F)$. It is a variant for the cd-topology of a result of Thomason ([38], Prop. 1.36). However, it does not follow formally from this result because the use of our non-classical definition of points of X_{cd}. Notice, that this definition was motivated in part by the necessity to make the Godement resolution T^*F and, hence, the hypercohomological spectrum $\mathbb{H}(X_{cd},F)$ functorial in X.

3. LOCAL HOMOTOPY THEORY FOR X_{cd}

In §§3, 4 we shall assume that X is a locally noetherian scheme.

3.1. In this section, we shall develop formalism of local sheaves of spectra $\Gamma_Y(F)$, (resp. $\Gamma_y(F)$) associated with a sheaf of spectra F on the cd-topology and a locally closed subscheme $Y \hookrightarrow X$ (resp. a point $y \in X$). Notice, that the sheaves $\Gamma_y(F)$, $y \in X$, are more complicated in our setting, than the corresponding sheaves on the Zariski topology. More precisely, they are not constant on the closure $\bar{y}$ of y in X. Nevertheless, they are still acyclic as we will see in §4, and the acyclicity is the main property of these sheaves which will be used for the construction of spectral sequence 0.5.1 and its variants.

3.2. Let X be a scheme, $i: Y \hookrightarrow X$ a closed subscheme, $\varphi: X' \longrightarrow X$ an etale morphism, $Y' = X' \times_X Y$. Assume that φ induces an isomorphism $\varphi_{Y'}: Y' \xrightarrow{\sim} Y$. Denote $U = X - Y$, $U' = X' \times_X U = X' - Y'$, $\varphi_U: U' \longrightarrow U$. Let $j: U \hookrightarrow X$, $j': U' \hookrightarrow X'$ and $i': Y' \hookrightarrow X'$ be the canonical embeddings.

Let $F: X_{cd} \longrightarrow FSp$ be a presheaf of fibrant spectra on X_{cd}. For a map $u: E \longrightarrow E'$ of spectra denote by $hf(u)$ its homotopy fibre. Consider the diagram

3.2.1.
$$\begin{array}{ccccc}
hfF(j) & \longrightarrow & F(X) & \xrightarrow{F(j)} & F(U) \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle F(\varphi)} & & \downarrow{\scriptstyle F(\varphi_{U'})} \\
hfF(j') & \longrightarrow & F(X') & \xrightarrow{F(j')} & F(U')
\end{array}$$

where all maps are natural.

3.3. <u>Definition</u>: (1) We say that the presheaf F has the *excision property for the pair* $(\varphi: X' \longrightarrow X,\ Y \hookrightarrow X)$ with $Y' = X' \times_X Y \xrightarrow{\sim} Y$ as above, if the canonical map $\lambda: hfF(j) \xrightarrow{\sim} hfF(j')$ induced by φ is a weak homotopy equivalence.

(2) We say that F has the *excision property for the cd-topology* if for any couple $(\varphi: X' \longrightarrow X,\ Y \hookrightarrow X)$ as above F has the excision property.*

3.4. If the pair $\{\varphi: X' \longrightarrow X,\ i: Y \hookrightarrow X\}$ is as above, the pair $\{\varphi: X' \longrightarrow X,\ j: U \longrightarrow X\}$ can be considered as a cd-covering of X. The fact that λ in 3.2.1 is a weak homotopy equivalence (= w.h.e.) is equivalent to the fact that diagram 3.2.1 is homotopically cartesian in the sense of ([33], §1; [8], §2), i.e. the canonical map $F(X) \longrightarrow F(X') \times^h_{F(U')} F(U)$ into the homotopy theoretical fibre product is a weak homotopy equivalence. Equivalently, it can be said that the presheaf F satisfies the Mayer-Vietoris property for the covering $\{\varphi: X' \longrightarrow X,\ j: U \hookrightarrow X\}$ as it is shown in §3.5 below.

3.5. Denote for simplicity $W = hfF(j)$, $W' = hfF(j')$. Then we have the commutative diagram of the homotopy groups:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_q(W) & \longrightarrow & \pi_q(F(X)) & \longrightarrow & \pi_q(F(U)) & \longrightarrow & \pi_{q-1}(W) & \longrightarrow & \cdots \\
 & & \wr\downarrow & & \downarrow & & \downarrow & & \wr\downarrow & & \\
\cdots & \longrightarrow & \pi_q(W') & \longrightarrow & \pi_q(F(X')) & \longrightarrow & \pi_q(F(U')) & \longrightarrow & \pi_{q-1}(W') & \longrightarrow & \cdots
\end{array}$$

It is known [65] that such a diagram is equivalent to the Mayer-Vietoris exact sequence for the homotopy groups of the cd-cover

* see the Note Added on p. 342.

$\{X',U\}$ of X:

3.5.1. $\cdots \longrightarrow \pi_q(F(X)) \longrightarrow \pi_q(F(U))\oplus\pi_q(F(X')) \longrightarrow \pi_q(F(U\times_X X')) \longrightarrow \cdots$

3.6. <u>Definition</u>: Let $F\colon (\mathrm{Sch}(X)_{fl} \longrightarrow \mathrm{FSp}$ be a contravariant functor from the category of X-schemes and flat morphisms $(\mathrm{Sch}/X)_{fl}$ to the category FSp of fibrant spectra. Assume that for any $X' \in \mathrm{Ob}(X'_{cd})$ and a closed subscheme $i'\colon Y' \hookrightarrow X'$ a *Gysin map* $Gy(i')\colon F(Y') \longrightarrow F(X')$ is defined, which is a morphism of spectra natural on the pair (X',Y').

We say that F has the *localization sequences* if the maps

3.6.1 $\quad F(Y') \xrightarrow{Gy(i')} F(X') \xrightarrow{F(j')} F(U')$

define a homotopy fibre sequence for any such pair (X',Y').

Notice that to homotopy fibre sequence 3.6.1 corresponds the Quillen type localization sequence of its homotopy groups

3.6.2 $\quad \cdots \longrightarrow \pi_q(F(Y')) \longrightarrow \pi_q(F(X')) \longrightarrow \pi_q(F(U')) \longrightarrow \cdots$

which justifies our terminology.

3.7. <u>Lemma</u>: Let $F\colon X_{cd} \longrightarrow \mathrm{FSp}$ be a presheaf of fibrant spectra on X_{cd} which has the localization sequences. Let $\varphi\colon X' \longrightarrow X$, Y, Y', U, U' be such as in 3.2. Then the pair $(\varphi\colon X' \longrightarrow X,\ Y \longrightarrow X)$ has the cd-excision property.

<u>Proof</u>: Consider the diagram

$$\begin{array}{ccccc} F(Y) & \xrightarrow{Gy(i)} & F(X) & \xrightarrow{F(j)} & F(U) \\ \downarrow{\scriptstyle F(\varphi_{Y'})} & & \downarrow{\scriptstyle F(\varphi)} & & \downarrow{\scriptstyle F(\varphi_{U'})} \\ F(Y') & \xrightarrow{Gy(i')} & F(X') & \xrightarrow{F(j')} & F(U') \end{array}$$

where all maps are the natural maps induced by i, i', j, j', φ, $\varphi_{Y'}$, and $\varphi_{U'}$. By our assumptions both rows are homotopy fibre sequences, i.e. $hf(F(j) \xleftarrow[w.h.e.]{\sim} F(Y)$ and $hf(j')) \xleftarrow[w.h.e.]{\sim} F(Y')$. Since $\varphi_{Y'}$ is an isomorphism, $F(\varphi_{Y'})$ is also an isomorphism, and it induces a weak homotopy equivalence

$$hf(F(j)) \xrightarrow[w.h.e.]{\sim} hf(F(j')).$$

3.8. <u>Examples</u>: (1) For a scheme X let $\underline{G}(X)$ (resp. $\underline{G}(X,\mathbb{Z}/\ell\mathbb{Z})$ be the fibrant K-theory (resp. K-theory with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients) spectrum of the category $\mathrm{Coh}(X)$ of coherent $\mathcal{O}_X$-Modules, (see §2.6). Then the contravariant functors $\underline{G}\colon X' \longrightarrow \underline{G}(X')$ and $\underline{G}(\mathbb{Z}/\ell\mathbb{Z})\colon X' \longrightarrow \underline{G}(X',\mathbb{Z}/\ell\mathbb{Z})$ on $(\mathrm{Sch}/X)_{fl}$ have the Gysin maps and the localization sequence ([33], §7, prop. 3.2, [38], §2.7). Therefore, they have the cd-excision property.

(2) Let $\underline{K}(X)$ (resp. $\underline{K}(\mathbb{Z}/\ell\mathbb{Z})$) be the fibrant K-theory (resp. K-theory with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients) spectrum of the category $LF(X)$ of locally free $\mathcal{O}_X$-Modules. Let $\underline{K}\colon X' \longrightarrow \underline{K}(X')$ and $\underline{K}(\mathbb{Z}/\ell\mathbb{Z})\colon X' \longrightarrow \underline{K}(X,\mathbb{Z}/\ell\mathbb{Z})$ be the corresponding contravariant functors on Sch/X. It has been proved recently by Thomason and Trobaugh [62] that if X is quasi-compact and quasi-separated, $\underline{K}$ and $\underline{K}(\mathbb{Z}/\ell\mathbb{Z})$ essentially have the cd-excision. More precisely, they constructed in [62] spectra $\underline{K}^B(X)$ and $\underline{K}^B(X,\mathbb{Z}/\ell\mathbb{Z})$, such that the families of their homotopy groups $K^B_n(X) = \pi_n(\underline{K}^B(X))$ and $K^B_n(X,\mathbb{Z}/\ell\mathbb{Z}) = \pi_n(\underline{K}^B(X,\mathbb{Z}/\ell\mathbb{Z}))$, $n \in \mathbb{Z}$, are Bass-type extensions of the usual K-groups $K_n(X)$ and $K_n(X,\mathbb{Z}/\ell\mathbb{Z})$ respectively to non-positive degrees $(K^B_n = K_n$ and

$K^B_n(\mathbb{Z}/\ell\mathbb{Z}) = K_n(\mathbb{Z}/\ell\mathbb{Z})$ for $n > 0$), and proved that they have the cd-excision property.

If X is <u>regular</u>, there exists canonical weak homotopy equivalences

3.8.1. $\underline{K}^B(X) \xrightarrow{\sim} \underline{K}(X) \xrightarrow{\sim} \underline{G}(X)$, $\underline{K}^B(X,\mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\sim} \underline{K}(X,\mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\sim} \underline{G}(X,\mathbb{Z}/\ell\mathbb{Z})$

([33], §4, Cor. 2; [38], [62]). Hence, these classes of examples coincide.

3.9. <u>Definition</u>: Let $i: Y \hookrightarrow X$ be a closed subscheme, $F: X_{cd} \to FSp$ a presheaf of fibrant spectra which has the excision property on X_{cd}. Define the *presheaf of local spectra of* F *modulo* Y, $\Gamma_Y(F): X_{cd} \to FSp$, as the presheaf whose value on an etale scheme $\varphi: X' \to X$ is given as the homotopy fibre of $F(j')$:

3.9.1: $\Gamma_Y(F)(X') = hf(F(j'): F(X') \to F(X'-Y'))$,

where $Y' = Y \times_X X'$, $U = X-Y$, $U' = X'-Y'$, and $j: U \hookrightarrow X$, and $j': U' \hookrightarrow X'$ are the natural open embeddings.

Notice, that this definition implies the canonical identification

3.9.2. $\Gamma_Y(X') = \Gamma_{Y'}(X')$.

3.10. Let Y be now a locally closed subscheme of X, i.e. there exists an open subscheme V of X such that

3.10.1. $Y = V \times_X \overline{Y}$

where $\overline{Y}$ is the closure of Y in X. Let $M(Y)$ be the category of all open $V \hookrightarrow X$ for which presentation 3.10.1 exists. It is easy to see that the category $M(Y)$ is pseudo-filtered and the dual category

$M(Y)^o$ is filtered (compare with Proposition 1.11(i)).

For all $V \in Ob(M(Y))$ define a presheaf of spectra $\Gamma^V_Y(F): X_{cd} \to FSp$ by the formula

3.10.2. $\Gamma^V_Y(F)(X') = hf(F(j'_V): F(X'\times_X V) \to F(X'\times_X V - X'\times_X V \times_X \overline{Y}))$

(Notice, that $X' \times_X V \times_X \overline{Y}$ is closed in $X' \times_X V$).

Define now the *presheaf* $\Gamma_Y(F)$ *of local spectra of* F *modulo* Y as a limit

3.10.3. $\Gamma_Y(F) = \varinjlim_{V \in Ob(M(Y)^o)} \Gamma^V_Y(F)$

Since the category of fibrant spectra is closed under filtered direct limits, $\Gamma_Y(F)$ is actually a presheaf of fibrant spectra.

Let now $V_1 \hookrightarrow V$ be a morphism in $M(Y)$. Then we have a commutative diagram

$$\begin{array}{ccccc} hf(F(j'_V)) & \longrightarrow & F(X'\times_X V) & \xrightarrow{F(j'_V)} & F(X'\times_X V - X'\times_X V \times_X \overline{Y}) \\ \downarrow \lambda & & \downarrow & & \downarrow \\ hf(F(j'_{V_1})) & \longrightarrow & F(X'\times_X V_1) & \xrightarrow{F(j'_{V_1})} & F(X'\times_X V_1 - X'\times_X V_1 \times_X \overline{Y}) \end{array}$$

rows of which are homotopy fibre sequences. Since F has the excision property on X_{cd}, in particular, on X_{Zar}, the right square is homotopy cartesian and, hence, the canonical map

3.10.4. $\lambda_{VV_1}: \Gamma^V_Y(F)(X') = hf(F(j'_V)) \xrightarrow{\sim} hf(F(j'_{V_1})) = \Gamma^{V_1}_Y(F)(X')$

is a weak homotopy equivalence.

By the definition of the inductive limit we have the canonical map

3.10.5. $\lambda_V: \Gamma_Y^V(F) \longrightarrow \Gamma_Y(F)$, for all $V \in Ob(M(Y))$

Since for a fixed $V_0 \in Ob(M(Y))$ the category of its open subschemes $V_1 \subset V_0$, containing in $M(Y)$, is cofinal in $M(Y)$, weak homotopy equivalences 3.10.4 for all open $V_1 \hookrightarrow V_0$ in $M(Y)$ imply that 3.10.5 is a weak homotopy equivalence, for all $V \in Ob(M(Y))$.

3.10.6. Lemma: Let X, Y, F be as above, $\varphi: X_1 \longrightarrow X$ an etale morphism, $Y_1 = \varphi^{-1}(Y)$. Then there exists a canonical weak homotopy equivalence

3.10.7. $\Gamma_Y(F)(X_1) \xrightarrow{\sim} \Gamma_{Y_1}(F)(X_1)$

Proof: Let $V \in Ob(M(Y))$, $V_1 = \varphi^{-1}(V)$. Then V_1 is open in X_1 and it is easy to see that $Y_1 = V_1 \times_X \overline{Y}_1$, i.e. $V_1 \in Ob(M(Y_1))$. By Definition 3.10.2 and equality 3.9.2 we can canonically identify

3.10.8. $\Gamma_Y^V(F)(X_1) = \Gamma_{Y_1}^{V_1}(F)(X_1)$

Combining 3.10.5 for Y_1 and V_1 and 3.10.8 we obtain a canonical weak homotopy equivalence (depending on V):

3.10.9. $\alpha_V: \Gamma_Y^V(F)(X') = \Gamma_{Y_1}^{V_1}(F)(X') \xrightarrow[\sim]{\lambda_{V_1}} \Gamma_{Y_1}(F)(X')$

Passing to the limit on V we obtain a canonical weak homotopy equivalence $\alpha = \varinjlim \alpha_V$:

3.10.10. $\alpha: \Gamma_Y(F)(X_1) = \varinjlim_{V \in Ob(M(Y)^o)} \Gamma_Y^V(F)(X_1) \xrightarrow{\sim} \Gamma_{Y_1}(F)(X_1)$

3.11. Proposition: (excision for $\Gamma_Y(F)$). Let Y be a subscheme of X locally closed in X, $\varphi: X_1 \longrightarrow X$ an etale map such that φ induces an isomorphism $\varphi|_{Y_1}: Y_1 = \varphi^{-1}(Y) \xrightarrow{\sim} Y$. Then for all $X' \in Ob(X_{cd})$ there exists a canonical weak homotopy equivalence:

3.11.1. $ex(X'): \Gamma_Y(F)(X') \xrightarrow{\sim} \Gamma_{Y_1}(F)(X' \times_X X_1)$

naturally depending on X'.

Proof: Pick $V \in Ob(M(Y))$, so that $Y = V \times_X \overline{Y}$. Let $V_1 = \varphi^{-1}(V)$. Then it is easy to see that $Y_1 = V_1 \times_X \overline{Y}_1$. It follows now from Definition 3.10.2 and diagram 3.2.1 applied to the pair $(X' \times_X V, X' \times_X X_1 \times_X V)$ that we have the canonical and naturally depending on X' map

3.11.2. $ex(X')_V: \Gamma_Y^V(F)(X') \longrightarrow \Gamma_{Y_1}^{V_1}(F)(X' \times_X X_1)$,

for any $X' \in Ob(X_{cd})$. Since the canonical map

$$\lambda_{V_1}(X' \times_X X_1): \Gamma_{Y_1}^{V_1}(F)(X' \times_X X_1) \xrightarrow{\sim} \Gamma_{Y_1}(F)(X' \times_X X_1)$$

is a weak homotopy equivalence by 3.10.5, it is sufficient to prove that 3.11.2 is a weak homotopy equivalence and to pass to the limit on $V \in Ob(M(Y))$ (as in the proof of Lemma 3.10.6). So replacing X (resp. X_1) by V (resp. V_1), we may assume that Y (resp. Y_1) is closed in X (resp. X_1).

Let $Y' = X' \times_X Y$, $X_1' = X' \times_X X_1$, and $Y_1' = X_1' \times_{X'} Y' = X' \times_{X_1} Y_1$. The assertion that 3.11.2 is a weak homotopy equivalence follows now from the excision for the pair $(\varphi_{X'}: X_1' \longrightarrow X', Y' \hookrightarrow X')$ (see Definition 3.3), which satisfies to the conditions of 3.3 because $\varphi_{X'}$ induces an isomorphism $Y_1' \xrightarrow{\sim} Y'$.

3.12. Lemma: Let $V \subset X$ be an open subscheme of X; $Z = V \times_X \overline{Z}$ be a locally closed subscheme of X, $Z_1 \subset Z$ a closed subscheme of Z, $Z_2 = Z - Z_1$. Then the natural maps give the homotopy fibre sequence of presheaves of spectra on X_{cd}:

3.12.1. $\underline{\Gamma}_{Z_1}(F) \longrightarrow \underline{\Gamma}_Z(F) \longrightarrow \underline{\Gamma}_{Z_2}(F)$

Proof: Notice, that under the conditions of the lemma, $Z_1 = V \times_X \overline{Z}_1$ and $V' = V - V \times_X \overline{Z}_1$ is open in V, hence, in X. We also have $Z_2 = V' \times_X \overline{Z}_2$ and

3.12.2. $V - V \times_X \overline{Z} = (V - V \times_X \overline{Z}_1) - (V \times_X \overline{Z}_2) = V' - V' \times_X \overline{Z}_2$.

Let $\varphi: X' \longrightarrow X$ be an etale morphism. Consider a 3×3 diagram

$$\begin{array}{ccccc}
\underline{\Gamma}_{Z_1}(F)(X') & \longrightarrow & \underline{\Gamma}_Z(F)(X') & \longrightarrow & \underline{\Gamma}_{Z_2}(F)(X') \\
\downarrow \wr & & \downarrow & & \downarrow \\
\underline{\Gamma}_{Z_1}(F)(X') & \longrightarrow & F(X' \times_X V) & \longrightarrow & F(X' \times_X V - X' \times_X V \times_X \overline{Z}) \\
& & & & = F(X' \times_X V') \\
\downarrow & & \downarrow & & \downarrow \\
(pt) & \longrightarrow & F(X' \times_X V - X' \times_X V \times_X \overline{Z}) & \xrightarrow{\sim} & F(X' \times_X V - X' \times_X V \times_X \overline{Z}) = \\
& & & & = F(X' \times_X V' - X' \times_X V' \times_X \overline{Z}_2)
\end{array}$$

Columns of this diagram and the middle row are homtopy fibre sequences by Definition 3.10.2 and (the last column) by 3.12.2. Hence, the top row is a homotopy fibre sequence by ([2], 1.2).

3.13. Let, as in §1.30, X_p be the set of all points of X of codimension p in X, $Z_p(X)$ be the set of all closed subsets of

codimension $\geq p$. $Z_p(X)$ is a directed partially ordered set, which can be considered in the usual way as a filtered category.

For a presheaf of fibrant spectra $F: X_{cd} \longrightarrow FSp$ which has the cd-excision property, define

3.13.1. $$S_X^p(F) = \varinjlim_{Z \in Z_p(X)} \underline{\Gamma}_Z(F)$$

Since $Z_p(X)$ is filtered $S_X^p(F)$ is a presheaf of fibrant spectra. It is a homotopy-theoretical version of the filtration F^p on the cohomology groups considered by Grothendieck (see §1.30).

3.14. For a subscheme Z of X and $y \in Z$ let $N_{Zar}(y,Z)$ be the pseudo-filtered category of Zariski open neighbourhoods of y in Z. If $Z = X$ we shall write simply $N_{Zar}(y)$ instead of $N_{Zar}(y,X)$. Let $N_{Zar}(u,Z)^o$ and $N_{Zar}(y)^o$ be the dual categories. They are filtered.

If X is an irreducible scheme and x_0 the generic point of X the functor $Z \longrightarrow X-Z$ is an equivalence of categories $Z_1(X)$ and $N_{Zar}(x_0,X)^o$.

3.15. For a point $y \in X$ define the *presheaf* $\underline{\Gamma}_y(F): X_{cd} \longrightarrow FSp$ *of local spectra of* F *modulo* y by the formula:

3.15.1. $$\underline{\Gamma}_y(F) \underset{def}{=} \varinjlim_{V_y \in N_{Zar}(y)^o} \underline{\Gamma}_{\overline{y} \times_X V_y}(F) = \varinjlim_{U_y \in N_{Zar}(y,\overline{y})^o} \underline{\Gamma}_{U_y}(F),$$

i.e. for all $X' \in Ob(X_{cd})$ we define

3.15.2.
$$\underline{\Gamma}_y(F)(X') \underset{def}{=} \varinjlim_{V_y \in N_{Zar}(y)^o} hf(F(X' \times_X V_y) \longrightarrow F(X' \times_X V_y - X' \times_X V_y \times_X \overline{y})).$$

The presheaf $\pi_q(\Gamma_y(F))$ of the q-th homotopy groups of $\Gamma_y(F)$ we shall call the q-th *presheaf of local homotopy groups of* F *modulo* y.

Since the category FSp is closed under filtered direct limits, $\Gamma_y(F)$ is again a presheaf of fibrant spectra. The equivalence of categories $N_{Zar}(y,\bar{y})$ and $Z_1(\bar{y})$ discussed in 3.14 and the second of the equalities of 3.15.1 imply that

3.15.3. $$\Gamma_y(F) = \varinjlim_{Z\in Z_1(\bar{y})} \Gamma_{\bar{y}-Z}(F).$$

Let now $y = \coprod_{i\in I} y_i$ be a finite sum of points $y_i \in X_p$, $p \geq 0$. Define then $\Gamma_y(F)$ as the wedge

3.15.4. $$\Gamma_y(F) = \vee_{i\in I} \Gamma_{y_i}(F)$$

Let k φ: $X' \longrightarrow X$ be an etale morphism, $y \in X$ a point, $y' = \varphi^{-1}(y)$. It follows then from 3.10.7, and Definitions 3.15.1 and 3.15.4 that there exists a canonical and naturally depending on X' weak homotopy equivalence

3.15.5. $$\Gamma_y(F)(X') \xrightarrow{\sim} \Gamma_{y'}(F)(X')$$

The following excision property of the presheaves $\Gamma_y(F)$ will be crucial for our proof of their acyclicity in §4:

3.16. Proposition: (excision for $\Gamma_y(F)$). Let φ: $X_1 \longrightarrow X$ be an etale morphism, $y \in X$, and $y_1 \in \varphi^{-1}(y)$ be a point such that φ induces an isomorphism of the residue fields φ^a: $k(y) \xrightarrow{\sim} k(y_1)$ (i.e. $y \in cd(X_1/X)$ in the notations of §1.

Let F: $X_{cd} \longrightarrow FSp$ be a presheaf of fibrant spectra on X_{cd} which has the cd-excision. Then for any $X' \in Ob(X_{cd})$ there exists a canonical weak homotopy equivalence

3.16.1. $$ex_y(X'): \Gamma_y(F)(X') \xrightarrow{\sim} \Gamma_{y_1}(F)(X'\times_X X_1).$$

naturally depending on X'.

Proof: Since $k(y_1) \xrightarrow{\sim} k(y)$ there exist open subschemes $V \subset \bar{y}$ and $V_1 \subset \bar{y}_1$ such that φ induces an isomorphism φ: $V_1 \xrightarrow{\sim} V$. Hence, the categories of neighborhoods $N_{Zar}(y,V)$ and $N_{Zar}(y_1,V_1)$ are isomorphic. Since $N_{Zar}(y,V)$ is cofinal in $N_{Zar}(y_1,\bar{y}_1)$ and $N_{Zar}(y_1,V_1)$ is cofinal in $N_{Zar}(y_1,\bar{y}_1)$, we obtain:

3.16.2.
$$\Gamma_y(F) = \varinjlim_{V_y\in N_{Zar}(y,V)^o} \Gamma_{V_y}(F).$$
$$\Gamma_{y_1}(F) = \varinjlim_{V_{y_1}\in N_{Zar}(y_1,V_1)^o} \Gamma_{V_{y_1}}(F).$$

By Proposition 3.11 we have a canonical weak homotopy equivalence induces by φ:

3.16.3. $$ex_V(X'): \Gamma_V(F)(X') \xrightarrow{\sim} \Gamma_{V_1}(F)(X'\times_X X_1)$$

depending functorially on X'. Using 3.16.2 and the isomorphism $N_{Zar}(y,V) \xrightarrow{\sim} N_{Zar}(y_1,V_1)$, define the map $ex_y(X')$ in 1.16.1 as the filtered direct limit:

3.16.4. $$ex_y(X') = \varinjlim_{V\in Ob(N_{Zar}(y,V)^o)} ex_V(X')$$

It is clear that it has all the required properties.

3.17. Let $Y \in Z_p(X)$, $\{Y_i, i \in I\}$ be the set of irreducible components Y_i of Y and $y_i \in Y_i$ the generic point of Y_i. Let Z be a proper closed subscheme of Y, such that $Z \in Z_1(Y)$. Then $Z \in Z_{p+1}(X)$. Applying 3.12.1 we obtain a homotopy fibre sequence

3.17.1. $\Gamma_Z(F) \longrightarrow \Gamma_Y(F) \longrightarrow \Gamma_{Y-Z}(F)$.

By taking a limit of sequences 3.17.1 over the filtered category $Z_1(Y)$ and using 3.15.3 we obtain again homotopy fibre sequence:

3.17.2. $\varinjlim_{Z \in Z_1(Y)} \Gamma_Z(F) \longrightarrow \Gamma_Y(F) \longrightarrow \bigvee_{i \in I} \Gamma_{y_i}(F)$

Passing now to the filtered direct limit on $Y \in Z_p(X)$ we obtain a homotopy fibre sequence

3.17.3. $S_X^{p+1}(F) \longrightarrow S_X^p(F) \longrightarrow \bigvee_{y \in X_p} \Gamma_y(F)$.

3.18. Lemma: Let $F\colon X_{cd} \longrightarrow FSp$ be a presheaf which has the cd-excision property. Assume that F is additive. Then for any locally closed subscheme $Z \hookrightarrow X$ and any point $x \in X$ the presheaves $\Gamma_Z(X)$ and $\Gamma_x(F)$ are additive.

Proof: Assume first, that Z is closed in X. Let $U = X - Z$. For all $X_1, X_2 \in Ob(X_{cd})$ consider a commutaitve diagram with the natural maps

$$\begin{array}{ccccc}
 & & & & F((U\times_X X_1)\sqcup(U\times_X X_2)) \\
 & & & & \parallel \\
\Gamma_Z(F)(X_1 \sqcup X_2) & \longrightarrow & F(X_1 \sqcup X_2) & \longrightarrow & F(U\times_X(X_1 \sqcup X_2)) \\
\downarrow \alpha(\Gamma_Z(F)) & & \wr\downarrow \alpha(F) & & \wr\downarrow \alpha(F|_U) \\
\Gamma_Z(F)(X_1) \times \Gamma_Z(F)(X_2) & \longrightarrow & F(X_1) \times F(X_2) & \longrightarrow & F(U\times_X X_1) \times F(U\times_X X_2)
\end{array}$$

In this diagram rows are homotopy fibre sequences by the definition of $\Gamma_Z(F)$, and the vertical maps $\alpha(F)$ and $\alpha(F|_U)$ are weak homotopy equivalences. Hence, $\alpha(\Gamma_Z(F))$ is a weak homotopy equivalence.

Assume now that Z is locally closed, i.e. $Z = \bar{Z} \cap V$, where V is an open subscheme of X. Then the restriction $F|_V$ of F on V is additive, and Z is closed in V. Hence, as shown above, $\Gamma_Z^V(F|_V) \xrightarrow[\text{w.h.e.}]{\sim} \Gamma_Z(F)$ is additive. The additivity of $\Gamma_x(F)$, $x \in X$ follows from the additivity of $\Gamma_{\bar{x} \times_X V_x}(F)$, for all $V_x \in N_{Zar}(x,X)$, shown above, Definition 3.15.1 and the fact that filtered direct limits preserve weak equivalences in FSp.

4. ACYCLICITY OF THE LOCAL HOMOTOPY PRESHEAVES AND A CONSTRUCTION OF THE DESCENT SPECTRAL SEQUENCE

4.0. Let $F\colon X_{cd} \longrightarrow FSp$ be a presheaf of fibrant spectra on X_{cd} which has the excision property for the cd-topology. Then the presheaves of local spectra $\Gamma_Y(F)$ and $\Gamma_x(F)$ are well defined for any locally closed subscheme $Y \hookrightarrow X$ and a point $x \in X$.

Under these assumptions we shall show in 4.1-4.3 that the presheaves $\Gamma_x(F)$ are weakly homotopically equivalent to the direct images of certain sheaves on x_{cd} and this will imply their acyclicity with respect to the cd-hypercohomology described in §2. The acyclicity of these sheaves will allow us to prove that the canonical augmentation $\eta: F(X) \to \mathbb{H}(X_{cd}, F)$ is a weak homotopy equivalence of presheaves of spectra on X_{cd} using the inductive process of [38], 2.4. Then the hypercohomological spectral sequence 2.22.1 corresponding to $\mathbb{H}(X_{cd}, F)$ will give to us spectral sequence 0.5.1, its generalizations and variants.

4.1. Lemma: Let $F: X_{cd} \to FSp$ be a presheaf of fibrant spectra which has the cd-excision property, x a point of X. Then the canonical morphism of presheaves on X_{cd}

4.1.1. $\lambda_x: \Gamma_x(F) \xrightarrow{\sim} i_{x,\#} i_x^{\#}(\Gamma_x(F))$

is a weak homotopy equivalence.

Proof: Let $Y \to X$ be an etale X-scheme $y = Y \times_X x$. Then

4.1.2. $(i_{x,\#} i_x^{\#}(\Gamma_x(F)))(Y) = i_x^{\#}(\Gamma_x(F))(y) = \varinjlim_{Y' \in Ob(N_{cd}(y,Y)^o)} \Gamma_x(F)(Y')$

by Proposition 1.11(2).

Therefore, λ_x is the canonical map

4.1.3. $\Gamma_x(F)(Y) \longrightarrow \varinjlim_{Y' \in Ob(N_{cd}(y,Y)^o)} \Gamma_x(F)(Y')$

of a member of the inductive system into the inductive limit. Denote $y' = Y' \times_X x$. Using the canonical weak homotopy equivalences $\Gamma_x(F)(Y) \xrightarrow{\sim} \Gamma_y(F)(Y)$ and $\Gamma_x(F)(Y') \xrightarrow{\sim} \Gamma_{y'}(F)(Y')$ of 3.15.5 we can rewrite 4.1.3 as

4.1.4. $\Gamma_y(F)(Y) \longrightarrow \varinjlim_{Y' \in Ob(N_{cd}(y,Y)^o)} \Gamma_{y'}(F)(Y')$

Let $y = \coprod_{i=1}^{k} y_i$ be the decomposition of y into the sum of its irreducible components y_i. Denote $k(y) = \bigoplus_{i=1}^{k} k(y_i)$.

By Definition 1.8, for every cd-neighborhood $\varphi: Y' \to Y$ of y there exists $y_0' \in \varphi^{-1}(y) = y'$ such that the induced maps $y_0' \xrightarrow{\sim} y'$ and $\varphi^a: k(y) \xrightarrow{\sim} k(y_0')$ are isomorphisms. If, moreover, $y_0' = y'$ then by Definition 3.15.4 and Proposition 3.16 the natural map

4.1.5. $\Gamma_y(F)(Y) \longrightarrow \Gamma_{y'}(F)(Y')$

is a weak homotopy equivalence.

Replacing Y' by its open subscheme $Y'' = Y' - (y' - y_0')$ as in the step (2) of the proof of Theorem 1.27 we can achieve that for $\psi = \varphi_{Y''}: Y'' \to Y$, $y_0' = \psi^{-1}(y)$ and, hence,

4.1.6. $\Gamma_y(F)(Y) \xrightarrow{\sim} \Gamma_{y'}(F)(Y'')$

is a weak homotopy equivalence. It shows that the subcategory $A(y,Y)$ of all $Y' \in Ob(N_{cd}(y,Y))$ for which 4.1.5 is a weak homotopy equivalence is cofinal in $N_{cd}(y,Y)$. Therefore, 4.1.4 and 4.1.1 are weak homotopy equivalences. Q.E.D.

4.2. Corollary: Let $F: X_{cd} \to FSp$ be an additive presheaf which has the cd-excision property, x a point of X. Then

(1) the canonical maps

4.2.1. $\Gamma_x(F) \xrightarrow{\sim} i_{x,\#} i_x^{\#}(\Gamma_x(F)) \xrightarrow{\sim} i_{x,*} i_x^{*}(\Gamma_x(F))$

are weak homotopy equivalences.

(2) The presheaves of the local homotopy groups $\pi_q(\Gamma_x(F))$ are actually sheaves on X_{cd}, for all $q \in \mathbb{Z}$, and

4.2.2. $\quad H^p(X_{cd}, \pi_q(\Gamma_x(F)) = 0$, for all $p > 0$, and for all $q \in \mathbb{Z}$.

<u>Proof</u>: (1) Since F is an additive presheaf $\Gamma_x(F)$ is also an additive presheaf by Lemma 3.18. Hence, by Lemma 2.12(2) the canonical map

4.2.3. $\quad g_x: i_x^{\#}(\Gamma_x(F)) \longrightarrow i_x^{*}(\Gamma_x(F))$

is a weak homotopy equivalence. Since the functor $i_{x,\#}$ preserves weak homotopy equivalences by Lemma 2.9(3), this implies that the second map in 4.2.1 is a weak homotopy equivalence. The first map in 4.2.1 is a weak homotopy equivalence by Lemma 4.1.

(2) The aditivity of $\Gamma_x(F)$ and weak homotopy equivalence 4.2.1 imply that $\pi_q(\Gamma_x(F))$ is a sheaf by Lemma 2.9(2) applied to $E = i_x^{*}(\Gamma_x(F))$. The vanishing property 4.2.2 follows from this fact, equivalences 4.2.1 and Corollary 2.16.

4.3. <u>Proposition</u>. Let $F: X_{cd} \longrightarrow FSp$ be an additive presheaf which has the cd-excision property. Then the natural augmentation

4.3.1. $\quad \eta(\Gamma_x(F)): \Gamma_x(F)(X) \xrightarrow{\sim} \mathbb{H}(X_{cd}, \Gamma_x(F))$

is a weak homotopy equivalence, for all $x \in X$.

<u>Proof</u>: Consider the hypercohomological spectral sequence for $\Gamma_x(F)$:

$$E_2^{p,q} = H^p(X_{cd}, \tilde{\pi}_q(\Gamma_x(F))) \Rightarrow \pi_{q-p}(\mathbb{H}(X_{cd}, \Gamma_x(F)).$$

(see Theorem 2.22). Since $\pi_q(\Gamma_x(F))$ is a sheaf by 4.2(2) we can replace in it $\tilde{\pi}_q(\Gamma_x(F))$ by $\pi_q(\Gamma_x(F))$.

By 4.2.2 $E_2^{p,q} = 0$, $p > 0$. Hence, the spectral sequence degenerates and gives the isomorphism of groups

4.3.2. $\quad H^0(X_{cd}, \pi_q(\Gamma_x(F))) \longrightarrow \pi_q(\mathbb{H}(X_{cd}, \Gamma_x(F))$, for all $q \in \mathbb{Z}$.

By definition $H^0(X_{cd}, \pi_q(\Gamma_x(F))) = \pi_q(\Gamma_x(F)(X))$. Hence, 4.3.2 implies 4.3.1.

4.4. <u>Theorem</u>: Let X be a noetherian scheme of finite Krull dimension, $F: X_{cd} \longrightarrow FSp$ an additive presheaf of fibrant spectra on X_{cd}, which has the cd-excision property. Then

(1) The canonical augmentation

4.4.1. $\quad \eta(S^p(F)): S^p(F)(X) \longrightarrow \mathbb{H}(X_{cd}, S^p(F))$.

is a weak homotopy equivalence for all $p \geq 0$. In particular, for $p = 0$, $S^0(F) \xrightarrow{\sim} F$

4.4.2. $\quad \eta(F): F(X) \xrightarrow{\sim} \mathbb{H}(X_{cd}, F)$

is a weak homotopy equivalence.

(2) There exists a strongly convergent spectral sequence

4.4.3. $\quad E_2^{p,q} = H^p(X_{cd}, \tilde{\pi}_q(F)) \Rightarrow \pi_{q-p}(F(X))$, $p \geq 0$, $q \in \mathbb{Z}$.

<u>Proof</u>: We shall prove statement (1) by a descending induction on p.

If $p > \dim X$, $Z_p(X) = \emptyset$. Hence, $S^p(F) \longrightarrow (pt)$ and $\mathbb{H}(X_{cd}, pt) \xrightarrow{\sim} pt$. This is the basis of our induction.

Assume now that the statement is true for $S^{p+1}(F)$. Consider the diagram

$$\begin{array}{ccccc} S^{p+1}(F)(X) & \longrightarrow & S^{p}(F)(X) & \longrightarrow & \bigvee_{x\in X_p} \Gamma_x(F)(X) \\ \int\Big\downarrow \eta(S^{p+1}(F)) & & \Big\downarrow \eta(S^{p}(F)) & & \int\Big\downarrow \eta(\Gamma,p) \\ \mathbb{H}(X_{cd},S^{p+1}(F)) & \longrightarrow & \mathbb{H}(X_{cd},S^{p}(F)) & \longrightarrow & \mathbb{H}(X_{cd},\bigvee_{x\in X}\Gamma_x(F)) \end{array}$$

The top row is a homotopy fibre sequence by 3.17.2, and the bottom row is a homotopy fibre sequence because the functor $F \longrightarrow \mathbb{H}(X_{cd},F)$ preserves homotopy fibre sequences by 2.13. By Proposition 4.3 $\eta(\Gamma,p)$ is a weak homotopy equivalence. Hence, $\eta(S^p(F))$ is a weak homotopy equivalence ([2],1.2).

(2) Consider now the hypercohomological spectral sequence for $\mathbb{H}(X_{cd},F)$:

4.4.4. $\quad E_2^{p,q} = H^p(X_{cd},\tilde{\pi}_q(F)) \Rightarrow \pi_{q-p}(\mathbb{H}(X_{cd},F))$

which is strongly convergent by Theor. 2.22. The weak homotopy equivalence 4.4.2 allows us to replace in it $\pi_{q-p}(\mathbb{H}(X_{cd},F))$ by $\pi_{q-p}(F(X))$. Q.E.D.

4.5. Examples: Let F be one of the presheaves of spectra $\underline{G}$ or $\underline{G}(\mathbb{Z}/\ell\mathbb{Z})$ (resp. $\underline{K}^B$ or $\underline{K}^B(\mathbb{Z}/\ell\mathbb{Z})$), corresponding to the K-theories of coherent sheaves (resp. to the Bass extensions of the K-theories of locally free sheaves) as in §3.8. Then these presheaves satisfy the conditions of Theorem 4.4 (see §3.8). Therefore, the specialization of spectral sequence 4.4.3 for these presheaves give spectral sequence

0.5.1, its analogue with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients:

$0.5.1._{\mathbb{Z}/\ell\mathbb{Z}} \quad E_2^{p,q} = H^p(X_{cd},\tilde{G}_q(\mathbb{Z}/\ell\mathbb{Z}) \Rightarrow G_{q-p}(X,\mathbb{Z}/\ell\mathbb{Z}), \quad p \geq 0, \ q-p \geq 0;$

and analogues of 0.5.1 and $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ for $K_n^B(X)$ and $K_n^B(X,\mathbb{Z}/\ell\mathbb{Z})$:

4.5.1. $\quad E_2^{p,q} = H^p(X_{cd},\tilde{K}_q^B) \Rightarrow K_{q-p}^B(X), \quad p \geq 0, \ q \in \mathbb{Z};$

$4.5.1_{\mathbb{Z}/\ell\mathbb{Z}} \quad E_2^{p,q} = H^p(X_{cd},\tilde{K}_q^B(\mathbb{Z})\ell\mathbb{Z})) \Rightarrow K_{q-p}^B(X,\mathbb{Z}/\ell\mathbb{Z}), \quad p \geq 0, \ q \in \mathbb{Z}.$

Here $\tilde{G}_q(\mathbb{Z}/\ell\mathbb{Z})$, $\tilde{K}_q^B$, ... are sheafifications of the presheaves G_q: $X' \longrightarrow G_q(X') \underset{\text{def}}{=\!=} \pi_q(\underline{G}(X'))$, K_q^B: $X' \longrightarrow K_q^B(X') \underset{\text{def}}{=\!=} \pi_q(\underline{K}_q^B(X'))$, ... respectively.

Recall that $K_q^B = K_q$ and $K_q^B(\mathbb{Z}/\ell\mathbb{Z}) = K_q(\mathbb{Z}/\ell\mathbb{Z})$ for $q > 0$, so spectral sequences 4.5.1 and $4.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ give some information about the usual K-groups of a singular X as well.

If X is regular, then spectral sequence 0.5.1 coincides with 4.5.1, and $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ coincides with $4.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$.

The following lemma gives a description of the fibres of the sheaf $\tilde{K}_n^{cd}(\mathbb{Z}/\ell\mathbb{Z})$ on X_{cd}:

4.6. Lemma: Let $x \in X$ be a point. Assume that ℓ is an integer which is prime to the characteristic of the residue field $k(x)$ of x. Then for any separable finite field extension $k'/k(x)$ we have

4.6.1. $\quad \tilde{K}_n^{cd}(\mathbb{Z}/\ell\mathbb{Z})_x(k') = K_n(k',\mathbb{Z}/\ell\mathbb{Z}).$

Proof: Let $\mathcal{O}_{x,X}^h$ be the henselization of the local ring $\mathcal{O}_{x,X}$ of x on X with respect to its maximal ideal $\mathfrak{m}_x$, and let $\mathcal{O}_{x'}^h$, be the unique local henselian ring which is etale over $\mathcal{O}_{x,X}^h$ and has the residue field k' ([EGA], IV, §18; [SGA 1], II).

Then for the sheaf $\tilde{K}_n^{cd}$ we have by Proposition 1.13.

4.6.2. $\tilde{K}_n^{cd}(\mathbb{Z}/\ell\mathbb{Z})_x(k') = K_n(\mathcal{O}_{x'}^h, \mathbb{Z}/\ell\mathbb{Z})$.

On the other hand by the rigidity theorem of Gabber [12] (see also [20], [36]) we have for ℓ prime to the char k(x) and the henselian local ring $\mathcal{O}_{x'}^h$:

4.6.3. $K_n(\mathcal{O}_{x'}^h, \mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\sim} K_n(k', \mathbb{Z}/\ell\mathbb{Z})$.

Equality 4.6.1 follows from 4.6.2 and 4.6.3.

4.7. Remark: The filtration by codimension of points of X was used in the proof of Theorem 4.4 only to establish required homotopy equivalence 4.4.2. This proof does not give, however, a comparison of spectral sequences 0.5.1 and $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ with the Quillen spectral sequence

4.7.1. $E_1^{p,q}(X) = \coprod_{x\in X_p} G_{q-p}(k(x)) \Rightarrow G_q(X), \quad p \geq 0, \ q - p \geq 0,$

and its analogue $4.7.1_{\mathbb{Z}/\ell\mathbb{Z}}$ for $G_q(X, \mathbb{Z}/\ell\mathbb{Z})$ arising from this filtration ([33], §7, theor. 5.4), or comparison of the corresponding filtrations on their common abutments $G_q(X)$ or $G_q(X, \mathbb{Z}/\ell\mathbb{Z})$. Such comparisons for our spectral sequences are open questions.

The coincidence of the Quillen and the Brown-Gersten spectral sequences from their E_2 terms upward for a smooth scheme X of finite type over a field is proved in ([33], §§7.5.6 - 7.5.11) and ([19], §2). One of the crucial ingredients of the proofs was the Gersten conjecture, proved in ([33], §7) under these assumptions.

On the other hand, the conjecture of Serre [51] and Grothendieck [51], [23] on the Zariski local triviality of rationally trivial principal homogeneous spaces modulo the results of [30], [31] is equivalent to the bijectivity of the canonical map

4.7.2. $H^1(X_{Zar}, r_*(H)) \xrightarrow{\sim} H^1(X_{cd}, H)$

for a regular X and a reductive X-group H (see 1.44.3). This conjecture and bijection 4.7.2 have been partially proved in [30]-[32] (see §1.44 for details). It can be considered as a group-theoretical analogue of the Gersten conjecture.

These results and the conjecture of Serre and Grothendieck motivate the following conjecture:

4.8. Conjecture: Let X be a regular scheme. Then spectral sequence 0.5.1, Brown-Gersten spectral sequence 0.4.2 and Quillen spectral sequence 4.7.1 coincide beginning from their E_2-terms upward. Similarly, their analogues with finite coefficients $0.4.2_{\mathbb{Z}/\ell\mathbb{Z}}$, $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ and $4.7.1_{\mathbb{Z}/\ell\mathbb{Z}}$ coincide beginning from their E_2-terms upward.

Notice, that at least in the case of finite coefficients, the sheaf $\tilde{K}_n^{cd}(X, \mathbb{Z}/\ell\mathbb{Z})$ is much easier to compute than the sheaf $\tilde{K}_n^{Zar}(X, \mathbb{Z}/\ell\mathbb{Z})$ due to Lemma 4.6. Hence, if Conjecture 4.8 is true, our spectral sequence $0.5.1_{\mathbb{Z}/\ell\mathbb{Z}}$ can be used to compute terms of two other spectral sequences involved.

However, one can show that for a singular X the canonical map

$$H^i(X_{Zar}, r_*(H)) \longrightarrow H^i(X_{cd}, H)$$

is not always bijective already for i = 2 and the multiplicative group scheme $H = G_m$. Since $K_1(R) = G_m(R) = R^*$ for a local ring R, this implies that spectral sequences 0.4.2 and 0.5.1 are different, in general, for a singular X. Spectral sequences 0.4.2 and 4.7.1 are known to be different, in general, for a singular X as well.

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NOTE ADDED IN PROOF:

(1) To make our definition of a neighbourhood of a point of x_{cd} (Def. 1.8) compatible with that in ([SGA 4], IV, §6) it is necessary to reformulate it by including the data of an X-section:

1.8 Definition. Let x be a point of X. A pair (U,s) consisting of an etale X-scheme $\varphi: U \longrightarrow X$ and an X-section $s: x \longrightarrow U$ of φ, is called a neighbourhood of the point x_{cd} of the site X_{cd}.

(2) The cd-excision condition for a presheaf $F: X_{cd} \rightarrow FSp$ in §3.3 must be formulated and actually is used in §§3.4 in the localized form, i.e. as the excision for any pair $[\varphi: V' \rightarrow V, Y \hookrightarrow V]$, consisting of an etale morphism $\varphi: V' \longrightarrow V$ in X_{cd} and a closed subscheme $Y \hookrightarrow V$ such that the induced map $\varphi_{Y'}: Y' \longrightarrow Y$ is an isomorphism, where $Y' = V' \times_V Y$.

With these definitions all statements and proofs remain the same.

TORSION ALGEBRAIC CYCLES ON VARIETIES OVER LOCAL FIELDS

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ABSTRACT. In this article we study the torsion in the second Chow group of a smooth, projective scheme over a Henselian discrete valuation ring with finite or separably closed residue field. We show that the prime to p torsion of this group injects into the prime to p torsion of the special fibre (where p is the characteristic of the residue field). Using this result, we prove the finiteness of the prime to p torsion in the second Chow group of certain varieties over p-adic fields. We also prove similar results for other K-cohomology groups.

0. INTRODUCTION

In a previous paper with Colliot-Thélène [CTR1], we used the work of Merkur'ev-Suslin [MS], Suslin [Su] and the Riemann hypothesis for etale cohomology over finite fields as proved by Deligne [D] to prove some finiteness theorems for the torsion in the second Chow group $CH^2(X)$ for certain varieties X over various fields k of arithmetic interest. Except when the field k is finite, all these varieties have the property that $H^2(X, \mathcal{O}_X) = 0$, and we were quite worried that this assumption might be indispensable even for the study of torsion cycles (one knows how important this assumption is in the study of

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