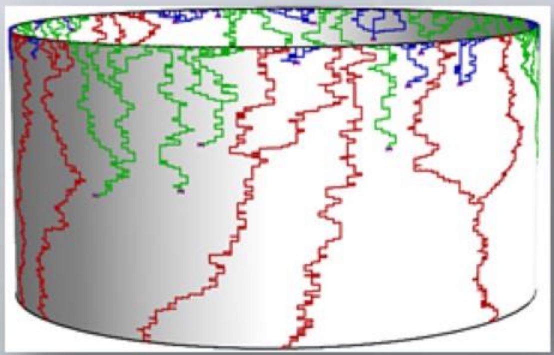
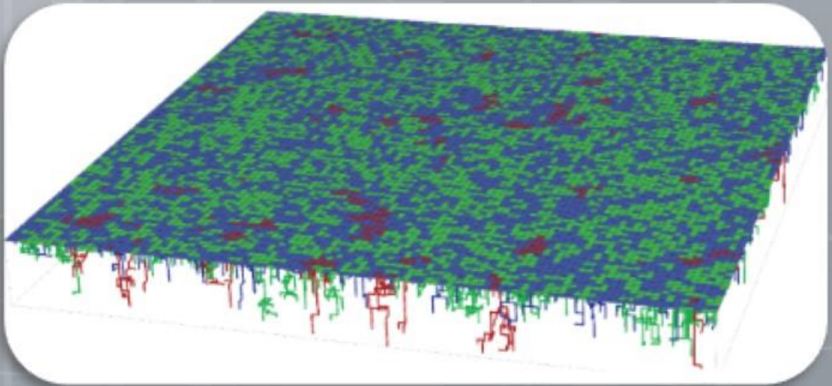


SPA 2014

Buenos Aires

# Information Percolation for the Ising model




**Eyal Lubetzky**

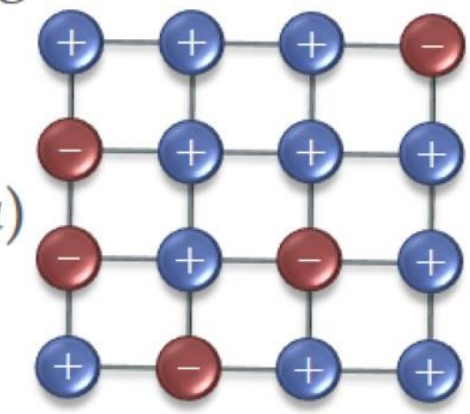
Microsoft Research

# Definition: the classical Ising model

- Underlying geometry: $\Lambda$ = finite 2D grid.
- Set of possible configurations:

$$\Omega = \{\pm 1\}^{\Lambda}$$

(each *site* receives a plus/minus *spin*)



- Probability of a configuration $\sigma \in \Omega$ given by the *Gibbs distribution*:

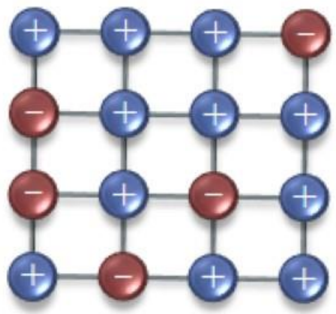
$$\mu(\sigma) = \frac{1}{Z(\beta)} \exp\left( \beta \sum_{x \sim y} \sigma(x)\sigma(y) \right)$$

Partition function

Inverse temperature $\beta \geq 0$

# The classical Ising model

- $\mu(\sigma) \propto \exp\left(\beta \sum_{x\sim y} \sigma(x)\sigma(y)\right)$ for $\sigma \in \Omega = \{\pm 1\}^{\Lambda}$
  - Larger $\beta$ favors configurations with aligned spins at neighboring sites.
  - Spin interactions: local, justified by rapid decay of magnetic force with distance.



- The *magnetization* is the (normalized) sum of spins:

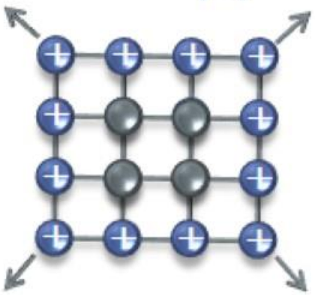
$$M(\sigma) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma(x)$$

  - Distinguishes between disorder ($M \approx 0$) and order.
- Symmetry: $\mathbb{E}[M(\sigma)] = 0$. What if we *break the symmetry*?

# The Ising phase-transition

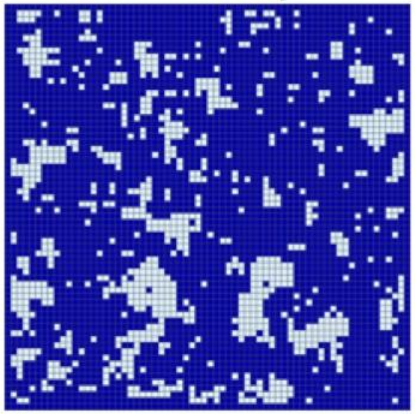
- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$

  ($\approx$ a magnetic field with effect $\to 0$).



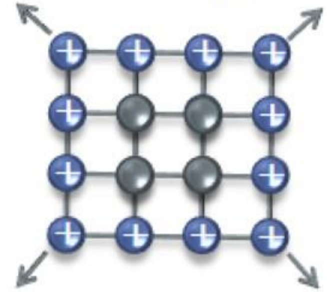
- What is the typical $M(\sigma)$ for large $|\Lambda|$ ?

  Does the effect of *plus* boundary vanish in the limit?



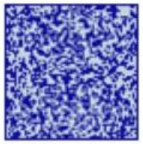
# The Ising phase-transition (ctd.)

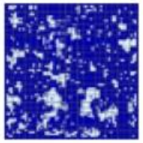
- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$



- Phase-transition at some critical $\beta_c$:

$$\lim_{|\Lambda|\to\infty} \mathbb{E}^+[M(\sigma)] = \begin{cases} 0 & \text{if } \beta < \beta_c \\ m_\beta > 0 & \text{if } \beta > \beta_c \end{cases}$$



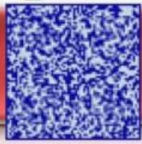


*all-plus boundary*

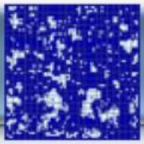
*spontaneous magnetization*

# Static vs. stochastic Ising

- Expected behavior for the Ising distribution:



| $\beta < \beta_c$ | $\beta_c$ | $\beta > \beta_c$ |
|---|---|---|
| $\mathbb{E}^+[M(\sigma)] \xrightarrow[\lvert\Lambda\rvert\to\infty]{} 0$ | | $\mathbb{E}^+[M(\sigma)] \xrightarrow[\lvert\Lambda\rvert\to\infty]{} c_\beta > 0$ |



- Expected behavior for the mixing time of dynamics:

| $\beta < \beta_c$ | $\beta_c$ | $\beta > \beta_c$ |
|---|---|---|
| logarithmic | power law | exponential *Free* b.c. |

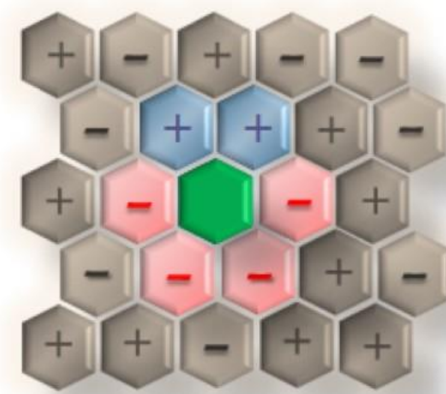
# Glauber dynamics for Ising

(*a.k.a.* the *Stochastic Ising model*)

- Introduced in 1963 by Roy Glauber. (*heat-bath* version; famous other flavor: *Metropolis*)

Time-dependent statistics of the Ising model
**RJ Glauber** – Journal of mathematical physics, 1963 Cited by 2749

- One of the most commonly used samplers for the Ising distribution $\mu$:
  - Update sites via IID Poisson(1) clocks
  - Each update replaces a spin at $x \in V$ by a new spin $\sim \mu$ given spins at $V \setminus \{x\}$.




- How long does it take it to converge to $\mu$ ?

$\frac{1}{Z(\beta)} e^{B-2\beta}$

$\frac{1}{Z(\beta)} e^{B+2\beta}$

# Measuring convergence to equilibrium

- Mixing time : (according to a given metric).
  Standard choice: $L^1$ (total-variation) mixing time to within distance $\varepsilon$ is defined as

$$t_{\mathrm{mix}}(\varepsilon) = \inf\left\{t \ : \max_{x_0} \left\|p^t(x_0\,,\cdot) - \mu\right\|_{\mathrm{tv}} \le \varepsilon\right\}$$

( where $\|\mu - \nu\|_{\mathrm{tv}} = \sup_{A\subset\Omega}[\mu(A) - \nu(A)]$ )

- Dependence on $\varepsilon$ : (*cutoff phenomenon* [DS81], [A83],[AD86])
  We say there is ***cutoff*** $\Leftrightarrow$ $t_{\mathrm{mix}}(\varepsilon) \sim t_{\mathrm{mix}}(\varepsilon') \quad \forall \text{ fixed } \varepsilon, \varepsilon'$


# Glauber dynamics for 2D Ising

- Fast mixing at **high** temperatures:
  - [Aizenman, Holley '84]
  - [Dobrushin, Shlosman '87]
  - [Holley, Stroock '87, '89]
  - [Holley '91]
  - [Stroock, Zegarlinski '92a, '92b, '92c]
  - [Lu, Yau '93]
  - [Martinelli, Olivieri '94a, '94b]
  - [Martinelli, Olivieri, Schonmann '94]

$\beta < \beta_c$

$t_{\text{mix}} \asymp \log n$

- Slow mixing at **low** temperatures:
  - [Schonmann '87]
  - [Chayes, Chayes, Schonmann '87]
  - [Martinelli '94]
  - [Cesi, Guadagni, Martinelli, Schonmann '96]

$\beta > \beta_c$ *free* b.c.

$t_{\text{mix}} = e^{(\tau_\beta + o(1))n^{d-1}}$

$\beta_c$

- **Critical** power-law:
  - *simulations*: [Ito '93],[Wang, Hatano, Suzuki '95],[Grassberger '95],…: $n^{2.17\ldots}$
  - *lower bound*: [Aizenman, Holley '84], [Holley '91]
  - *upper bound (polynomial mixing)*: [L., Sly '12]

$n^{c_1} \le t_{\text{mix}} \le n^{c_2}$

# Glauber dynamics for 2D Ising

| | $\beta < \beta_c$ | $\beta_c$ | $\beta > \beta_c$ (*free* b.c.) |
|---|---|---|---|
| $t_{\text{mix}}$: | $O(\log n)$ | $n^c$ sim: $n^{2.17...}$ | $e^{(\tau_\beta + o(1))n^{d-1}}$ |

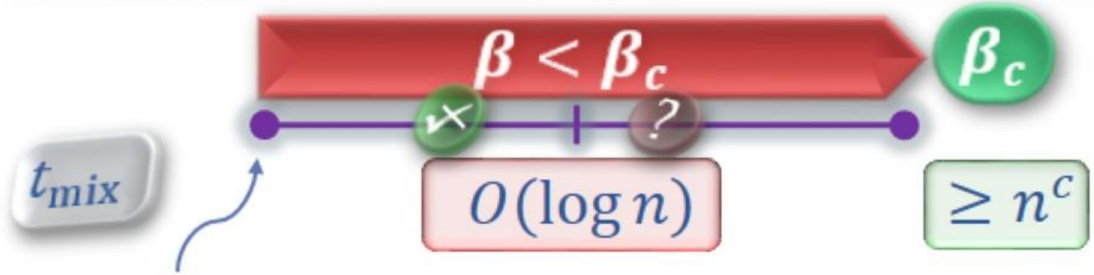
- High temperature in 2D:
  - [L., Sly '13]: ***cutoff*** for any $\beta < \beta_c = \frac{1}{2}\log(1+\sqrt{2})$:

$$t_{\text{mix}}(\varepsilon) = \tfrac{1}{2}\lambda_\infty^{-1}\log n + O(\log\log n)$$

  - Method caveat: needs *strong spatial mixing*; *e.g.*, breaks on 3D Ising for $\beta$ close to $\beta_c$.

# High temperature unknowns (I)

- *High temperature* ⇝ *Infinite temperature:*
  Qualitatively, $\beta < \beta_c$ believed to behave ≈ as $\beta = 0$.




[Martinelli, Olivieri '94], [Aizenman, Holley '84]

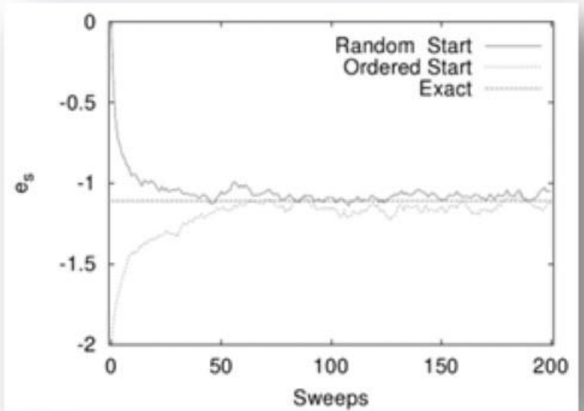
  - $\beta = 0$: (*independent spins*) one of the first examples of **cutoff**:
    $t_{\text{mix}}(\varepsilon) = c \log n + O(1)$ [Aldous '83], [Diaconis Shahshahani '87] [Diaconis, Graham, Morisson '90]
  - ⇒ expect **cutoff** $\forall \beta < \beta_c$ (conj. [Peres '04]) & with $O(1)$-window

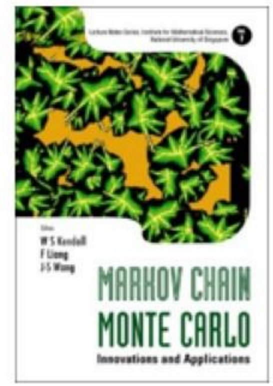
- Concretely: for 3D Ising (*e.g.* on a torus) at $\beta = 0.99\, \beta_c$ :
  - ? does the dynamics exhibit cutoff?
    if so, where & what is the window?

# High temperature unknowns (II)

- *Warm (random) start vs. cold (ordered) start:* random start is better than ordered

  - *e.g.*




- Concretely: for 3D Ising at $\beta = 0.01$ :
  - ? what is $t_{\text{mix}}^{(\text{U})}(\varepsilon) = \inf\left\{t \;:\; \left\| \frac{1}{|\Omega|}\sum_{x_0} p^t(x_0,\cdot) - \mu \right\|_{\text{tv}} \le \varepsilon \right\}$ ? how does it compare with $t_{\text{mix}}(\varepsilon)$ ?

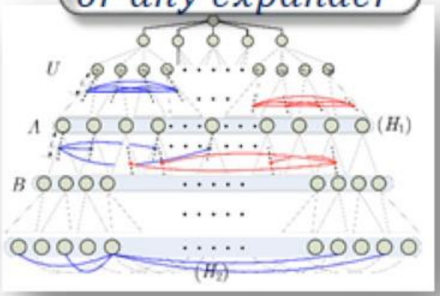
CFTP

# High temperature unknowns (III)

- *Universality of cutoff:*
  on any locally finite geometry there should be cutoff if the temperature is high enough (function of max-degree)
  - $\exists c_0 > 0$: The Ising model on any graph $G$ on $n$ vertices with maximal degree $d$ at $\beta < c_0/d$ has $t_{\text{mix}} = O(\log n)$
    [Dobrushin '71],[Holley '72],[Dobrushin-Shlosman '85], [Aizenman-Holey'87]
  - $\Rightarrow$ expect **cutoff** $\forall \beta < \kappa/d$, and with $O(1)$-window.

- Concretely: for Ising on a binary tree at $\beta = 0.01$ :
  - ? does the dynamics exhibit cutoff?
    if so, where & what is the window?

*or any expander*

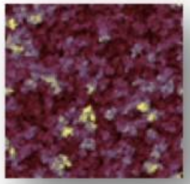


# Recipe for stochastic Ising analysis

- Traditional approach to sharp mixing results
  1. Establish spatial properties of static Ising measure
  2. Use to drive a multi-scale analysis of dynamics.

---

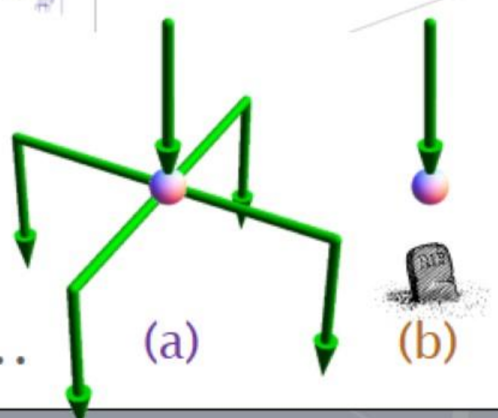
- Example: best-known results on 2D Ising (torus $\mathbb{Z}_n^2$):
  - [L., Sly '13]: ***cutoff*** at $\beta < \beta_c = \frac{1}{2}\log(1+\sqrt{2})$
    - used log-Sobolev ineq. & strong spatial mixing.
  - [L., Sly '12]: ***power-law*** at $\beta_c$
    - used SLE behavior of critical interfaces.
  - [L., Martinelli, Sly, Toninelli '13]: at $\beta > \beta_c$ quasi-polynomial mixing under all-plus b.c.
    - uses interface convergence to Brownian bridges



# New framework for the analysis

- Traditional approach to sharp mixing results
  1. Establish spatial properties of static Ising measure
  2. Use to drive a multi-scale analysis of dynamics.
- New approach: study these *simultaneously* examining ***information percolation*** clusters in the space-time slab:

- track update lineage back in time.
- update either (a) branches out, or (b) terminates (“*oblivious*”)
- analyze RED/GREEN/BLUE clusters…



# Results: cutoff up to $\beta_c$ in 3D Ising

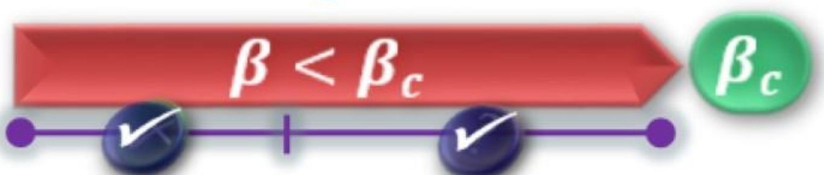
- Confirm Peres's conj. on $\mathbb{Z}_n^d$ for any $d$, with $O(1)$-window.
- <u>THEOREM:</u> ([L.-Sly '14+])

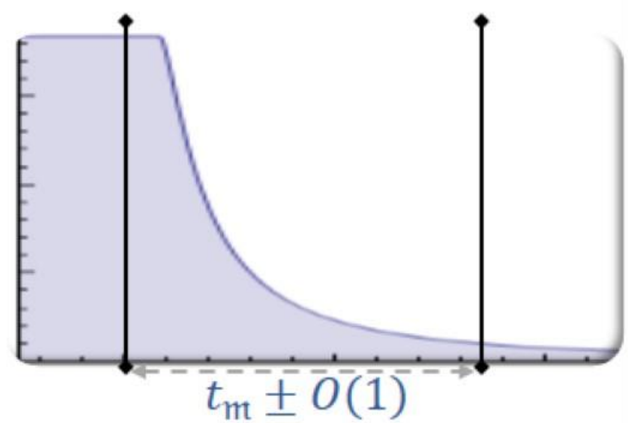
> $\forall d \geq 1$ and $\beta < \beta_c$ there is **cutoff** with an $O(1)$-window at
> $$t_{\mathrm{m}} = \inf\left\{t : \mathbb{E}_+[M(\sigma_t)] \leq \sqrt{n^d}\right\}$$

*cutoff window:* $O(\log(1/\varepsilon))$

[recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]

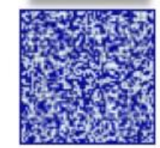
- Examples:
  - $d = 1$: $t_{\mathrm{m}} = \frac{1}{2(1-\tanh(2\beta))}\log n$.
  - $\beta = 0$: $t_{\mathrm{m}} = \frac{1}{2}\log n$ (matching [Aldous '83])



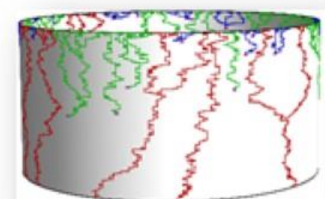



# Results: initial states

- *Warm start is twice faster*:
  - All-plus starting state is worst (up to an additive $O(1)$) [but twice faster than naïve monotone coupling bound].
  - Uniform initial state ≈ twice faster than all-plus.
  - Almost ∀ deterministic initial state ≈ as bad as all-plus.



- Example: the 1D Ising model ($\mathbb{Z}_n$):
  <u>THEOREM:</u> ([L.-Sly '14+])



Fix $\beta > 0$ and $0 < \varepsilon < 1$ ; set $t_{\mathfrak{m}} = \frac{1}{2(1-\tanh(2\beta))} \log n$.

1. (*Annealed*) $t_{\mathrm{mix}}^{(\mathrm{U})}(\varepsilon) \sim \frac{1}{2} t_{\mathfrak{m}}$
2. (*Quenched*) $t_{\mathrm{mix}}^{(x_0)}(\varepsilon) \sim t_{\mathrm{mix}}^{(+)}(\varepsilon) \sim t_{\mathfrak{m}}$ for almost $\forall x_0$

# Results: universality of cutoff

- **Paradigm:** cutoff for *any locally finite geometry* at high enough temperature (including expanders, trees, …)

$\beta < \kappa/d$ Cutoff

- THEOREM: ([L.-Sly '14+])

$\exists\, \kappa > 0$ so that, if $G$ is any $n$-vertex graph with degrees $\leq d$ and $\beta < \kappa/d$, then $\exists$ cutoff with an $O(1)$-window at

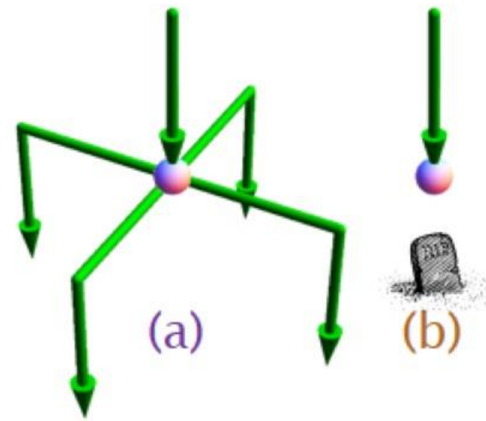
$$t_{\mathrm{m}} = \inf\left\{t : \sum_x \mathbb{E}_+\left[M\big(\sigma_t(x)\big)^2\right] \leq 1\right\}.$$

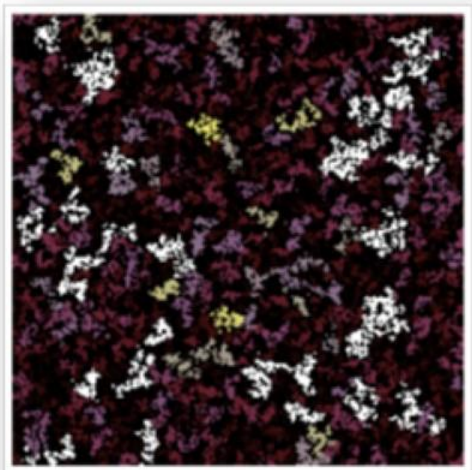
- Moreover:

$$t_{\mathrm{mix}}^{(\mathrm{U})} \leq \left(\tfrac{1}{2} + \varepsilon_\beta\right)t_{\mathrm{m}} \text{ yet } t_{\mathrm{mix}}^{(x_0)} \geq \left(1 - \varepsilon_\beta\right)t_{\mathrm{m}} \text{ a.e. } x_0.$$

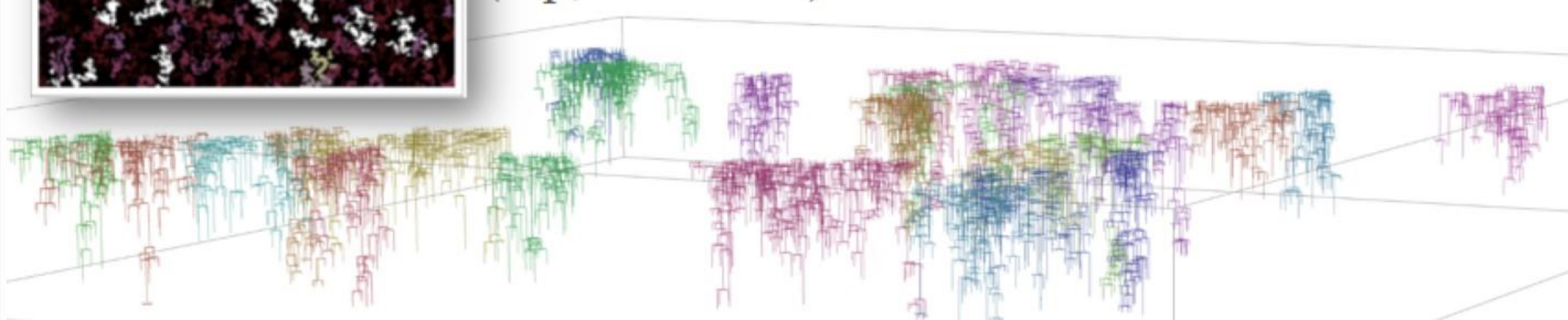
# The new framework (revisited)

- ***Information percolation*** clusters in the space-time slab:
  - track update lineage back in time.
  - update either (a) branches out, or (b) terminates ("*oblivious*")



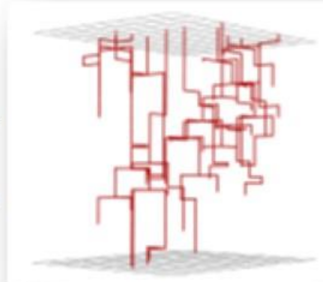



$\mathbb{Z}_{200}^2$ cluster (top/side view)



# Information percolation clusters

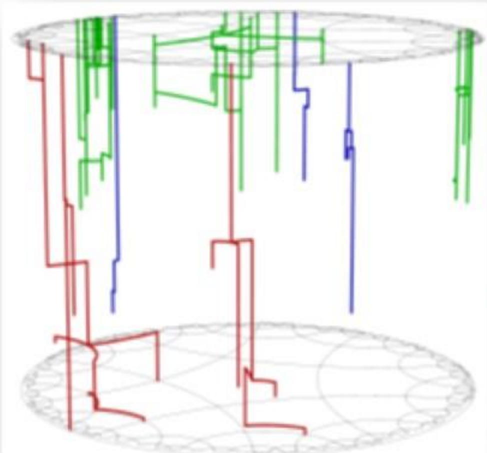
BLUE:
dies out quickly in space & time.

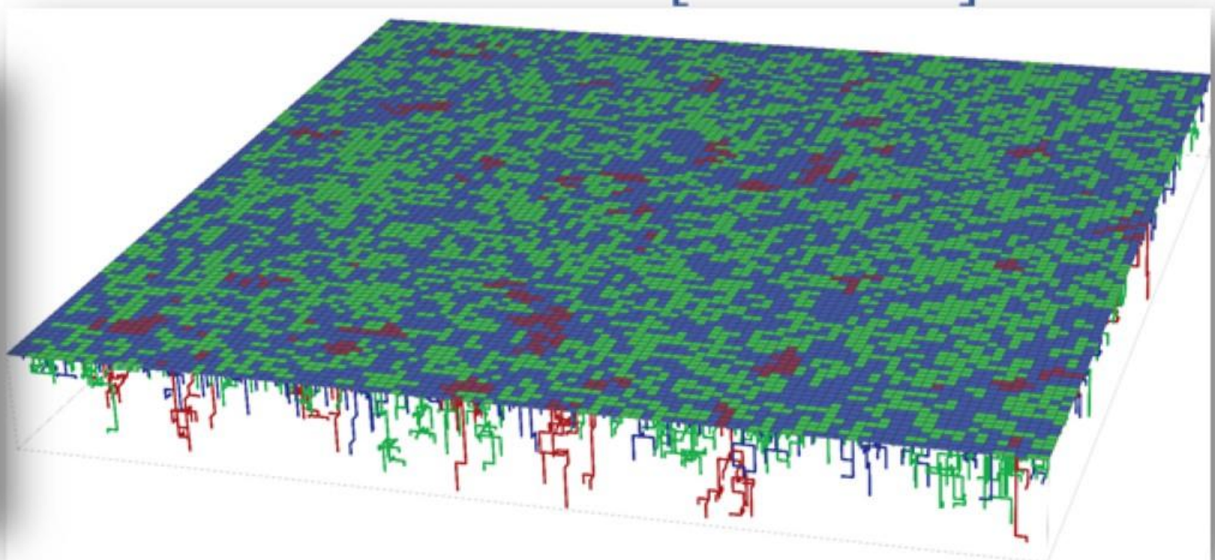


RED:
top spins are affected by initial state.

GREEN:
o/w.

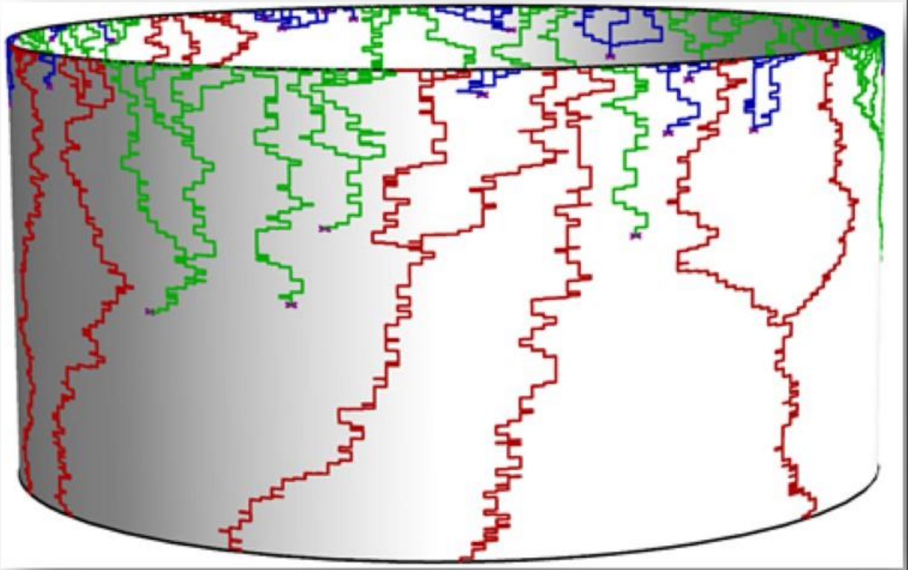
- Rough idea: condition on GREEN, let the effect of RED clusters vanish among BLUE (show $\mathbb{E}\left[2^{|R\cap R'|} \mid G\right] \to 1$).





# Example: the framework in 1D

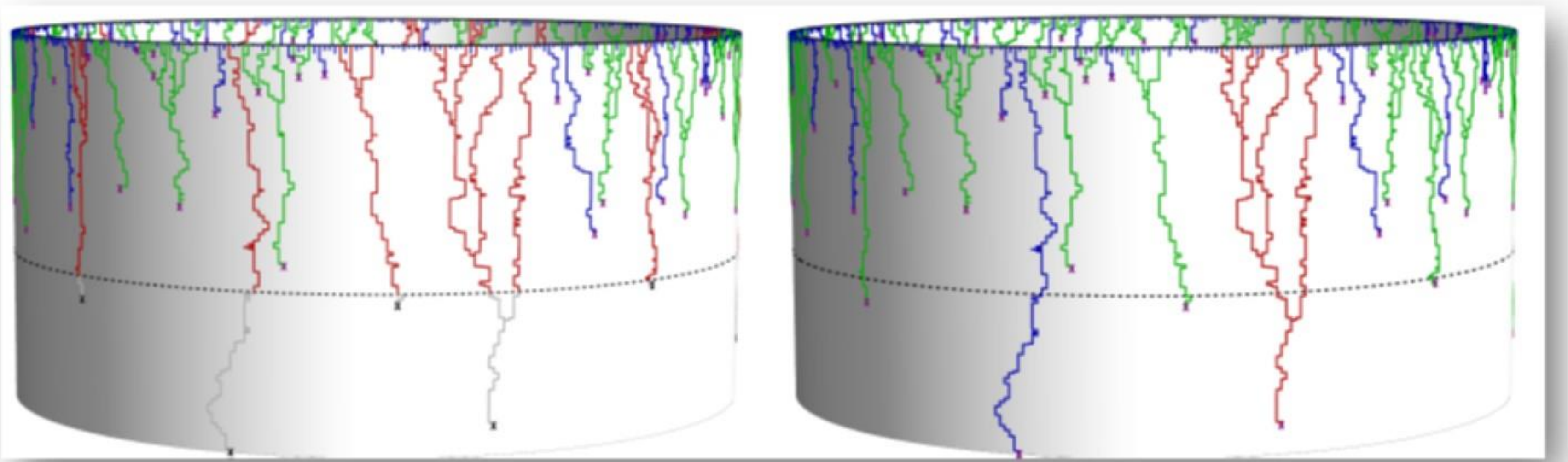
- In 1D: $\theta = \mathbb{P}(\text{oblivious update}) = 1 - \tanh 2\beta$
- Update history: continuous-time RW killed at rate $\theta$.
- $\mathbb{P}(\text{surviving to time } t_{\mathfrak{m}})$ is $\asymp 1/\sqrt{n}$.
- Cutoff at $t_{\mathfrak{m}} = \frac{1}{2\theta} \log n$
- Effect of the initial state on the final state is in terms of the bias of the cont.-time RW…



the 3 cluster classes (R/G/B) in $\mathbb{Z}_{256}$

# Example: random initial state

- Handling a uniform (IID) starting configuration:
  - Compare the dynamics directly with Ising measure: develop history to time $-\infty$ (*coupling from the past*).
  - Redefine RED clusters (coalesce before time 0).



# Losing red clusters in a blue sea

- **LEMMA:** ([Miller, Peres '12])

Let $\mu$ be a measure on $\sigma \in \Omega = \{\pm 1\}^n$ as follows:

1. draw a random variable $R \subseteq [n]$ via a law $\tilde{\mu}$ ;
2. let $\sigma_R \sim$ some law $\phi_R$ and $\sigma_{R^c} \sim$ IID Bernoulli $\begin{cases} +1 & 1/2 \\ -1 & 1/2 \end{cases}$

$$\Rightarrow \|\mu - \nu\|^2_{L^2(\nu)} \le \mathbb{E}\left[2^{|R \cap R'|}\right] - 1$$

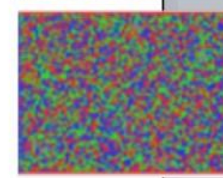
$\nu$=uniform measure

$R, R'$ IID

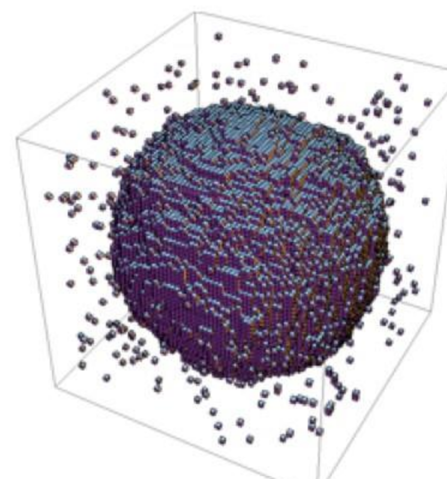
- the set $R$ embodies the nontrivial part of $\mu$
- it has a negligible effect on provided the exponential moment can be controlled…

# Open problems

- High temperature regime for other spin-systems (Potts / Independent sets /Legal colorings / Spin glass,…):
  - asymptotic mixing on the lattice up to $\beta_c$
  - cutoff on a transitive **expander**
  - asymptotic mixing from random starting states (*e.g.*, compare ordered/disordered start in Potts)

- 3D Ising:
  - no cutoff at criticality
  - power-law behavior at criticality
  - sub-exponential upper bound at low temperatures under all-plus b.c.



# Thank you