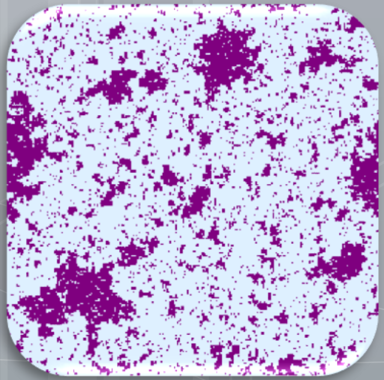
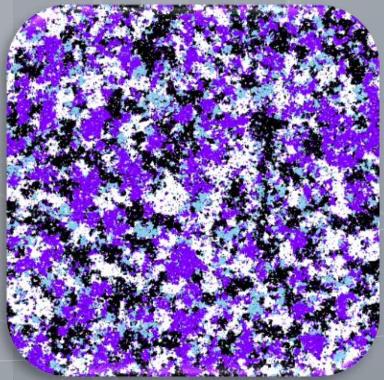
Yale University

Math Colloquium

Sep 2016

The stochastic Ising and Potts models at criticality





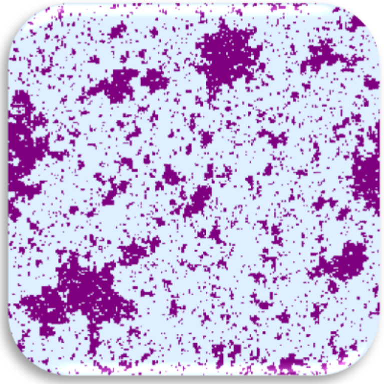
Eyal Lubetzky

Courant Institute (NYU)

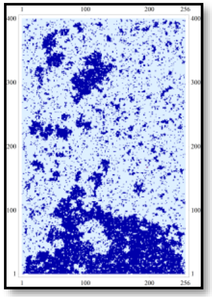
The Ising model

Wilhelm Lenz
1888–1957

- Introduced by Wilhelm Lenz in 1920 as a model of *ferromagnetism*:
  - Place iron in a magnetic field: increase field to maximum , then slowly reduce it to zero.
  - There is a critical temperature T_c (the Curie point) below which the iron retains residual magnetism.
- Magnetism caused by charged particles spinning or moving in orbit in alignment with each other.
- How do local interactions between nearby particles affect the global behavior at different temperatures?



The Ising model



- Gives random binary values (spins) to vertices accounting for nearest-neighbor interactions.
- Initially thought to be over-simplified to capture ferromagnetism, but turned out to have a crucial role in understanding phase transitions & critical phenomena.

Biology
Sociology
Chemistry
Economics
Physics
…

- One of the most studied models in Math. Phys.: more than 10,000 papers over the last 30 years…

Google scholar allintitle: ising

Scholar Articles excluding patents Custom range...

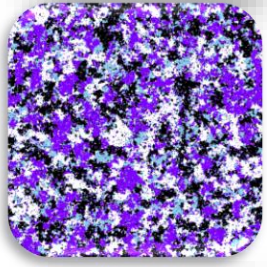
About 13,300 results 1986 — 2016

Ordered phase of short-range **Ising** spin-glasses
DS Fisher, DA Huse - Physical Review Letters, 1986 - APS
We propose a new picture of the **Ising**-spin-glass phase, based on an Ansatz for the scaling of low-lying large-scale-droplet excitations. We find behavior very different from the infinite-range model. The truncated spatial correlations decay as a power of distance, the ac nonlinear ...
Cited by 375 - Related articles - All 3 versions

The Potts model

- Proposed in 1951 by C. Domb to his Ph.D. student R. Potts.
- Generalization of the Ising model to allow $q > 2$ states per site.
  - Special case $q = 4$ was first studied in 1943 by Ashkin and Teller.
- Rich critical phenomena: first/second order phase transitions depending on q and the dimension.

Cyril Domb
1920-2012



Renfrey Potts
1925-2005

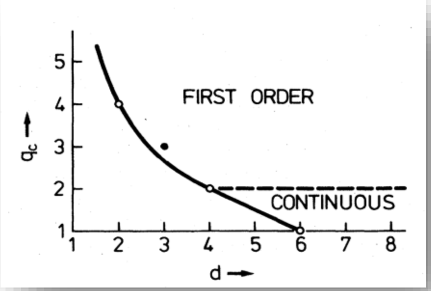
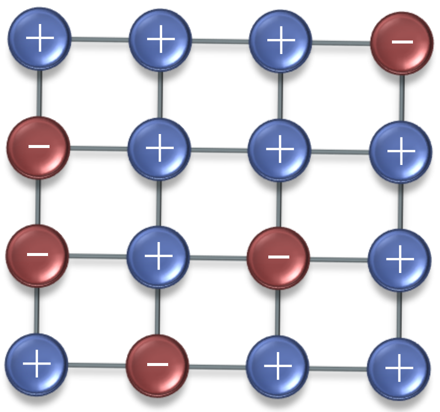


figure taken from:

The Potts model
FY Wu – Reviews of modern physics, 1982 Cited by 2964

Definition: the classical Ising model

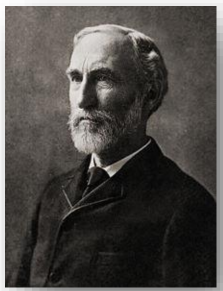
- Underlying geometry: Λ = finite 2D grid.
- Set of possible configurations:
$$\Omega = \{\pm 1\}^{\Lambda}$$
(each *site* receives a plus/minus *spin*)
- Probability of a configuration $\sigma \in \Omega$ given by the ***Gibbs distribution***:



$$\mu_{\mathrm{I}}(\sigma) = \frac{1}{Z(\beta)} \exp\left(\tfrac{1}{2}\beta \sum_{x\sim y} \sigma(x)\sigma(y)\right)$$

Partition function

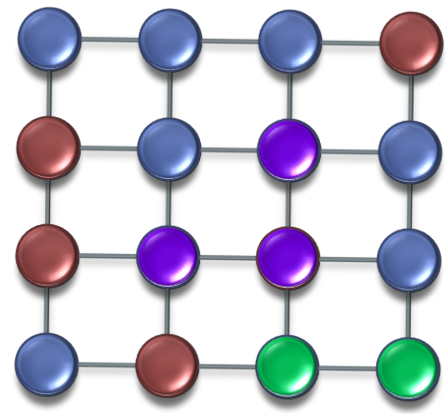
Inverse temperature $\beta \geq 0$



Josiah W. Gibbs
1839–1903

Definition: the q–state Potts model

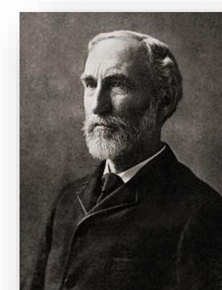
- Underlying geometry: Λ = finite 2D grid.
- Set of possible configurations:
  $$\Omega = \{1, \dots, q\}^{\Lambda}$$
  (each *site* receives a *color*)
- Probability of a configuration $\sigma \in \Omega$ given by the ***Gibbs distribution***:



$$\mu_{\mathrm{P}}(\sigma) = \frac{1}{Z(\beta)} \exp\left(\beta \sum_{x \sim y} \mathbb{1}_{\{\sigma(x) = \sigma(y)\}}\right)$$

Partition function

Inverse temperature $\beta \geq 0$



Josiah W. Gibbs
1839–1903

Glauber dynamics for Ising/Potts

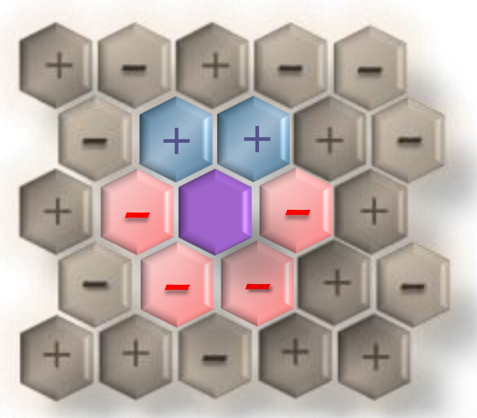
- MCMC sampler introduced in 1963 by Roy Glauber (Nobel in Physics 2005).

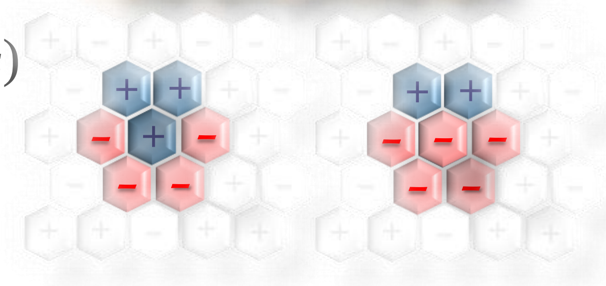
R.J. Glauber

Time-dependent statistics of the Ising model
RJ Glauber – Journal of mathematical physics, 1963 Cited by 3186

- One of the most commonly used samplers for the Ising/Potts model:
  - Update sites via IID Poisson(1) clocks
  - Each update replaces a spin at $x \in V$ by a new spin $\sim \mu$ given spins at $V \setminus \{x\}$.

  (*heat-bath* version; famous other flavor: *Metropolis*)

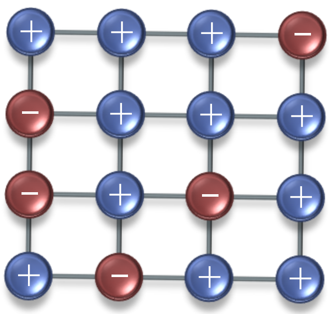




- Time it takes to converge to μ ?

The classical Ising model

- $\mu_{\mathrm{I}}(\sigma) \propto \exp\left(\frac{1}{2}\beta \sum_{x\sim y} \sigma(x)\sigma(y)\right)$ for $\sigma \in \Omega = \{\pm 1\}^{\Lambda}$
  - Larger β favors configurations with aligned spins at neighboring sites.
  - Spin interactions: local, justified by rapid decay of magnetic force with distance.



- The *magnetization* is the (normalized) sum of spins:

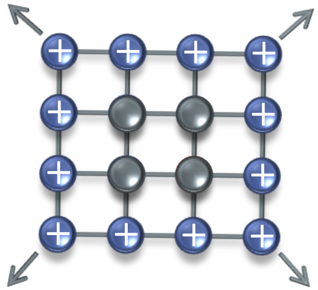
$$M(\sigma) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma(x)$$

  - Distinguishes between disorder ($M \approx 0$) and order.
- Symmetry: $\mathbb{E}[M(\sigma)] = 0$. What if we *break the symmetry*?

The Ising phase-transition

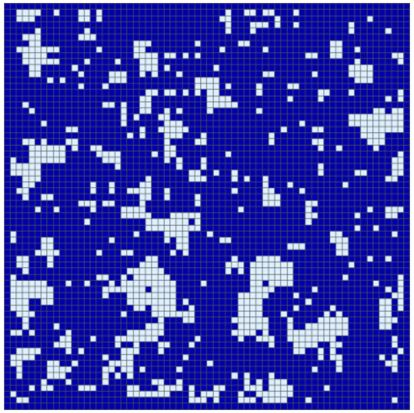
- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$

  ($\approx$ a magnetic field with effect $\to 0$).



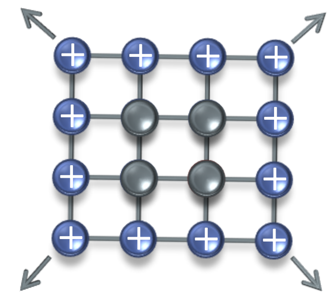
- What is the typical $M(\sigma)$ for large $|\Lambda|$?

  Does the effect of *plus* boundary vanish in the limit?



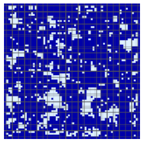
The Ising phase-transition (ctd.)

- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$



- Expect: phase-transition at some critical β_c:

$$\lim_{|\Lambda|\to\infty} \mathbb{E}^{+}[M(\sigma)] = \begin{cases} 0 & \text{if } \beta < \beta_c \\ m_\beta > 0 & \text{if } \beta > \beta_c \end{cases}$$

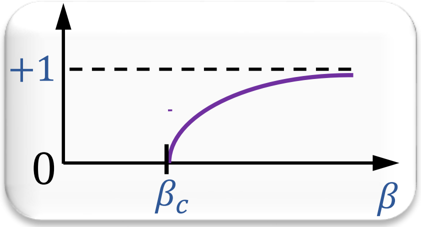


all-plus boundary

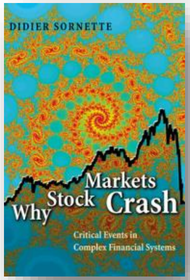
spontaneous magnetization

Example: evolution of Ising model

- The magnetization phase-transition at β_c :



- Replace *magnetization* ⇜⇝ *price* to find this diagram in "Why Stock Markets Crash" / D. Sornette (2001) [Chapter 5 "Modeling bubbles and crashes"]

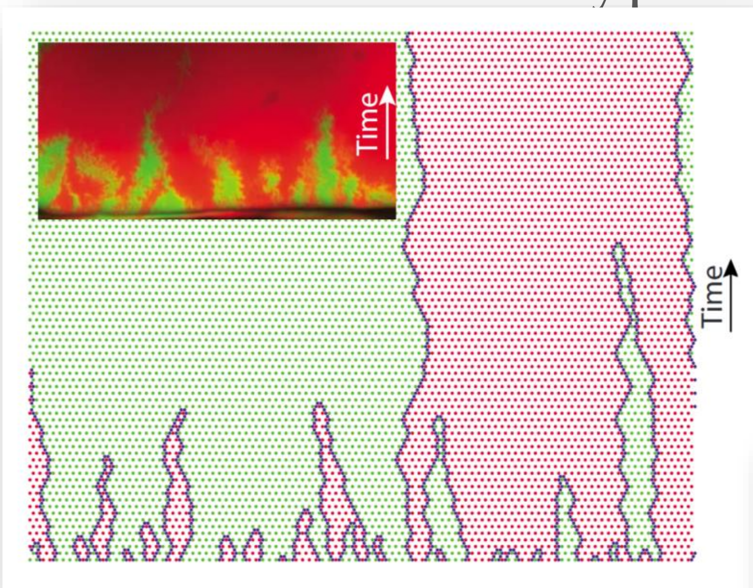


D. Sornette

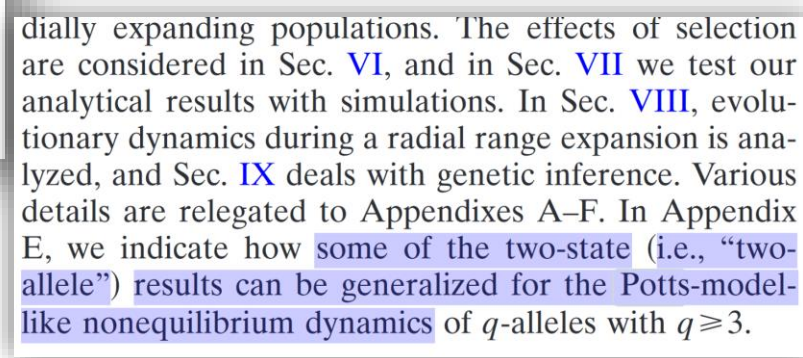
- Such applications of the Ising Model emphasize the dimension of ***time***:
  - How does the system evolve?
  - From a given starting state, how long does it take for certain configurations to appear?

Example: evolution of Potts model

- Bacteria growth with mutation modelled by an evolution of a Potts-type interface.



K.S. Korolev, O.M. Avlund,
O. Hallatschek, and D.R. Nelson,
Genetic demixing and evolution in linear stepping stone models.
Reviews of Modern Physics 82(2) (2010), 1691–1718.



dially expanding populations. The effects of selection are considered in Sec. VI, and in Sec. VII we test our analytical results with simulations. In Sec. VIII, evolutionary dynamics during a radial range expansion is analyzed, and Sec. IX deals with genetic inference. Various details are relegated to Appendixes A–F. In Appendix E, we indicate how some of the two-state (i.e., "two-allele") results can be generalized for the Potts-model-like nonequilibrium dynamics of q-alleles with $q \geqslant 3$.

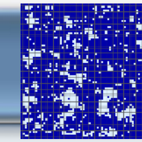
Static vs. stochastic Ising

- Behavior for the Ising distribution:

$\beta < \beta_c$

β_c

$\beta > \beta_c$



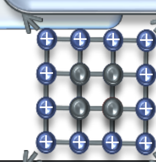
$\mathbb{E}^+[M(\sigma)] \xrightarrow[|\Lambda|\to\infty]{} 0$

$\mathbb{E}^+[M(\sigma)] \xrightarrow[|\Lambda|\to\infty]{} c_\beta > 0$

- Expected behavior for the inverse-gap of dynamics:

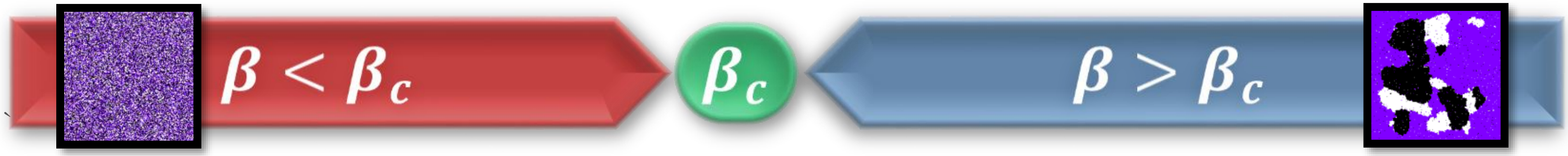
$\beta < \beta_c$	β_c	$\beta > \beta_c$
constant	power law	exponential *Torus/free*

  - Symmetry breaking: power law *Plus b.c.*

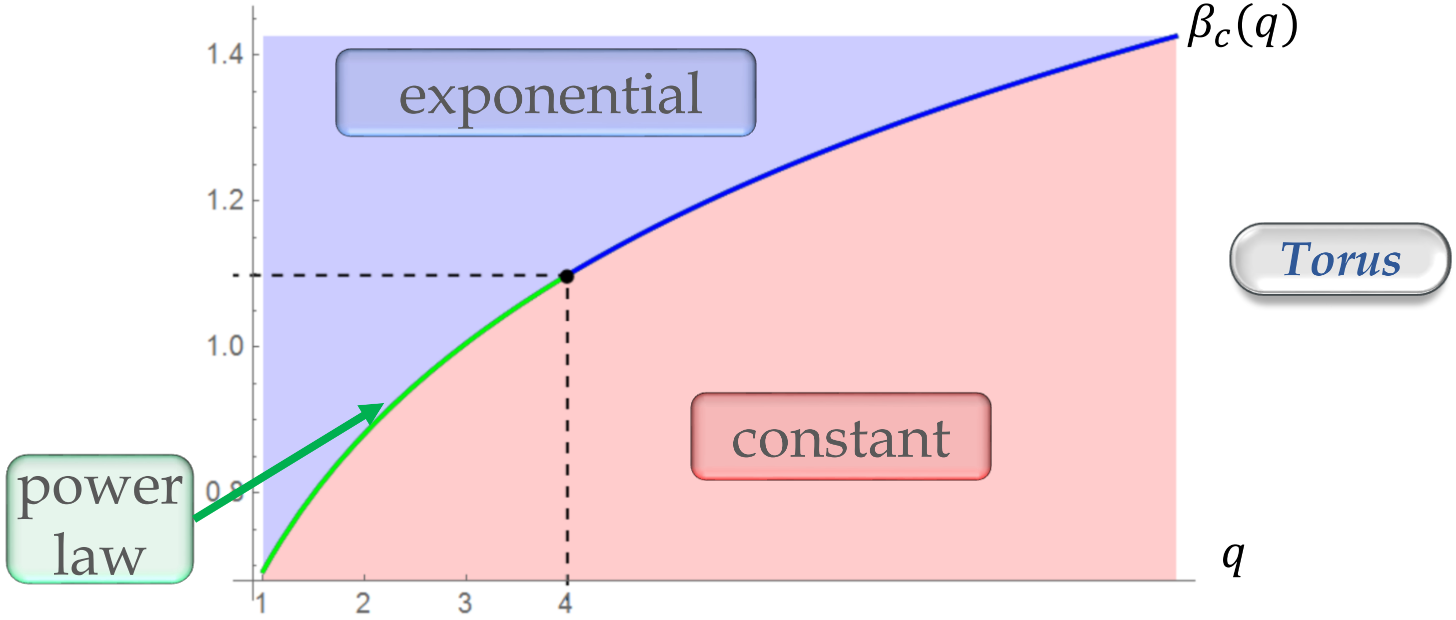


Static vs. stochastic Potts

- Static behavior for the Potts distribution:



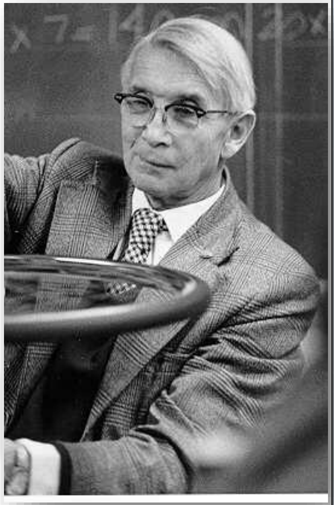
- Expected behavior for the inverse-gap of dynamics:



The 1D Ising model

- Ph.D. in Physics in 1924 from U. Hamburg under the supervision of Lenz.
- Studied the 1D model of Lenz in his thesis:

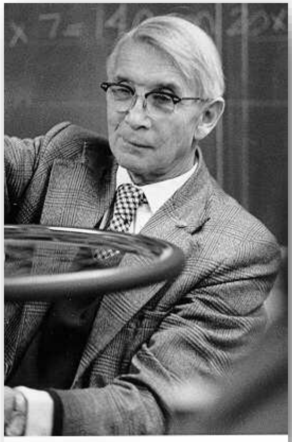
  Beitrag zur theorie des ferromagnetismus
  E Ising - Zeitschrift für Physik A Hadrons and Nuclei, 1925 - Springer Cited by 2173



Ernst Ising
1900-1998

  - Exact solution for the 1D model.
  - Unfortunately: no phase-transition…
    - $\mathbb{E}^+[M(\sigma)] \xrightarrow[|\Lambda|\to\infty]{} 0$ for any $\beta \geq 0$.
    - Intuition: (+ + + + + + + +) similar to (+ + + + − − − −)
  - Heuristic arguments why there would not be a phase-transition in higher dimensions either.

After solving the 1D model



- Ising [letter to S. Brush in 1967]:
  "...I discussed the result of my paper widely with Prof. Lenz and with Dr. Wolfgang Pauli, who at that time was teaching in Hamburg. There was some disappointment that the linear model did not show the expected ferromagnetic properties..."
- Left research after a few years at the German General Electric Co. and turned to teaching in public schools.
- Survived WW2 in Luxembourg isolated from scientific life. Came to the US in 1947 and only then "...did I learn that the idea had been expanded."

Meanwhile, on 2D Ising

- Heisenberg (1928) proposed his own theory of ferromagnetism, motivated by Ising's result.
- Followed by other models attempting to explain order/disorder in metallic alloys.
- In 1936 Rudolf Peierls published the paper

W. Heisenberg
1901-1976

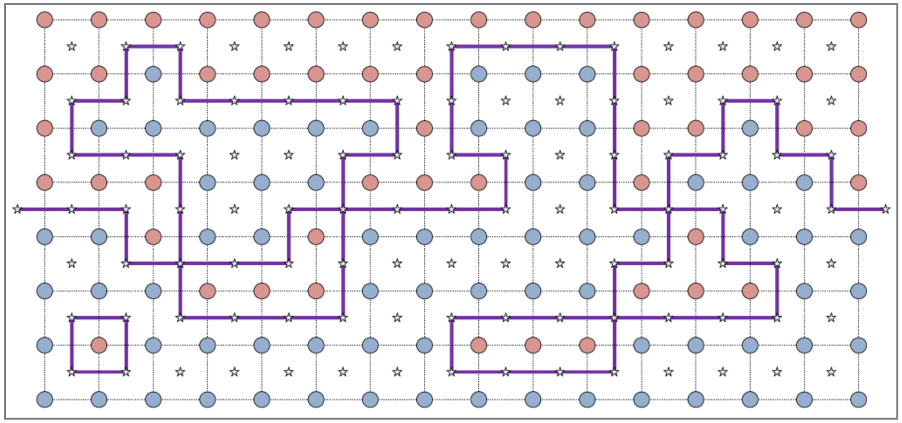
R. Peierls
1907-1995

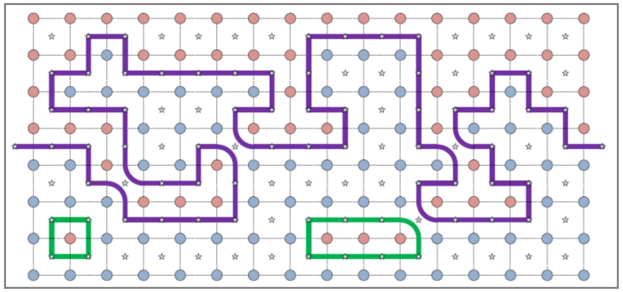
On Ising's model of ferromagnetism
R Peierls - Mathematical Proceedings of the Cambridge ..., 1936
Ising* discussed the following **model** of a **ferromagnetic** body: A
of moment yn to be arranged in a regular lattice; each of them is s
orientations, which we call positive and negative. Assume further t
Cited by 327 - Related articles - All 3 versions

arguing that the 2D and 3D Ising models *do* have spontaneous magnetization at *low enough temperature* (contrary to Ising's prediction).

Peierls' phase transition argument

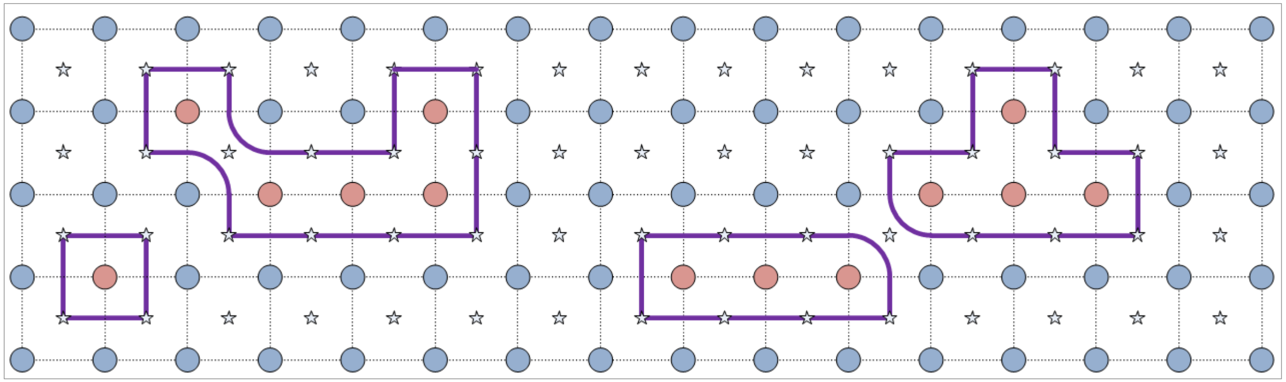
- Peierls' combinatorial argument is simple and robust.
- Key idea: represent Ising configurations as *contours* in the *dual graph*: the edges are dual to disagreeing edges.





Peierls' phase transition argument

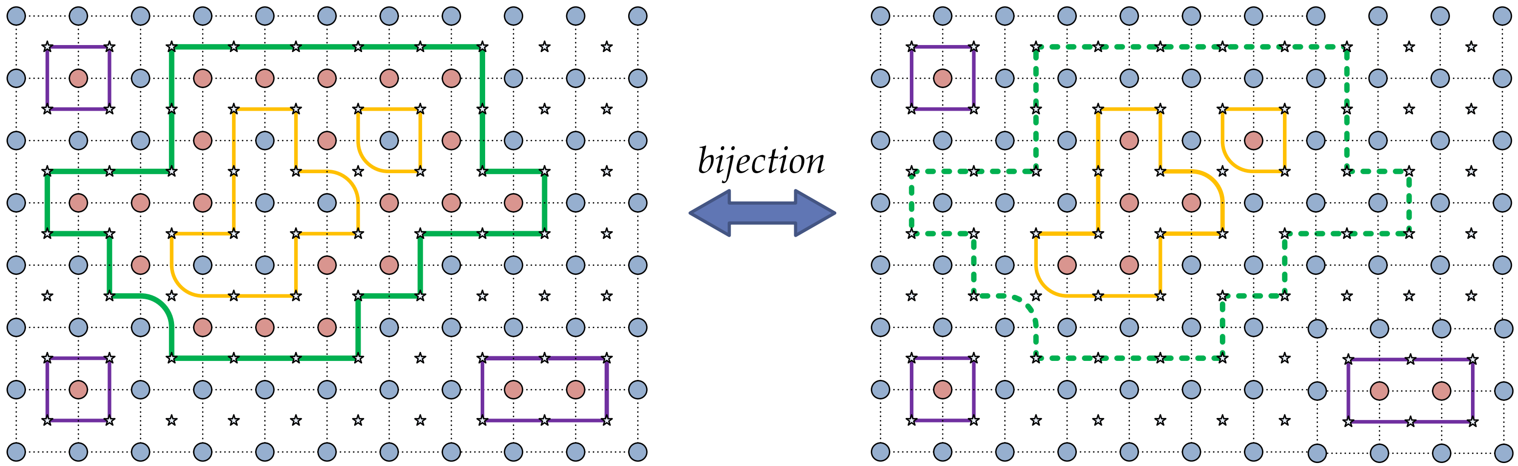
- When all boundary spins are + 's the Peierls contours are all closed [marking "islands" containing of − 's].



- The proof will follow a first moment argument on the number of sites inside such a − component.

Peierls' phase transition argument

- Setting: $\Lambda \subset \mathbb{Z}^2$ is an $n \times n$ box with all-plus boundary.
- Fix a contour C of length ℓ.
- For any σ containing C flip *all spins in the interior of* C :

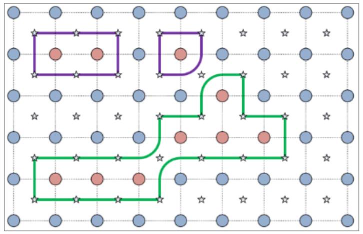


$$\mathbb{P}(\sigma) = \frac{1}{Z(\beta)} e^{B} \qquad \mathbb{P}(\sigma') = \frac{1}{Z(\beta)} e^{B+\beta\ell}$$

- $\Rightarrow \mathbb{P}(C \text{ belongs to contours}) \le e^{-\beta\ell}$.

Peierls' phase transition argument

- For a fixed contour C of length ℓ :
  - $\Rightarrow \mathbb{P}(C \text{ belongs to contours}) \leq e^{-\beta\ell}$.
  - C can contain $\leq \ell^2$ sites (isoperimetric).



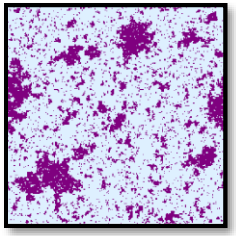
- $O(N\,3^{\ell})$ possible such contours, where $N = |\Lambda| = n^2$.

- Summing we get:

$$\mathbb{E}[\#\{x : \sigma(x) = -1\}] \lesssim N \sum_{\ell} \ell^2 \left(3e^{-\beta}\right)^{\ell} < \varepsilon N$$

where $\varepsilon < 1/2$ for a suitably large β. ■

Landmarks for 2D Ising



- Critical point candidate $\beta_c = \log(1 + \sqrt{2}) \approx 0.88$ found by Kramers and Wannier in 1941 via duality.
- 2D Ising model *exactly solved* in 1944 in seminal work of Lars Onsager (Nobel in Chemistry 1968).
  - Proof analyzed the $2^n \times 2^n$ transfer matrix using the theory of Lie algebras.

L. Onsager
1903-1976

- Understanding of critical geometry boosted by advent of SLE [Schramm '00] and breakthrough results of Smirnov.

O. Schramm
1961-2008

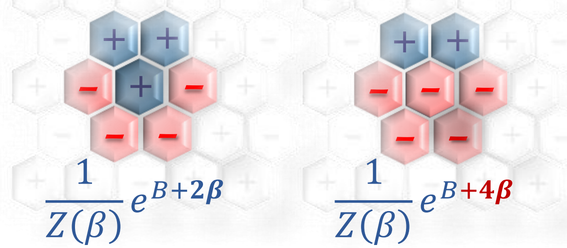
S. Smirnov

- In parallel: extensive study of the dynamical model…

Dynamics for Ising/Potts models

- **Glauber dynamics** dynamics (1963):
  - Update sites via IID Poisson(1) clocks
  - Each update replaces the state of $x \in V$ by a new $S \sim \mu_{\mathrm{P}}(\cdot \mid \textit{other states of } V \setminus \{x\})$.

  Local: $\mathbb{P}(S = s) \propto \exp(\beta\, \#\{y \sim x : \sigma(y) = s\})$
  - Advantage: easy to simulate.
  - Drawback: susceptible to bottlenecks in energy landscape.



- **Swendsen–Wang** dynamics (1987) :
  - Draw percolation clusters as per the Fortuin–Kasteleyn representation (1969) of the Potts model.
  - Let each cluster choose its color i.i.d. uniformly.

  Global (able to treat the Ising model double-well).

Measuring convergence to equilibrium

- **Spectral gap** in the spectrum of the generator:

$$\text{gap} = \text{ smallest positive eigenvalue of the heat-kernel } H_t \text{ of the dynamics.}$$
$$= \inf \frac{\mathcal{E}(f,f)}{\mathrm{Var}_\mu(f)} \text{ for } \mathcal{E}(f,f) \text{ the Dirichlet form.}$$

- **Mixing time** (according to a given metric):
Standard choice: L^1 (total-variation) mixing time to within distance ε is defined as

$$t_{\mathrm{mix}}(\varepsilon) = \inf\left\{t \;:\; \max_{x_0} \left\|p^t(x_0,\cdot) - \mu\right\|_{\mathrm{tv}} \le \varepsilon\right\}$$

where $\|\mu - \nu\|_{\mathrm{tv}} = \sup_{A \subset \Omega} [\mu(A) - \nu(A)]$

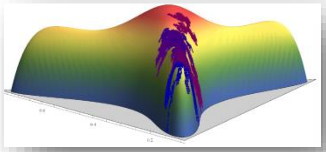
Believed picture for Ising on $\mathbb{Z}_n^d$

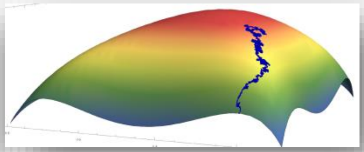
- For some critical inverse-temperature β_c:

$\beta < \beta_c$ | β_c | $\beta > \beta_c$ *free or periodic b.c.*

gap^{-1}: $O(1)$ | $\asymp n^z$ *universal* | $\asymp e^{\tau_\beta\, n^{d-1}}$

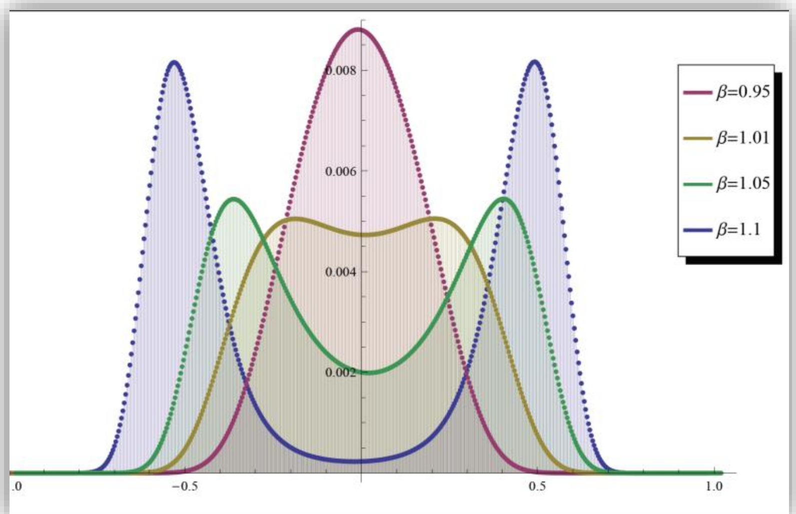
- Analogous picture verified for:
  - Ising model on complete graph
    [Ding, L., Peres '09a, '09b], [Levin, Luczak, Peres '10]
  - Regular tree [Berger, Kenyon, Mossel, Peres '05], [Ding, L., Peres '10]
  - Potts model on complete graph
    [Cuff, Ding, L., Louidor, Peres, Sly '12],
    [Galanis, Stefankovic, Vigoda '15] , [Blanca, Sinclair '15]





Critical slowdown

- Intuition: low temperature
  - Exponential mixing due to a bottleneck between the "mostly-plus" and the "mostly-minus" states



Stationary magnetization

- Intuition: high temperature
  - At $\beta = 0$ there is complete independence.
  - For very small $\beta > 0$ a spin is likely to choose the same update given 2 very different neighborhoods (weak "communication" between sites).
  - States can be coupled quickly, hence rapid mixing.
- Intuition: critical power-law
  - Doubling the box incurs a constant factor in mixing…

Glauber dynamics for 2D Ising

- Fast mixing at **high** temperatures:
  - [Aizenman, Holley '84]
  - [Dobrushin, Shlosman '87]
  - [Holley, Stroock '87, '89]
  - [Holley '91]
  - [Stroock, Zegarlinski '92a, '92b, '92c]
  - [Lu, Yau '93]
  - [Martinelli, Olivieri '94a, '94b]
  - [Martinelli, Olivieri, Schonmann '94]

$\beta < \beta_c$

$\mathrm{gap}^{-1} = O(1)$

$t_{\mathrm{mix}} = O(\log n)$

- Slow mixing at **low** temperatures:
  - [Schonmann '87]
  - [Chayes, Chayes, Schonmann '87]
  - [Martinelli '94]
  - [Cesi, Guadagni, Martinelli, Schonmann '96]

$\beta > \beta_c$

free or periodic b.c.

$\mathrm{gap}^{-1} = e^{(\tau_\beta + o(1))n^{d-1}}$

- **Critical** power-law?
  - *lower bound*: [Aizenman, Holley '84], [Holley '91]
  - *simulations*: [Ito '93], [Wang, Hatano, Suzuki '95], [Grassberger '95], [Nightingale, Blöte '96], [Wang, Hu '97],…

β_c

no sub-exponential upper bounds

$\mathrm{gap}^{-1} \geq n^c$

sim: $n^{2.17\ldots}$

Glauber dynamics for 2D Ising

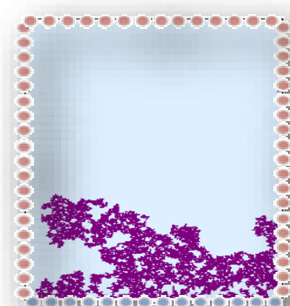
$\beta < \beta_c$ | β_c | $\beta > \beta_c$ | *free or periodic b.c.*

gap^{-1}: $O(1)$ | $\geq n^c$ sim: $n^{2.17\ldots}$ | $e^{(\tau_\beta + o(1))n^{d-1}}$

- Further progress ([L., Sly '12]): ***power-law*** at β_c:

$$\boxed{\mathrm{gap}^{-1} \leq n^C}$$

  - analysis used SLE-type behavior of critical interfaces due to [Duminil-Copin, Hongler, Nolin '09] (cf. [Chelkak, Smirnov '09], [Camia, Newman '09]).



And the critical Potts model?

- Believed to have the same critical slowdown phenomenon as the Ising model in the *continuous phase transition* regime, *vs.* slow critical mixing elsewhere.
- On $\mathbb{Z}_n^2$, this corresponds to:
  - Conjectured: n^z inverse-gap at criticality for $1 < q \le 4$.
  - Conjectured: $\exp(c\, n)$ inverse-gap at criticality for $q > 4$.
- *For q* **large enough:** [Borgs, Chayes, Frieze, Kim, Tetali, Vu '99] (refined in [Borgs, Chayes, Tetali '12]): $\boxed{\mathrm{gap}^{-1} = \exp\left(n^{1+o(1)}\right)}$

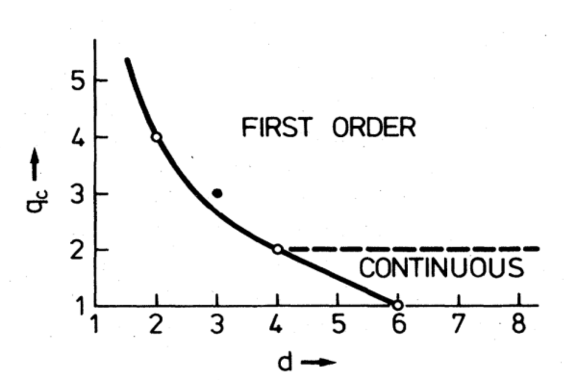
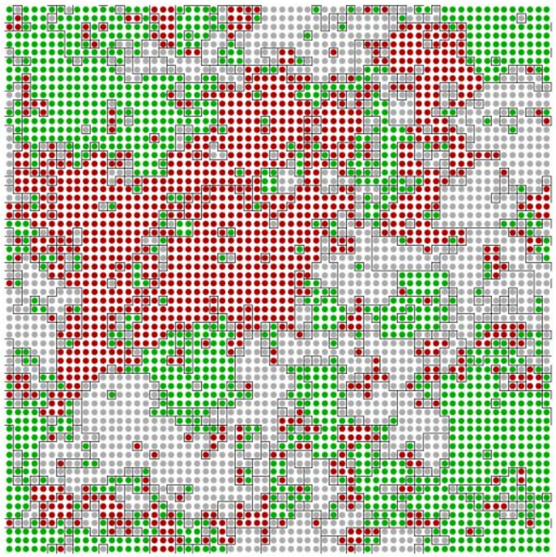


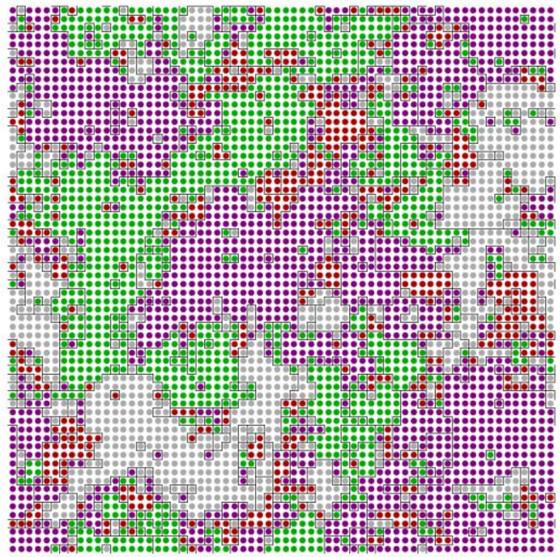
figure from:

The Potts model
FY Wu – Reviews of modern physics, 1982 Cited by 2964

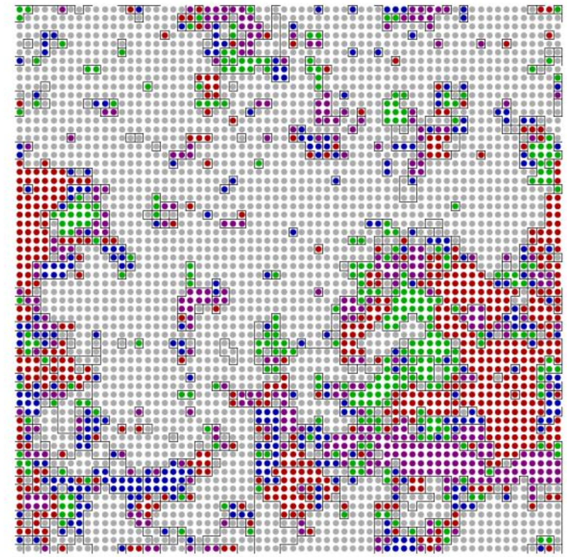
Critical Potts model on $(\mathbb{Z}/n\mathbb{Z})^2$



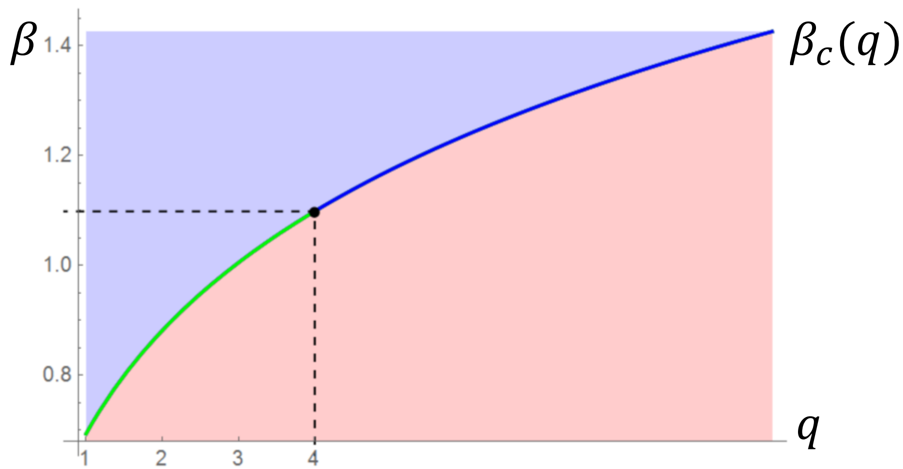
$q = 3$



$q = 4$



$q = 5$



Mixing of critical 2D Potts/FK

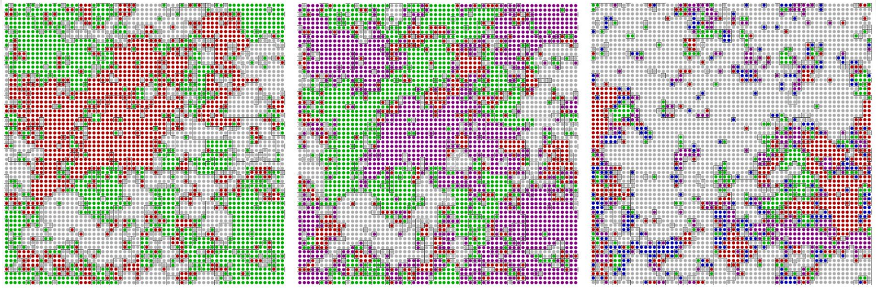
- **THEOREM:** ([Gheissari, L.])

Glauber dynamics and Swendsen–Wang dynamics for the critical Potts model on the torus $(\mathbb{Z}/n\mathbb{Z})^2$ both satisfy:

(a) $q = 3$: $\quad \text{gap}^{-1} \leq n^C$.

(b) $q = 4$: $\quad \text{gap}^{-1} \leq n^{C \log n}$.

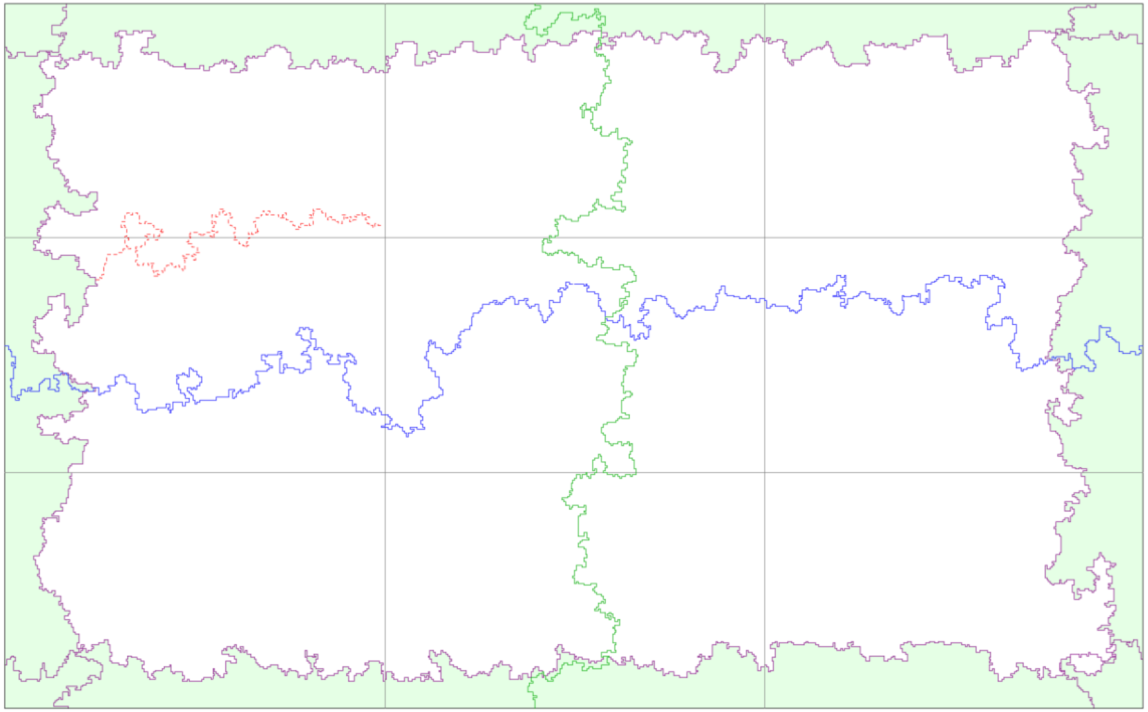
(c) $q \geq 5$, $\pi^0_{\mathbb{Z}^2} \neq \pi^1_{\mathbb{Z}^2}$: $\quad \text{gap}^{-1} \geq \exp(c\, n)$.



- Proofs build on ideas from [L., Sly '12] and recent results by [Duminil-Copin, Sidoravicius, Tassion '15].

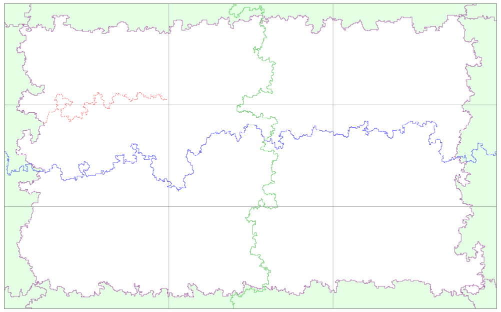
$q \geq 5$: exponential lower bound

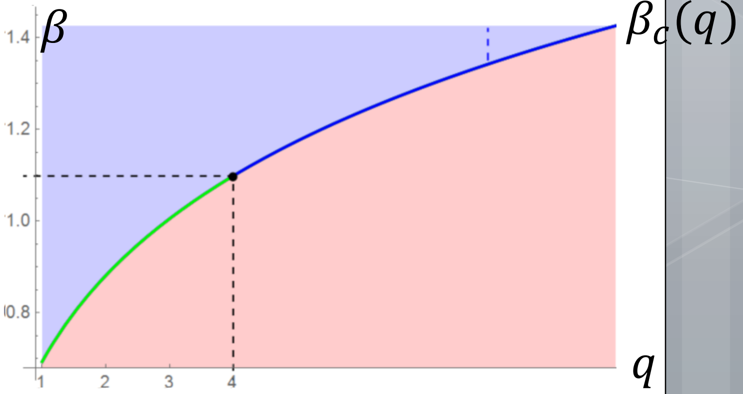
- Establish a bottleneck w.r.t. six crossings:



use exponential decay of correlations in $\pi^1_{\mathbb{Z}^2}$ due to [Duminil-Copin, Sidoravicius, Tassion '15].

Exponential lower bounds (cont.)





- Analysis of Potts model at $\beta = \beta_c$ extends to the regime $\beta > \beta_c$, where the exponentially slow behavior of gap^{-1} was established for *sufficiently large* q by [Borgs, Chayes, Frieze, Kim, Tetali, Vu '99] [Borgs, Chayes, Tetali '12]

- <u>THEOREM:</u> ([Gheissari, L.])

Glauber dynamics for the Potts model on the torus $(\mathbb{Z}/n\mathbb{Z})^2$ at $q > 1$ and $\beta > \beta_c$ has

$$\text{gap}^{-1} \geq \exp(c\, n)\ .$$

Beyond the torus

free ≡ periodic ???

- Unlike low temperature Ising, free boundary conditions do ***not*** induce the same slow mixing behavior as the torus.
- <u>**THEOREM:**</u> ([Gheissari, L.])

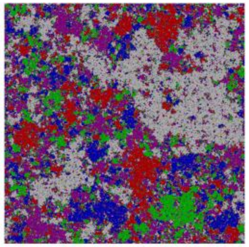
Fix q large enough. Swendsen–Wang for the critical Potts model on an $n \times n$ **box with free boundary conditions** has

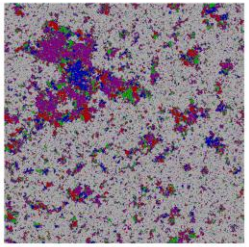
$$\text{gap}^{-1} \leq \exp\left(n^{o(1)}\right),$$

whereas Glauber dynamics has

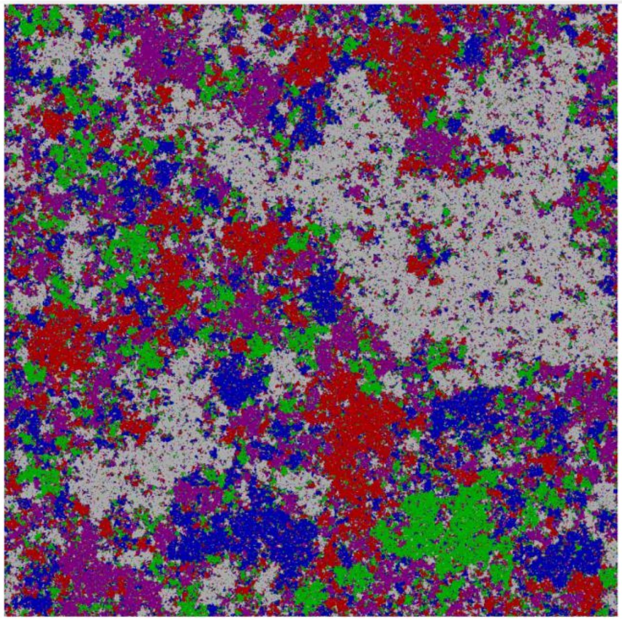
$$\text{gap}^{-1} \leq \exp\left(n^{1/2+o(1)}\right).$$

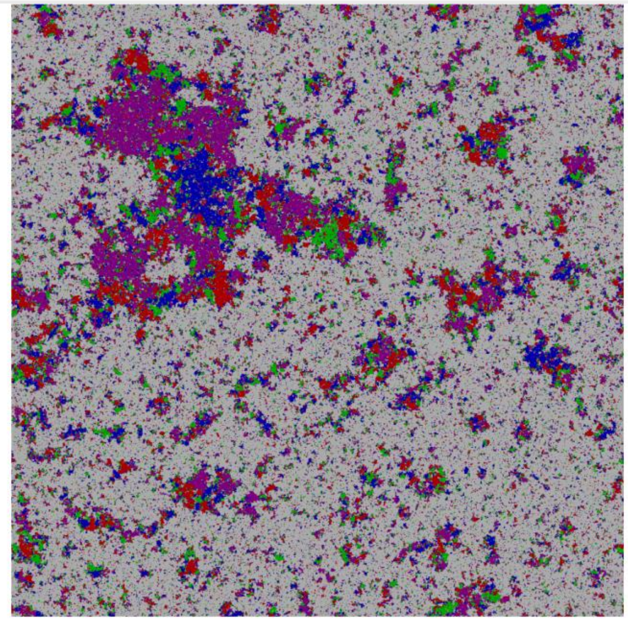
- Proofs build on cluster expansion and a powerful multi-scale approach due to [Martinelli, Toninelli '11].





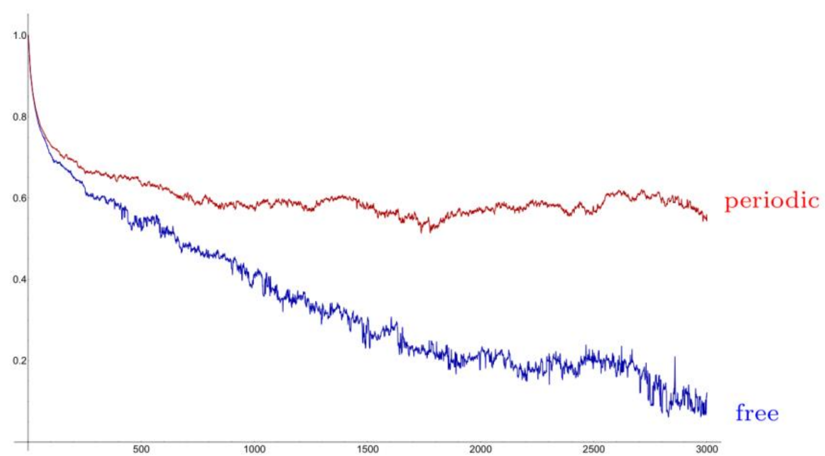
The effect of boundary conditions





Free boundary conditions

Periodic boundary conditions



Thank you