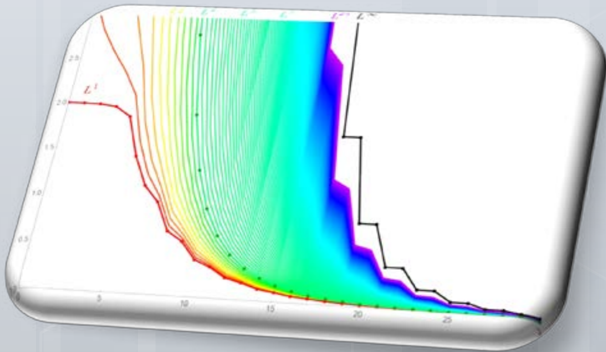
Feb 2017

Simons workshop

Expanders & extractors



Random walks on Ramanujan graphs, digraphs & complexes

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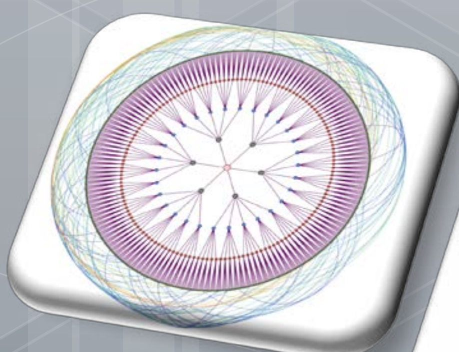
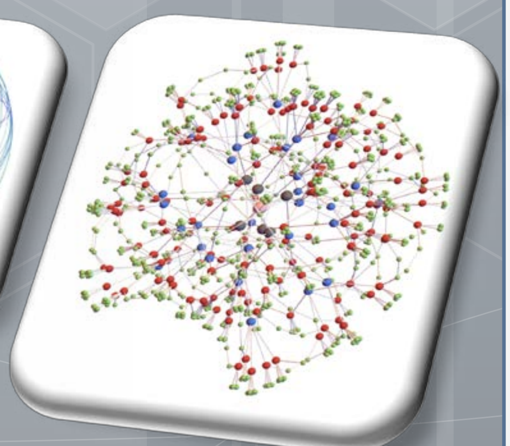
Based on joint works with
Y. Peres and with
A. Lubotzky and O. Parzanchevski





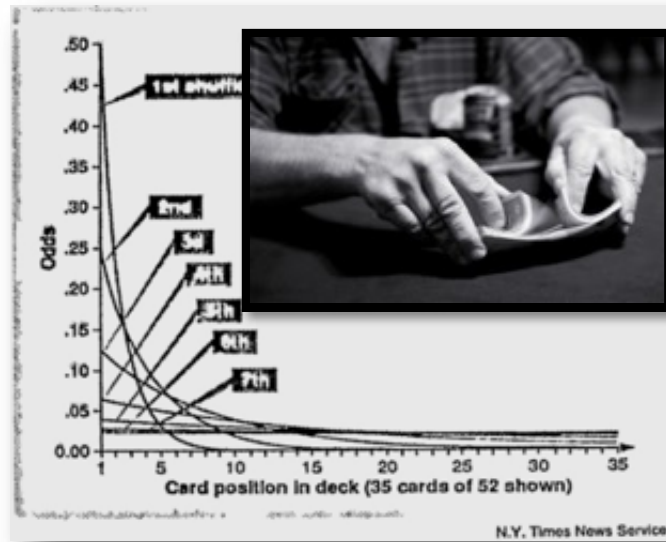
Open problems

- Consider the Cayley graph $(\mathrm{PSL}_2(\mathbb{F}_q), S \cup S^{-1})$ for
$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$
  - d-regular *expander* for $d = 4$. Profile of distances?
  - Rate of convergence of *simple random walk* (SRW) to uniform distribution? Convergence type ("gradual"/ "abrupt")?

- Consider the 3-uniform hypergraph whose hyperedges are the triangles in the graphs constructed by [LSV'05].
  - Profile of distances (loose / tight paths, etc.)?
  - Rate of convergence of SRW to uniform distribution?

Similar question: shuffling cards

- How many shuffles are needed to mix a deck of cards?
  (e.g., can we say where A♥ is, does it precede K♣, …)
  - [Aldous, Diaconis '86]:
    ”For card players, the question is not 'exactly how close to uniform is the deck after a million riffle shuffles?', but 'is 7 shuffles enough?'”
  - *Is there a sharp transition (**cutoff**)?*

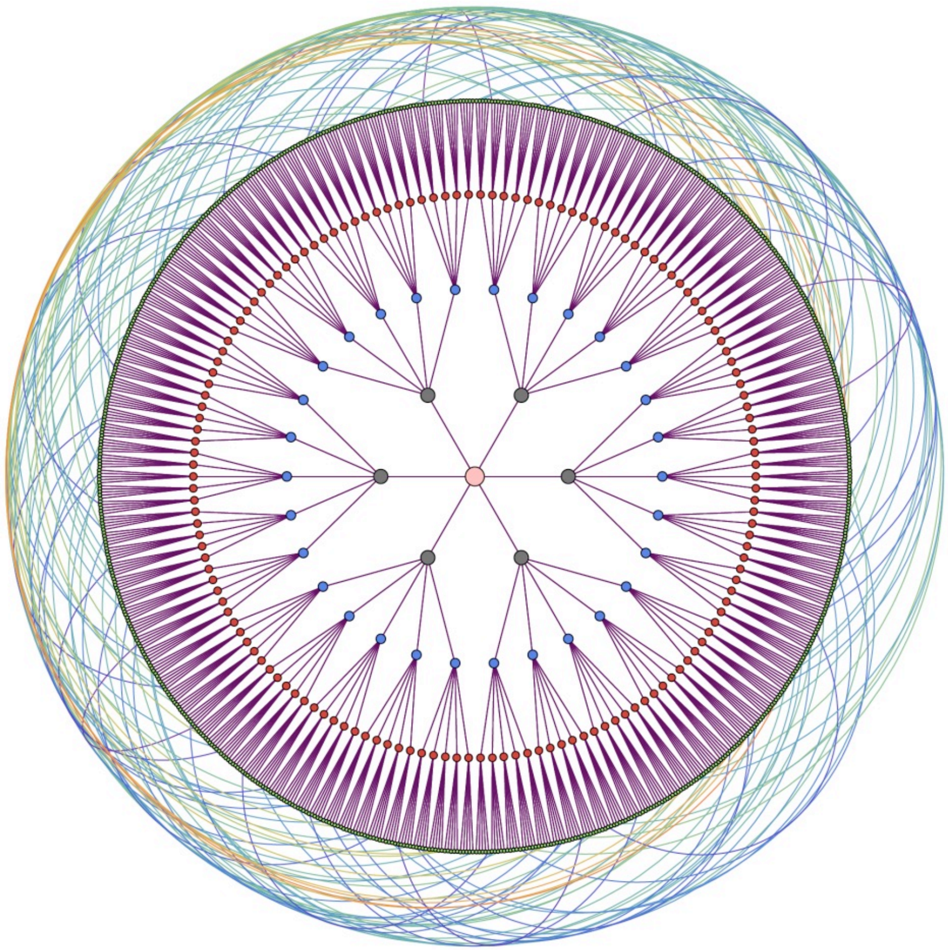


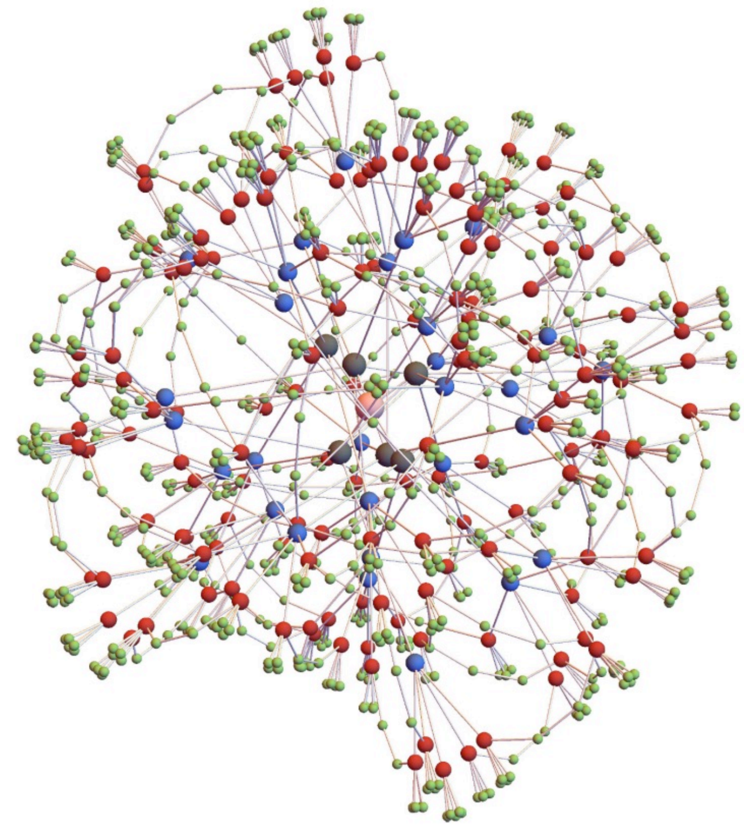
- Formally: understand the mixing time (t_{mix}) of the random walk on the symmetric group with a prescribed set of generators (e.g., all transpositions).

Expander graphs

- d-regular expander: family of connected graphs G_n with $\sup_n \lambda_2(G_n) < d$, where $d \geq 3$.
- Alon–Milman ('85) and Alon ('86) proved equivalence to lack of sparse cuts ("Cheeger's inequality").
- Alon–Boppana Theorem ('86; Nilli '91): For any d-regular graph, $\lambda_2 \geq 2\sqrt{d-1} - c_d (\log n)^{-1}$.
- **Ramanujan graph**: connected d-regular graph G s.t. $\forall$ eigenvalue λ of G has $|\lambda| = d$ or $|\lambda| \leq 2\sqrt{d-1}$.
  - Lubotzky, Phillips, Sarnak ('88), Margulis ('88): explicit d-reg Ramanujan graphs for $d - 1 = p^k$.
  - Marcus, Spielman, Srivastava ('13): $\exists$ biprtite Ramanujan graphs for all d.

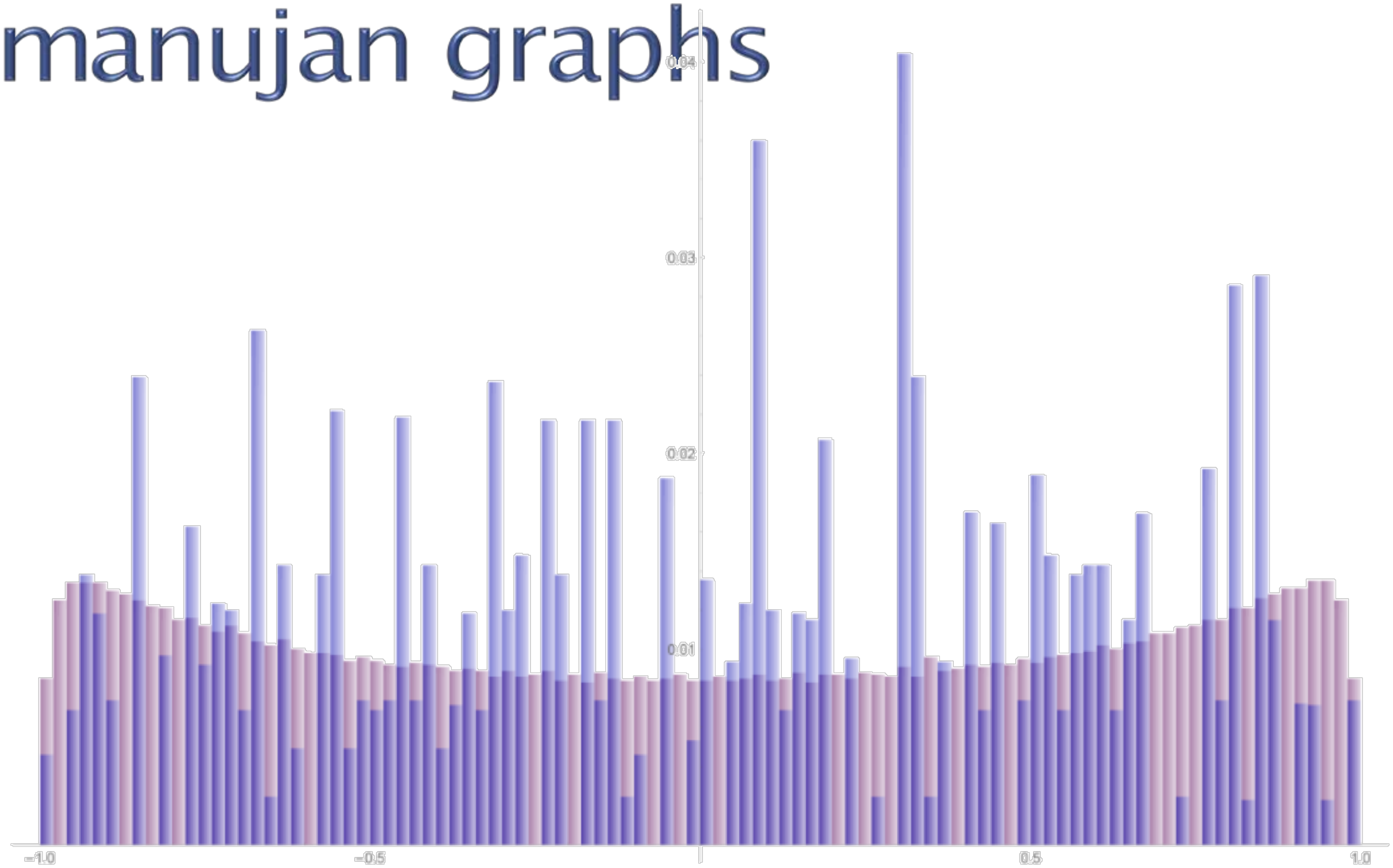
Ramanujan graphs





Ball of radius 4 in the 6-regular LPS-expander on $n = 12180$ vertices
(Cayley graph on $\mathrm{PSL}(2,\mathbb{F}_{29})$ *via* $S \cup S^{-1}$ *for* $S = \{\begin{pmatrix}1 & 0\\ 0 & 13\end{pmatrix}, \begin{pmatrix}1 & 27\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 5\\ 5 & 1\end{pmatrix}\}$*.)*

Ramanujan graphs



Normalized eigenvalues of Ramanujan graphs on $n = 12180$ vertices:

- front: 6-regular LPS-expander on $\mathrm{PSL}(2, \mathbb{F}_q)$ for $q = 29$
  (every nontrivial eigenvalue has multiplicity at least $(q-1)/2$)
- back: 1000-lift of the 3-regular Petersen graph.

Link to Quantum Computing

- Recent applications: see Peter Sarnak's letter on the Solvay–Kitaev Theorem and Golden Gates (http://publications.ias.edu/sarnak/paper/2637)
  - O. Parzanchevski's talk from yesterday…

- New understanding of distances in arithmetic Ramanujan graphs (see the letter above).
  - Recent work of N.T. Sardari constructed an infinite family of $(p+1)$-regular LPS Ramanujan graphs with diameter at least $\frac{4}{3}\log_p n$.
  - N.T. Sardari's talk from yesterday…

Mixing time and cutoff

- Total-variation distance:

$$\|\mu - \nu\|_{\text{tv}} = \sup_A [\mu(A) - \nu(A)] = \tfrac{1}{2}\ \|\mu - \nu\|_{L^1}\ .$$

- For a finite Markov chain with transition kernel P and stationary distribution π, let

$$D_{\text{tv}}(t) = \max_x \|P^t(x,\cdot) - \pi\|_{\text{tv}}\ .$$

- Total variation mixing time:

$$t_{\text{mix}}(\varepsilon) = \min\{t : D_{\text{tv}}(t) \le \varepsilon\}\ .$$

- **Cutoff phenomenon**: (discovered in [DS'81], [A'83], [AD'86])
  A sequence of chains exhibits *cutoff* if

$$t_{\text{mix}}(\varepsilon) = \big(1 + o(1)\big) t_{\text{mix}}(\varepsilon')$$

  for every fixed $0 < \varepsilon, \varepsilon' < 1$.

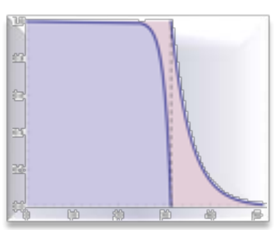
Walking on groups

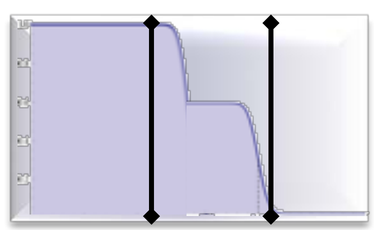
- Revisiting 1st example: what is t_{mix} of SRW on the Cayley graph $(\mathrm{PSL}_2(\mathbb{F}_q), S \cup S^{-1})$ for $S = \left\{\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\right\}$?
  - d-regular *expander* for $d = 4$ (the spectrum of the adjacency matrix supported on $(-d+\varepsilon, d-\varepsilon) \cup \{d\}$).
- On any expander $\exists c, C > 0$ such that $\forall\ 0 < \varepsilon < 1$ fixed,

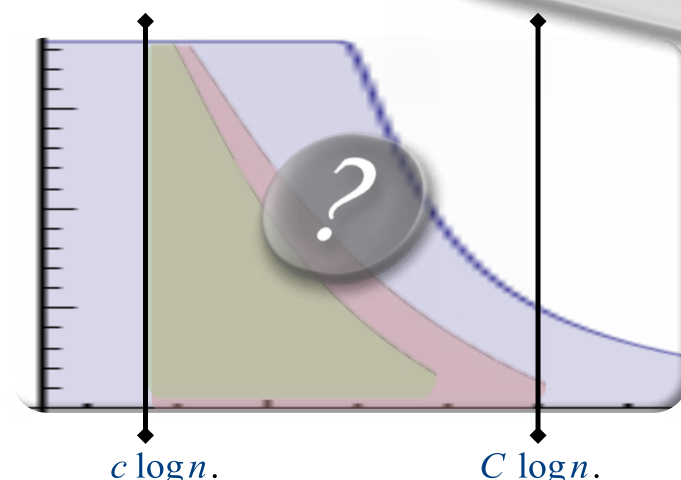
$$\big(c + o(1)\big)\log n \le t_{\mathrm{mix}}(\varepsilon) \le \big(C + o(1)\big)\log n$$

“not too gradual”…

  - *Is there cutoff (convergence ≈ step function)?*
  - *Multiple step functions?*







Cutoff history for random walks

- Discovered:
  - Random transpositions on S_n [Diaconis, Shahshahani '81]
  - RW on the hypercube, Riffle-shuffle [Aldous '83]
  - Named "Cutoff Phenomenon" in top-in-at-random shuffle analysis [Diaconis, Aldous '86]
- Until 2010: ***only one*** example of cutoff for RW on a ***bounded degree graph*** (lamplighter on $\mathbb{Z}_n^2$ [Peres & Revelle '04]).
  - <u>CONJECTURE</u> [Durrett '07]: on almost every 3-regular graph there is cutoff.

- Until 2015: ***no*** example of a transitive *expander* with cutoff
  - <u>CONJECTURE</u> [Peres '04]: on every transitive expander SRW has cutoff (*<u>no examples</u>*!)

SRW on d–regular graphs

- [Friedman '08]: proved "Alon's conjecture": for $d \geq 3$, almost every random d-reg graph is *weakly-Ramanujan*:

$$\lambda_2 = 2\sqrt{d-1} + o(1).$$

- [L., Sly '10]: confirmed the conjecture of Durrett:

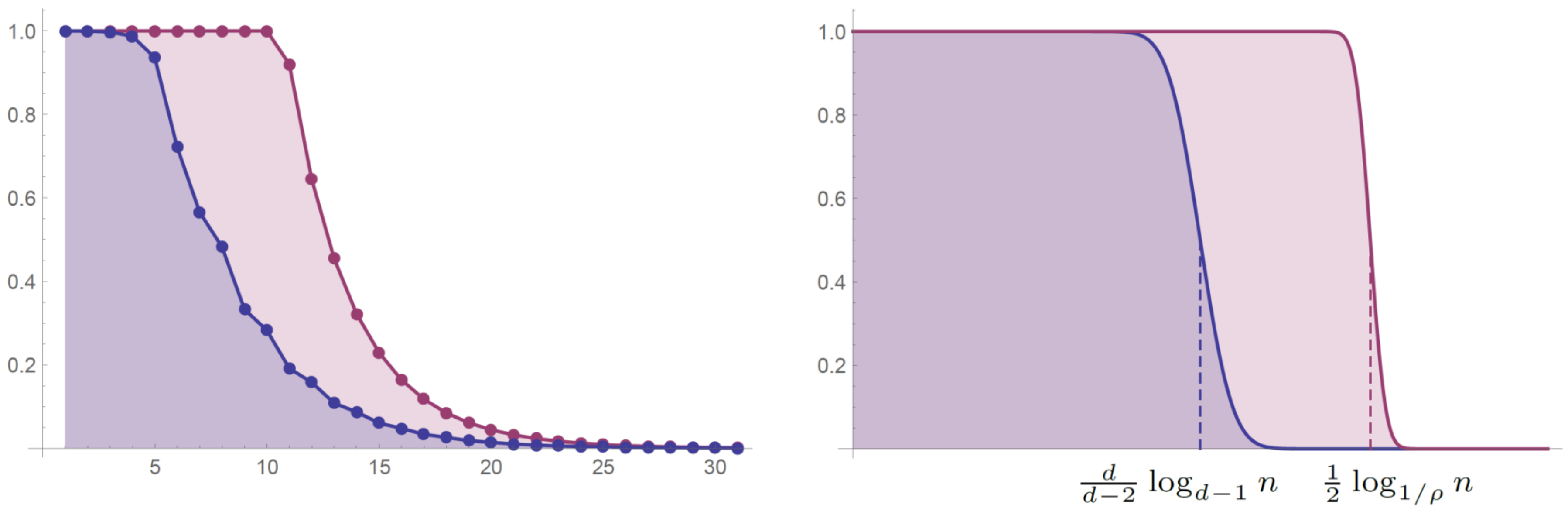
For $d \geq 3$, on almost every n-vertex d-regular graph:

- SRW *exhibits cutoff at* $\frac{d}{d-2} \log_{d-1} n$.
- NBRW *exhibits cutoff at* $\log_{d-1} n$.

Fastest possible…

- [L., Sly '11]: explicit (non-transitive) constructions of expanders with cutoff and ones without cutoff.
- Cutoff for Ramanujan graphs conjectured by Shayan Oveis Gharan ('15).

SRW on Ramanujan graphs



Distance of SRW from equilibrium in L^1 and L^2 (capped at 1).

(left: LPS-expander on $\mathrm{PSL}(2, \mathbb{F}_{29})$ *; right: asymptotic behavior.)*

Cutoff on all Ramanujan graphs

- **Theorem** (L., Peres '16):

On any sequence of d-regular non-bipartite Ramanujan graphs, SRW *exhibits* ***cutoff****: If G_n is such a graph on n vertices then for every initial vertex x, the* SRW *has*

$$t_{\mathrm{mix}}(\varepsilon) = \left(\tfrac{d}{d-2} + o(1)\right) \log_{d-1} n \ ,$$

for every fixed $0 < \varepsilon < 1$.

- Extensions:
  - Result holds also for weakly-Ramanujan graphs.
  - Can allow expanders with $n^{o(1)}$ large eigenvalues (and the rest as in the weakly-Ramanujan case).

Cutoff in L^p–distance

- For $1 \le p \le \infty$, the L^p-distance mixing time of a Markov chain with transition kernel P from its stationary distribution π is defined as

$$t_{\mathrm{mix}}^{(L^p)}(\varepsilon) = \min\{t : D_p(t) \le \varepsilon\}$$

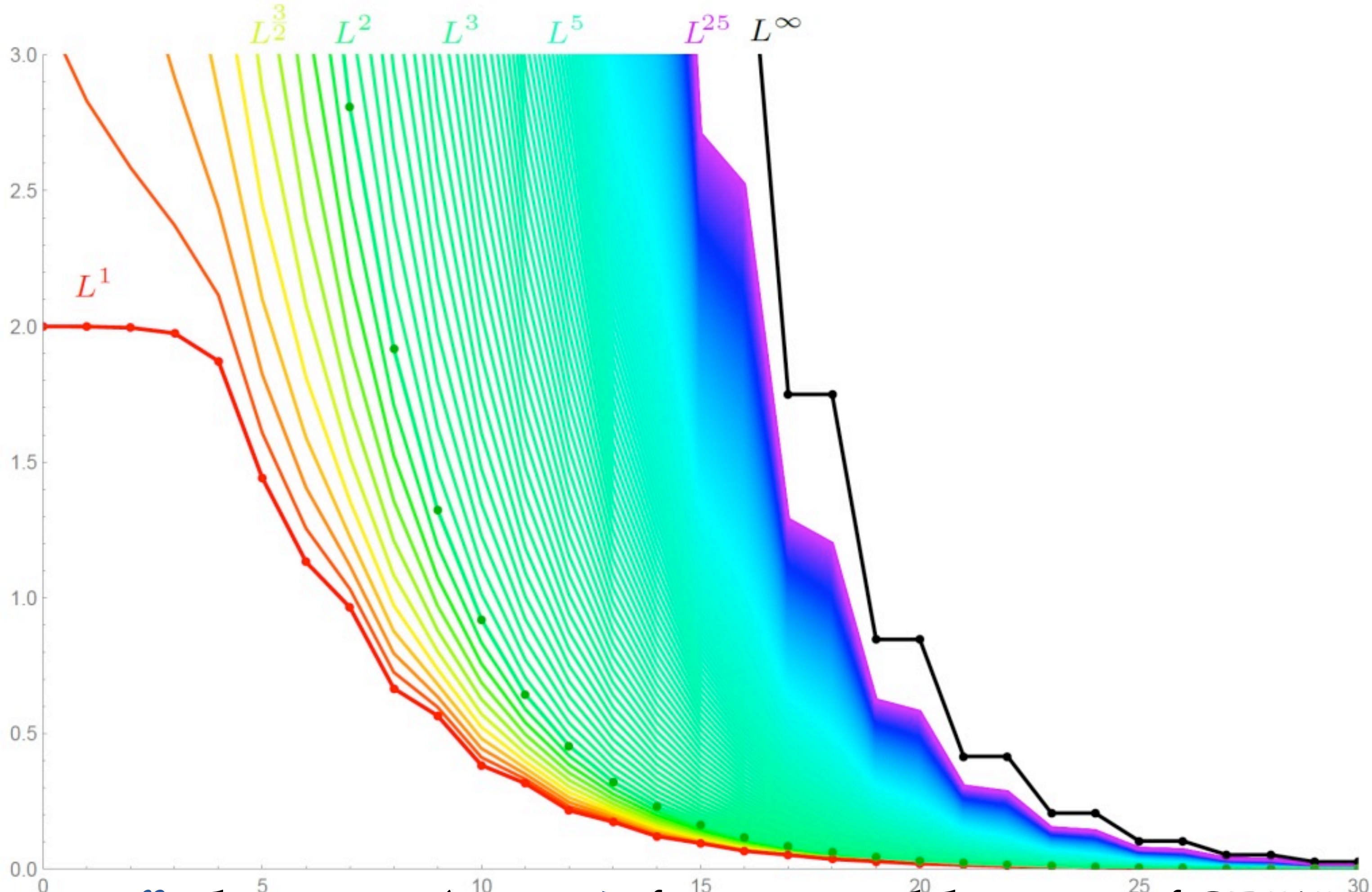
where

$$D_p(t) = \max_x \|P^t(x,\cdot)/\pi - 1\|_{L^p(\pi)}.$$

- **THEOREM** (L., Peres '16):

Let $d \ge 3$ and $p > 1$. Of all connected d-regular graphs, non-bipartite Ramanujan graphs have the fastest L^p-mixing time for SRW. *Moreover, on such graphs, L^p-**cutoff** occurs.*

Cutoff in L^p picture



L^p-distance ($p \geq 1$) from equilibrium of SRW on the LPS-expander on $\mathrm{PSL}(2, \mathbb{F}_{29})$

Graph distances

- COROLLARY (L., Peres '16):

Let G_n be a d-regular weakly Ramanujan sequence of graphs on n vertices. Then for every vertex x in G_n ,

$$\#\left\{y : \left|\frac{\mathrm{dist}(x,y)}{\log_{d-1} n} - 1\right| > \varepsilon\right\} = o(n)\,,$$

for every fixed $\varepsilon > 0$.
In particular, $\mathrm{diam}(G) \le \big(2 + o(1)\big)\log_{d-1} n$.

Shortest possible…

- Diameter bound: new proof of best known bound due to Chung, Faber, Manteuffel ('94).
- Typical distance: proved independently for Ramanujan graphs by N.T. Sardari using Chebyshev polynomials.

More detailed information

- **Corollary** (L., Peres '16):

Let G_n be a d-regular weakly Ramanujan graph on n vertices. Then $\forall$ directed edge $\vec{e}_1$ in G_n and all but $o(n)$ directed edges $\vec{e}_2$ there $\exists$ **path** *from $\vec{e}_1$ to $\vec{e}_2$ of length*

$$\big(1 + o(1)\big) \log_{d-1} n \ .$$

In particular, if the girth is $\geq (1+\varepsilon) \log_{d-1} n$, then for all but $o(n)$ directed edges $\vec{e}_2$ there $\exists$ **simple cycle** *through $\vec{e}_1, \vec{e}_2$ of length at most*

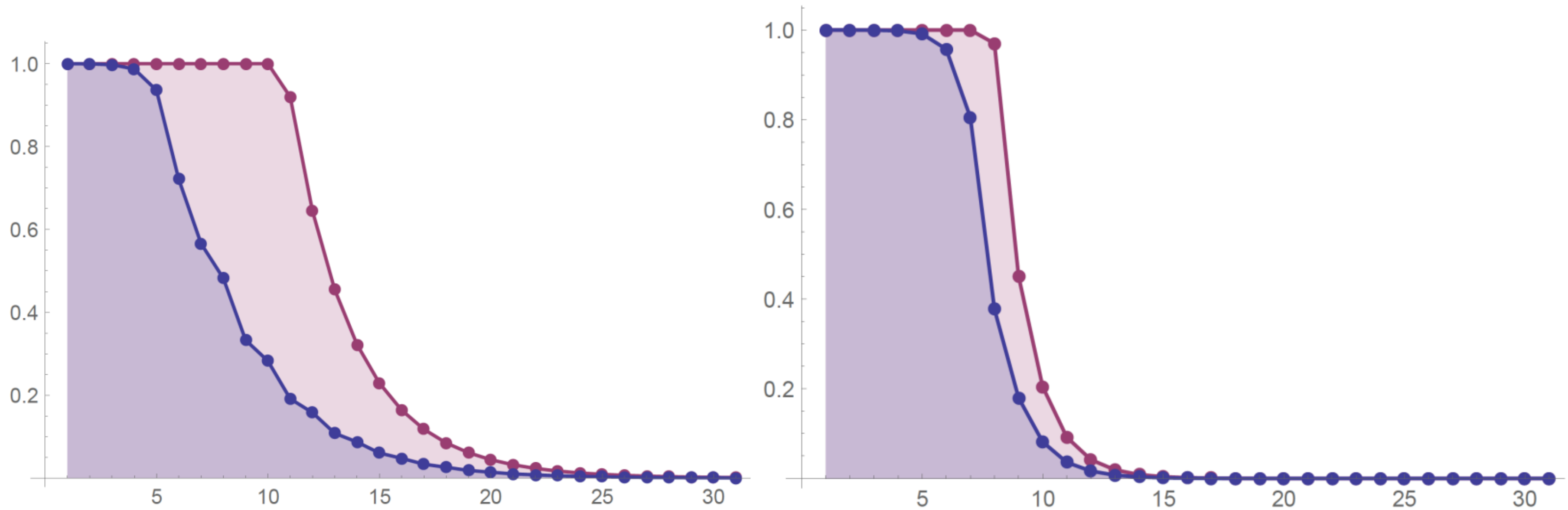
$$\big(2 + o(1)\big) \log_{d-1} n \ .$$

- (e.g.: the bipartite LPS Ramanujan graphs)

Key to the proof: NBRW analysis

- NBRW: essential in proofs that random d-regular graphs and random lifts are weakly Ramanujan: Friedman ('08), Friedman–Kohler ('14), Bordenave ('15).
- L^1-mixing for SRW reduces to L^1-mixing of NBRW using a covering tree argument.
- For random walks on Ramanujan graphs:
  - SRW: spectral analysis (L^2 distance) fails to give a sharp bound for L^1 : the two cutoff locations differ…
  - NBRW: L^1 and L^2 cutoff locations **coincide**!

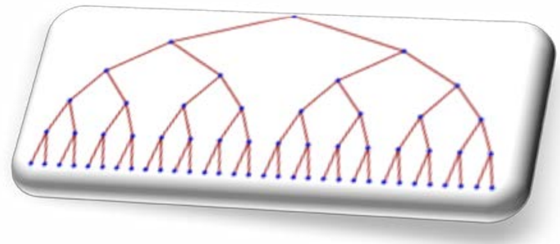
Mixing times of SRW vs. NBRW



Distance of from equilibrium in L^1 and L^2 (capped at 1) on LPS graph on $\mathrm{PSL}(2,\mathbb{F}_{29})$ (left: SRW; right: NBRW).

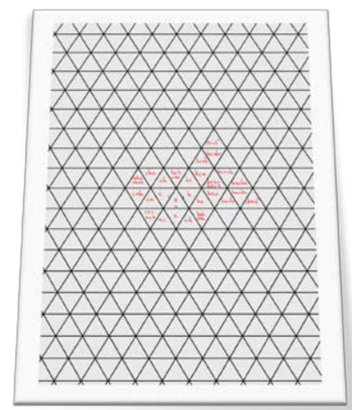
A general principle?

- Pulling the walk to the ***universal cover*** (the d-reg tree): the NBRW can only "descend": never creates cycles ©.



- Proposed high dimensional generalization:

*If $T:\mathfrak{C}\to\binom{\mathfrak{C}}{k}$ is an operator on cells $\mathfrak{C}\subset\mathcal{B}$ satisfying some combinatorial condition © then on any Ramanujan complex it exhibits **cutoff** at the **fastest possible** timepoint.*



- The generalized condition © : **collision-free:**

$\forall\, x,y\in\mathcal{B}$ *there is at most one* m *so that* $y\in T^m(x)$

Definition: Ramanujan complex

- $G = \mathbf{G}(F)$: simple algebraic group of rank r over a non-Archimedean local field F with residue field of order q.
- $\mathcal{B} = \mathcal{B}(G)$: the associated Bruhat–Tits building: a d-dimensional contractible simplicial complex.
- $\sigma_0 \in \mathcal{B}$: an r-dimensional cell (a "chamber").
- $\mathcal{I}$: the *Iwahori* subgroup of G (the point-wise stabilizer of σ_0)

- Γ : a torsion-free cocompact discrete subgroup of G.
- The quotient $X = \Gamma \backslash \mathcal{B}$ is a finite r-dim simplicial complex.
  - it is a **Ramanujan complex** if and only if

every irreducible infinite-dim $\mathcal{I}$-spherical G-subrepresentation of $L^2(\Gamma \backslash G)$ is *tempered*

contains an $\mathcal{I}$-fixed vector

weakly-contained in $L^2(G)$

Cutoff for RWs on complexes

- **THEOREM** (E. Lubetzky, A. Lubotzky, O. Parzanchevski):

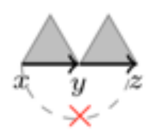
Let $T : \mathfrak{C} \to \binom{\mathfrak{C}}{k}$ *be a* k*-regular collision-free* G*-equivariant operator on cells in the Bruhat–Tits building* $\mathcal{B} = \mathcal{B}(G)$. *Let* $X = \Gamma \backslash \mathcal{B}$ *be any Ramanujan complex on* n *vertices.*

1. *Cutoff for the walk corr. to* T (*avg. over periodicity in step 1*):
$$|\, t_{mix}(\varepsilon) - \log_k n| \leq c_G \log_k \log n \;.$$
2. *Distances : if* $\rho(x,y) = \min\{m : y \in T^m(x)\}$ *then*
$$\#\{y : \; |\rho(x,y) - \log_k n| > c_G \log_k \log n\} = o(n)$$

for every fixed $\varepsilon > 0$.

Example: geodesic flow operators

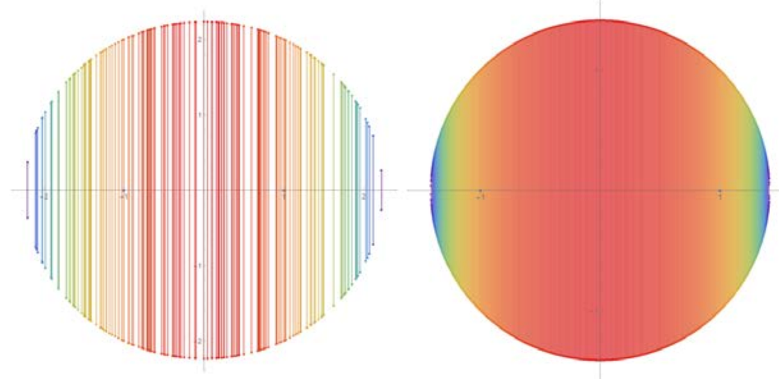
- RW operators that were studied in the context of zeta functions on Ramanujan complexes are collision-free.
  - e.g., in dimension $r = 2$: two operators on a subset $\mathfrak{X}$ of the cells of the 3-partite Ramanujan complex X:

j	k-branching geodesic flow operator T		k	cutoff location
1	$\mathfrak{X} = \{(x,y) \in X : \mathrm{col}(y) \equiv \mathrm{col}(x) + 1\}$ $T(x,y) = \{(y,z) \in \mathfrak{X} : \{x,y,z\} \notin X\}$		q^2	$\frac{1}{2}\log_q n$
2	$\mathfrak{X} = \left\{(x,y,z) \in X : \begin{array}{l} \mathrm{col}(y) \equiv \mathrm{col}(x) + 1 \\ \mathrm{col}(z) \equiv \mathrm{col}(y) + 1 \end{array}\right\}$ $T(x,y,z) = \{(y,z,w) \in \mathfrak{X} \,,\ w \neq x\}$		q	$\log_q n$

- Byproduct: confirm R.H. for the associated zeta functions over any group G (previously known for types $\tilde{A}_n$, $\tilde{C}_2$).

Proof ideas — dimension 1

- Analysis of NBRW on graph relied on two key points:
  1. *Spectral*: its nontrivial eigenvalues lie in the ball of radius $\sqrt{d-1}$.
  2. *Algebraic*: its matrix is unitarily similar to a block-diagonal matrix with blocks of size ≤ 2.



- *Spectral* property equivalent to saying the nonbacktracking matrix is the adjacency matrix of a **Ramanujan digraph.**
  - Proved by Hashimoto ('89) via the Ihara ('66) zeta function: the NBRW eigenvalues θ_i, θ_i' and the SRW eigenvalues λ_i are related by the quadratic equation $\theta^2 - \lambda_i\theta + d - 1 = 0$

Proof ideas — dimension 1 (ctd.)

- *Algebraic* property: proved in [L., Peres '16] for any graph.
- <u>THEOREM</u>:

The nonbacktracking operator is unitarily similar to

$$\Lambda = \operatorname{diag}\left(d-1, \begin{pmatrix} \theta_2 & \alpha_2 \\ 0 & \theta_2' \end{pmatrix}, \dots, \begin{pmatrix} \theta_n & \alpha_n \\ 0 & \theta_n' \end{pmatrix}, -1, \dots, -1, 1, \dots, 1\right)$$

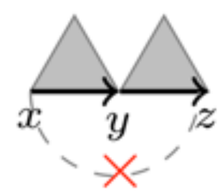
where $\theta_i\,, \theta_i' \in \mathbb{C}$ *are the solutions of* $\theta^2 - \lambda_i \theta + d - 1 = 0$, *and*

$$|\alpha_i| = \begin{cases} d-2 & |\lambda_i| \le 2\sqrt{d-1}\,, \\ \sqrt{d^2 - \lambda_i^2} & |\lambda_i| > 2\sqrt{d-1}\,. \end{cases}$$

Proof ideas — high dimensions

- Generalize the NBRW analysis to RW on a Ramanujan **digraph** with blocks of any bounded size (rather than 2).
- Show that if an operator is collision-free on the building $\mathcal{B}$ then its walk corresponds to a Ramanujan digraph whose adj. matrix is block-diagonal with bounded-size blocks.
- Give a sufficient and necessary condition for being collision-free on the building (then used for the flows).
- Example: the geodesic flow for $j = 1$ has blocks of size r (identifies with the NBRW decomposition at $r = 1$):

$$\begin{pmatrix} q^{\frac{r}{2}} z_1 & & & & \\ (q-1)\, q^{\frac{r-1}{2}} z_1 & q^{\frac{r}{2}} z_2 & & & \\ (q-1)\, q^{\frac{r}{2}} z_1 & (q-1)\, q^{\frac{r-1}{2}} z_2 & q^{\frac{r}{2}} z_3 & & \\ \vdots & \vdots & \ddots & \ddots & \\ (q-1)\, q^{r-1} z_1 & (q-1)\, q^{\frac{2r-3}{2}} z_2 & \cdots & (q-1)\, q^{\frac{r-1}{2}} z_r & q^{\frac{r}{2}} z_{r+1} \end{pmatrix}$$



Thank you