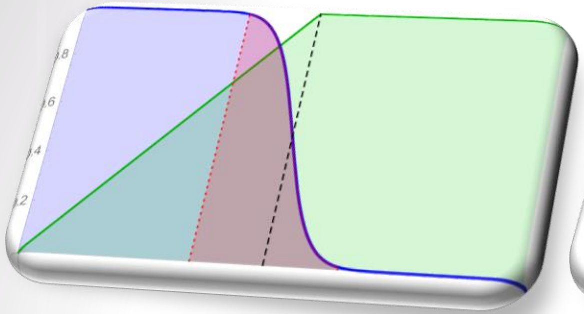
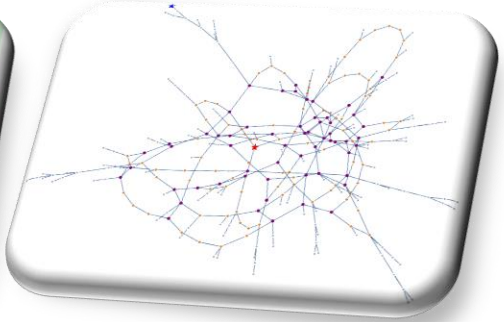
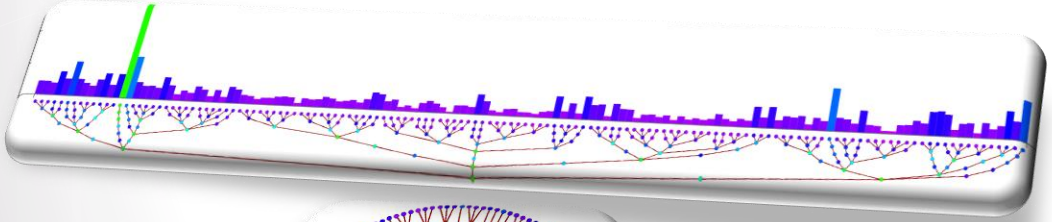
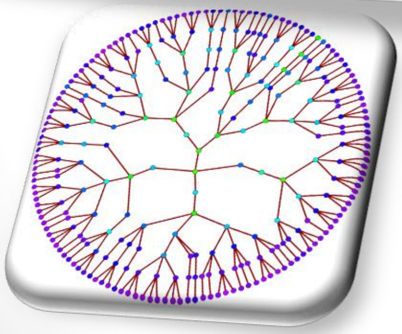




May 2015
SE Probability Conference
Duke University

Random walks on the Random graph





Eyal Lubetzky

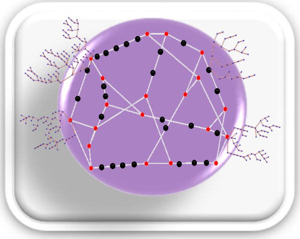
Courant Institute (NYU)

Joint work with
N. Berestycki, Y. Peres, A. Sly

In this talk

- Mixing time of **random walk** and specifically ***cutoff*** as a gauge for delicate properties of the geometry.
- Compare its behavior between

$\mathcal{G}(n,p)$
Erdős-Rényi
random graph



$\mathcal{G}(n,3)$
random regular graph
(and other degree distributions)

relatives

and the effect of the initial state on mixing.

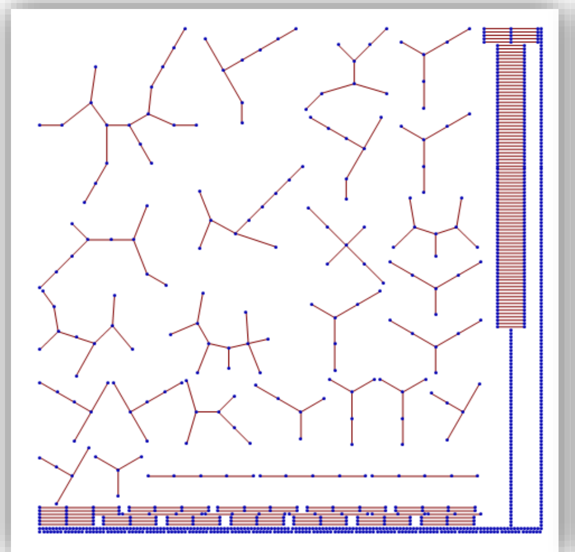
The Erdős-Rényi random graph

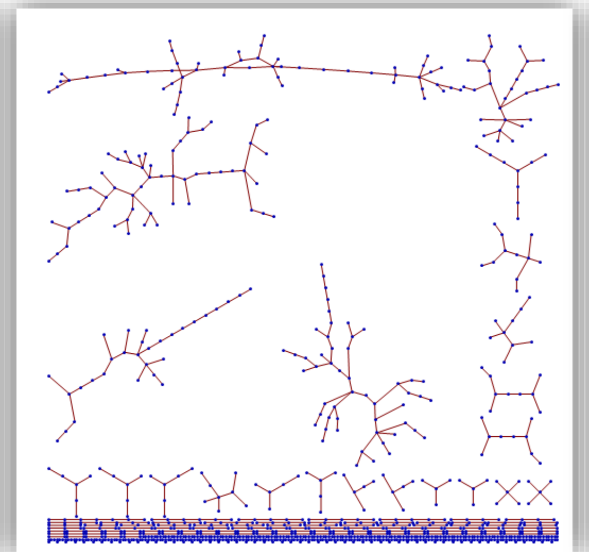
- $\mathcal{G}(n,p)$: indicators of the $\binom{n}{2}$ edges are IID $\mathrm{Bernoulli}(p)$.

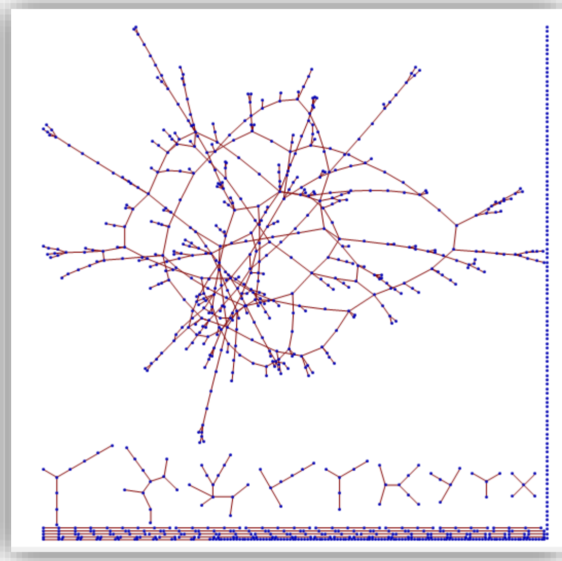
$n = 1000$, $p = 0.75/n$

$n = 1000$, $p = 1/n$

$n = 1000$, $p = 1.5/n$





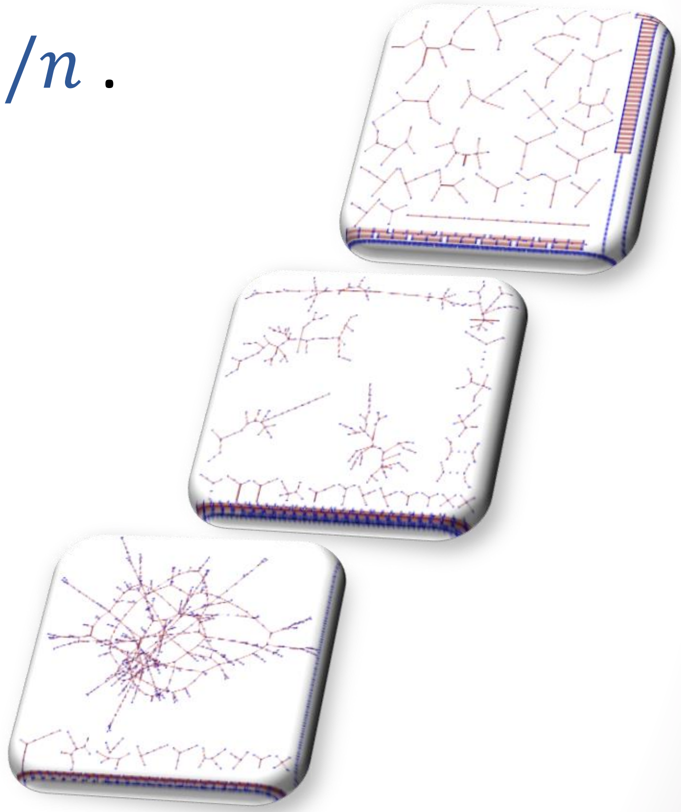


"This double "jump" of the size of the largest component… is one of the most striking facts concerning random graphs."
(Erdős and Rényi, 1960)

The Erdős-Rényi random graph

- Setting: $\mathcal{G}(n,p)$ around the critical point $p = 1/n$.
- "Double jump" phenomenon for order of $|\mathcal{C}_1|$:
  [Erdős-Rényi (1960's)], [Bollobás '84] , [Łuczak '90]
  - $\log n$ for $p = \lambda/n$ with $\lambda < 1$ fixed.
  - $n^{2/3}$ at and throughout *critical window*: $p = (1 \pm \varepsilon)/n$ for $\varepsilon = O(n^{-1/3})$.
  - n for $p = \lambda/n$ with $\lambda > 1$ fixed.
- Emerging from the critical window:
  - $p = (1+\varepsilon)/n$ for $n^{-1/3} \ll \varepsilon \ll 1$:

    $|\mathcal{C}_1| \sim 2\varepsilon n$ (giant component gradually forms)



Measuring convergence to equilibrium

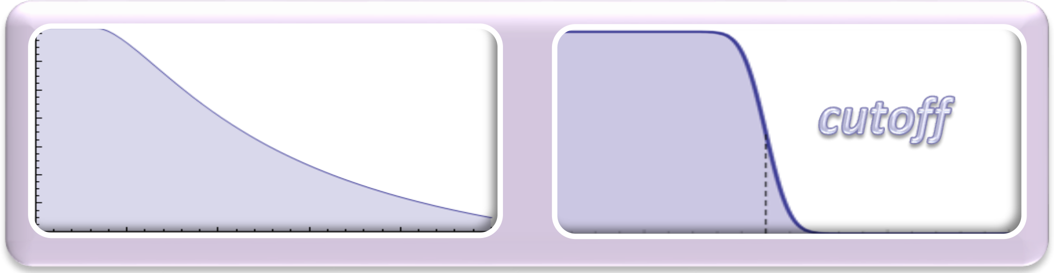
- <u>Total-variation mixing time</u> :
  - ➢ the mixing time of a Markov Chain on Ω with transition kernel P to within distance ε from its stationary distribution π is defined as

$$t_{\mathrm{mix}}(\varepsilon) = \inf\left\{t \;:\; \max_{x_0} \left\|P^t(x_0\,,\cdot) - \pi\right\|_{\mathrm{tv}} \le \varepsilon\right\}$$

(where $\|\mu - \nu\|_{\mathrm{tv}} = \sup_{A\subset\Omega}[\mu(A) - \nu(A)]$)

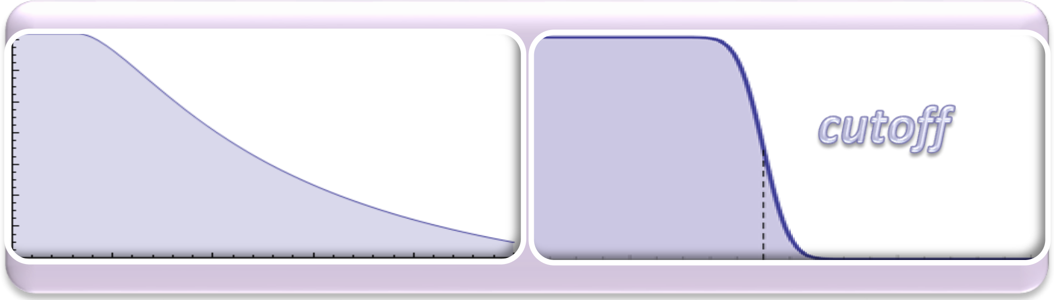
  - ➢ Analogous definition of $t_{\mathrm{mix}}^{(x_0)}(\varepsilon)$ for a prescribed starting state x_0.

- <u>Dependence on ε</u> : (*cutoff phenomenon* [DS81], [A83],[AD86])
  We say there is ***cutoff*** $\Longleftrightarrow$ $t_{\mathrm{mix}}(\varepsilon) \sim t_{\mathrm{mix}}(\varepsilon')$ $\forall$ fixed $\varepsilon, \varepsilon'$



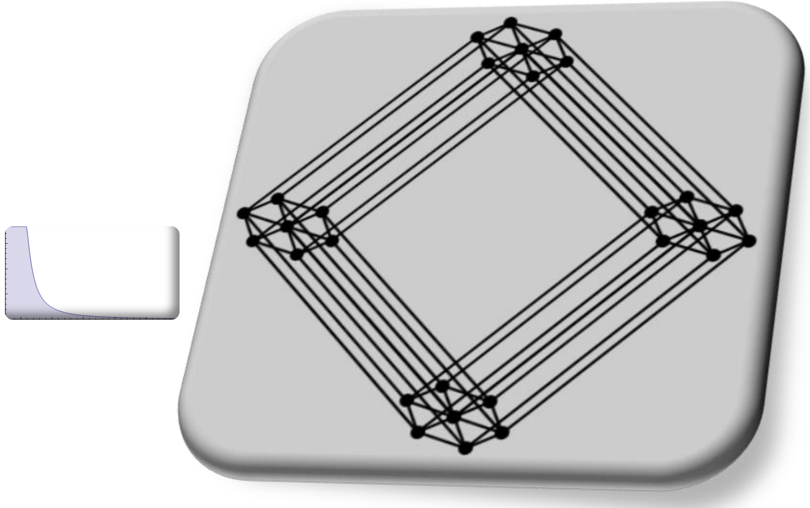
Cutoff History (RWs on graphs/groups)

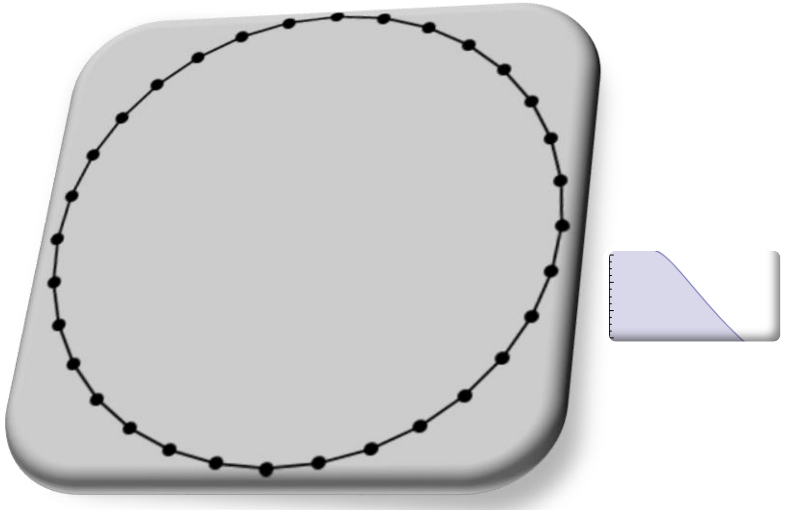
- Discovered:
  - Random transpositions on S_n [Diaconis, Shahshahani '81]
  - RW on the hypercube, Riffle-shuffle [Aldous '83]
  - Named "Cutoff Phenomenon" in top-in-at-random shuffle analysis [Diaconis, Aldous '86]
- Nearly 3 decades after its discovery: *only example* of cutoff for RW on a ***bounded-degree graph*** was the lamplighter on $\mathbb{Z}_n^2$ [Peres & Revelle '04].
  - Is this a phenomenon of (mainly) large degree graphs?



Basic examples: RWs on graphs

Lazy discrete-time simple random walk





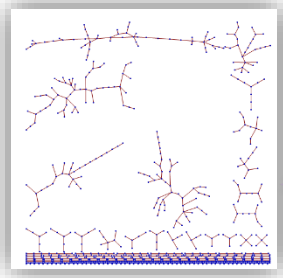
hypercube $\{0,1\}^n$:
☑ cutoff at $\frac{1}{2}n\log n \pm O(n)$
[Aldous ’83]

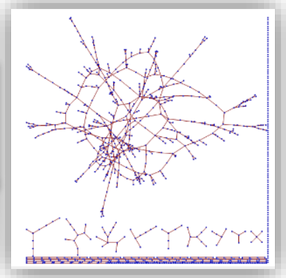
n-cycle:
☒ No cutoff.

- *What about mixing on $\mathcal{C}_1$ of $\mathcal{G}(n,p)$?*

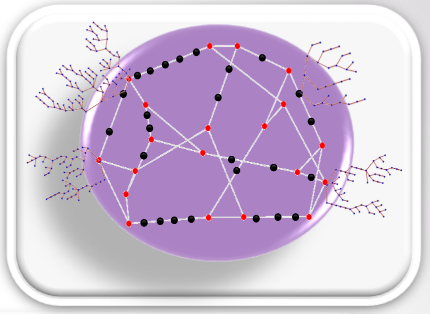
Mixing on the largest component

	Critical window $p = (1 \pm \varepsilon)/n$ $\varepsilon = O(n^{-1/3})$	Mildly supercritical $p = (1+\varepsilon)/n$ $n^{-1/3} \ll \varepsilon \ll 1$	**Supercritical** $p = (1+\varepsilon)/n$ $\varepsilon > \mathbf{0}$ fixed
$\lvert \mathcal{C}_1 \rvert$	$\asymp n^{2/3}$	$\sim 2\varepsilon n$	$\sim 2\varepsilon n$
Mixing time on $\mathcal{C}_1$	$\asymp n$ Nachmias, Peres ’08	$\asymp \varepsilon^{-3}\log^2(\varepsilon^3 n)$ Ding, L., Peres ’12	$\asymp \log^2 n$ Fountoulakis, Reed ’08 and independently Benjamini, Kozma, Wormald ’13

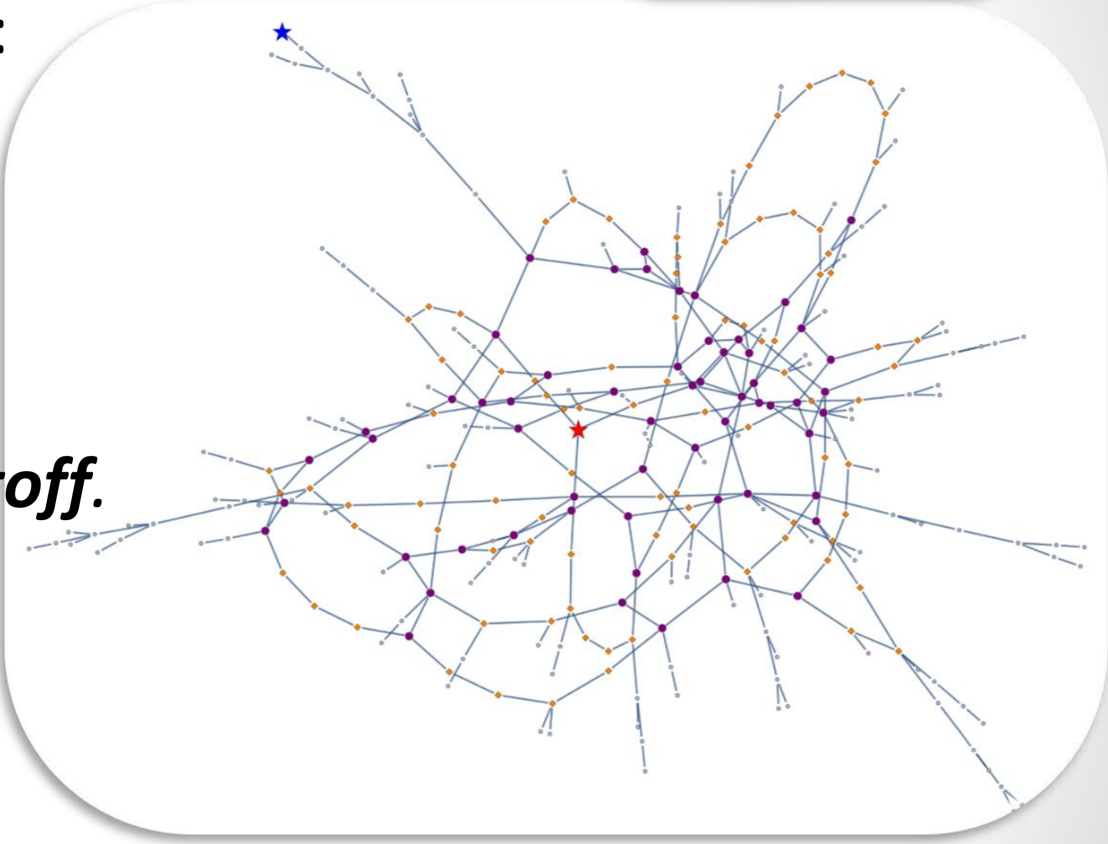




Bottlenecks slow the mixing on $\mathcal{C}_1$

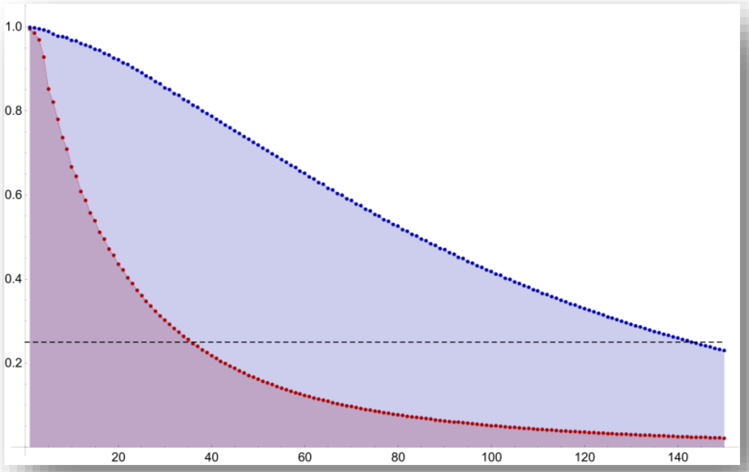


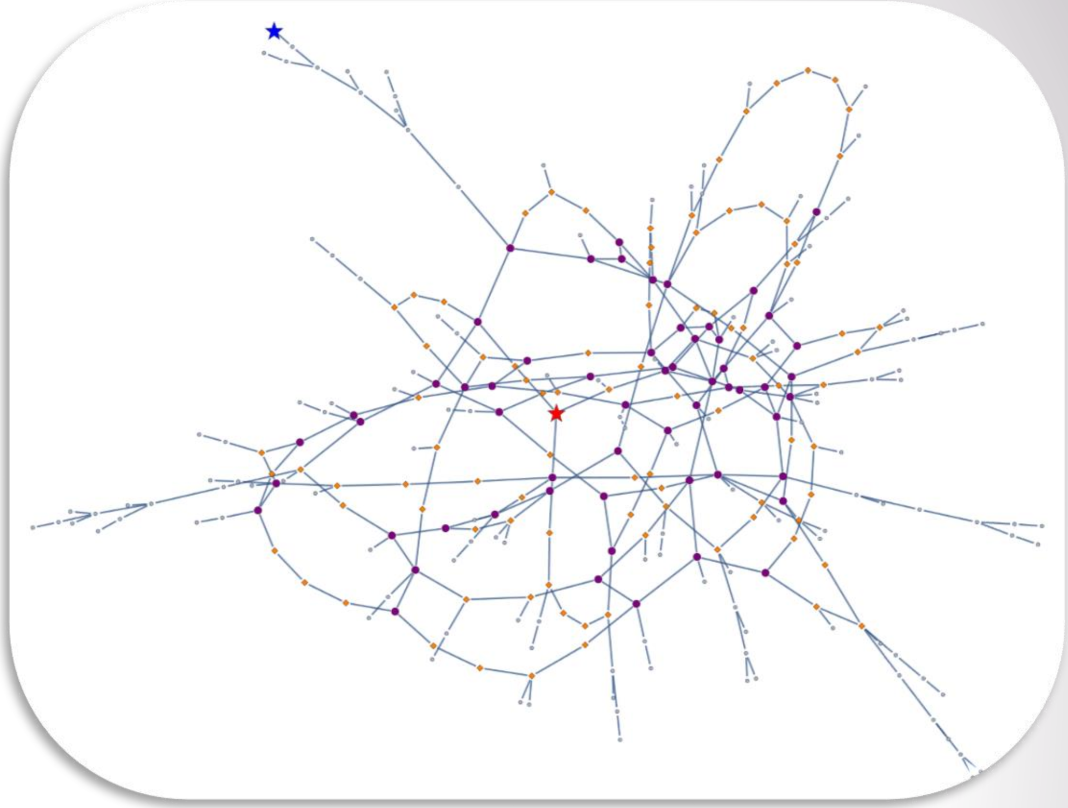
- Lower bound $t_{\mathrm{mix}} \geq C \log^2 n$ immediate:
  - w.h.p. $\mathcal{C}_1$ contains a path $\mathcal{P}$ of $c \log n$ degree-2 vertices.
  - escaping $\mathcal{P}$ starting from v_1 at its center takes $\left(\frac{c}{2}\log n\right)^2$ steps in expectation.
  - large hanging trees have a similar effect.
- Dominates mixing ($t_{\mathrm{mix}} \asymp \log^2 n$); ***no cutoff***.
- Such bottlenecks should be *rare*…
  - faster mixing from a typical initial vertex v_1?



- *Indeed*: *starting from a typical vertex* *<u>accelerates</u> the RW & <u>concentrates</u> it (**cutoff**)!*

New results: RW on a giant





- **THEOREM** [Berestycki, L., Peres, Sly]:

RW from a uniform vertex $v_1 \in \mathcal{C}_1$ w.h.p. satisfies

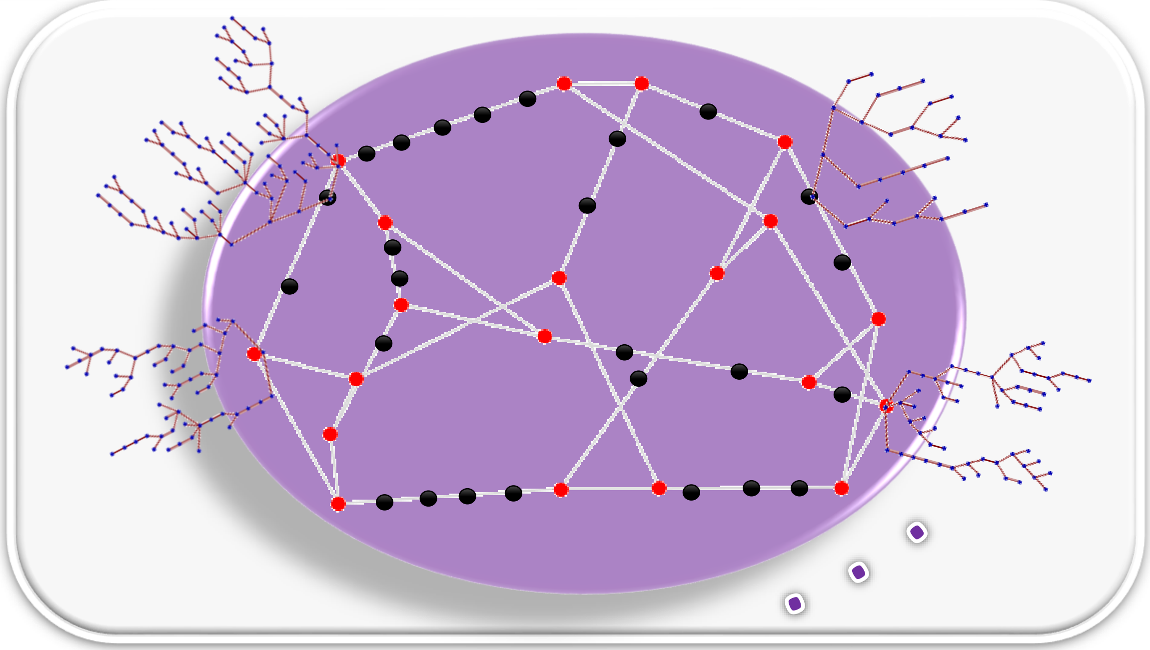
$$t_{\text{mix}}^{(v_1)}(\varepsilon) = \nu^{-1}\mathbf{d}^{-1}\log n \pm (\log n)^{1/2+o(1)}$$

  - $\mathcal{C}_1 =$ largest component of $\mathcal{G}(n, p = \lambda/n)$ [$\lambda > 1$ fixed].
  - $\nu =$ speed of RW on a Po(λ)-GW tree.
  - $\mathbf{d} =$ dimension of harmonic measure Po(λ)-GW tree.

Anatomy of a giant

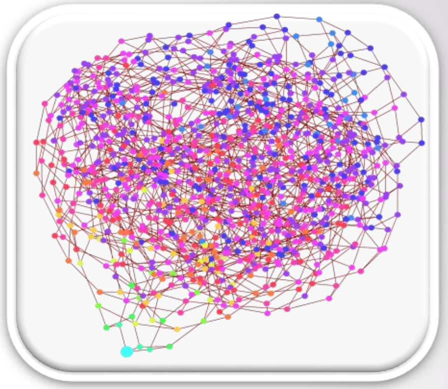
<u>THEOREM</u> [Ding, L., Peres '13]: giant of $\mathcal{G}(n, p = \lambda/n)$ is $\approx$

1. kernel : $\mathcal{K}$ random graph with (nice) given degrees ($D_i \sim \mathrm{Po}(\lambda - \varepsilon_\lambda \mid \cdot \geq 3)$ IID for $i = 1, \dots, N$)
2. 2-core : edges $\mapsto$ paths of lengths IID $\mathrm{Geom}(1 - \varepsilon_\lambda)$
3. giant : attach IID $\mathrm{Po}(\varepsilon_\lambda)$-Galton-Watson trees



a typical $v_1 \in \mathcal{C}_1$ will be "far" from the bottlenecks: what is t_{mix} from a typical vertex on an *expander?*

RWs on expanders



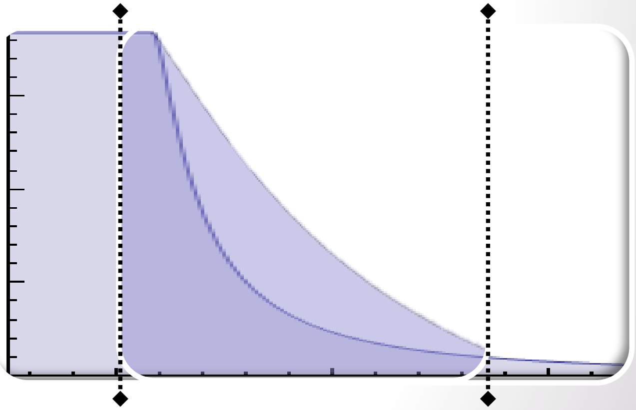
- **DEFINITION** [*regular expander*]:

  sequence of d-regular graphs ($d \geq 3$ fixed) such that the relaxation time (1/spectral-gap) of **SRW** is $O(1)$.

- Since $t_{\text{rel}} = O(1)$ the "*product condition*" of Peres (2004) holds and we expect **cutoff…**
- Specifically, convergence of RW on such a graph occurs along

  $$t \in [c \log n \,, c' \log n]$$

  (not too gradual: 'pre-cutoff').
- *Consider a random regular graph (an expander w.h.p.)*



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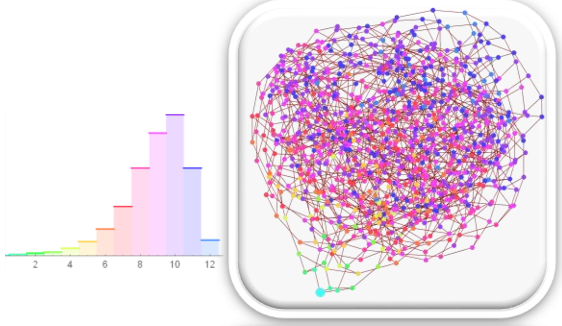
RWs on random regular graphs

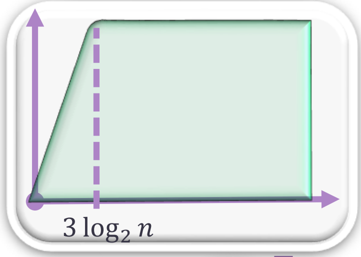
- $\mathcal{G}(n,d) =$ uniformly chosen d-regular n-vertex graph.
  Its study pioneered by Bollobás in early 80's.
- W.h.p. $G \sim \mathcal{G}(n,d)$ for $d \geq 3$ is an expander [Pinsker '73], [Broder, Shamir '87].
- **<u>THEOREM</u>** [Berestycki, Durret '08]:

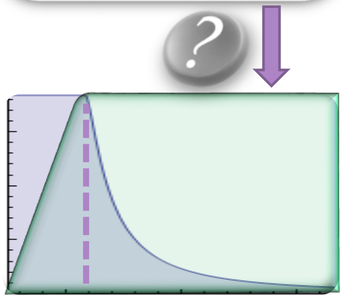
  RW on $\mathcal{G}(n,3)$ after $c \log_2 n$ steps is w.h.p. at distance $\sim (c/3 \wedge 1) \log_2 n$ from origin.

- **<u>CONJECTURE</u>** [Durrett '07]:

  Mixing time of the lazy RW on the random cubic graph $\mathcal{G}(n,3)$ is w.h.p. $\sim 6 \log_2 n$.







Cutoff for RW on $\mathcal{G}(n,d)$

- As Durrett and Peres conjectured, ∃ cutoff almost always:
- <u>**THEOREM**</u> [L., Sly ‘10]:

  Let $G \sim \mathcal{G}(n,d)$ for $d \geq 3$ fixed. The **SRW** on G w.h.p. has cutoff at $\frac{d}{d-2}\log_{d-1} n$ with window $\sqrt{\log n}$

  - *e.g., for* $d = 3$: $t_{\mathrm{mix}}(\varepsilon) = 3\log_2 n - \left(2\sqrt{6} + o(1)\right)\Phi^{-1}(\varepsilon)\sqrt{\log_2 n}$

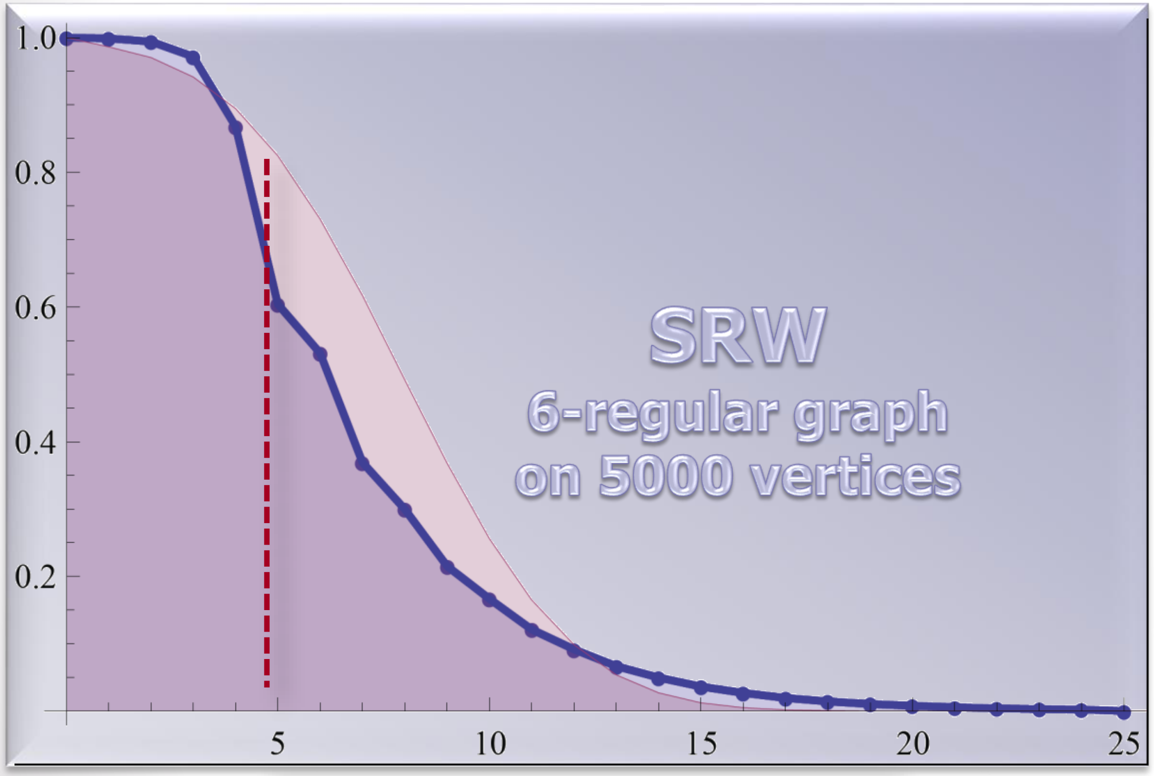
Fastest possible…

- **NBRW** (does not traverse same edge twice in a row) also has cutoff, earlier and with a **constant** window!
- <u>**THEOREM**</u> [L., Sly ‘10]:

  Let $G \sim \mathcal{G}(n,d)$ for $d \geq 3$ fixed. The **NBRW** on G w.h.p. has cutoff at $\log_{d-1}(dn)$ with window $O(1)$.

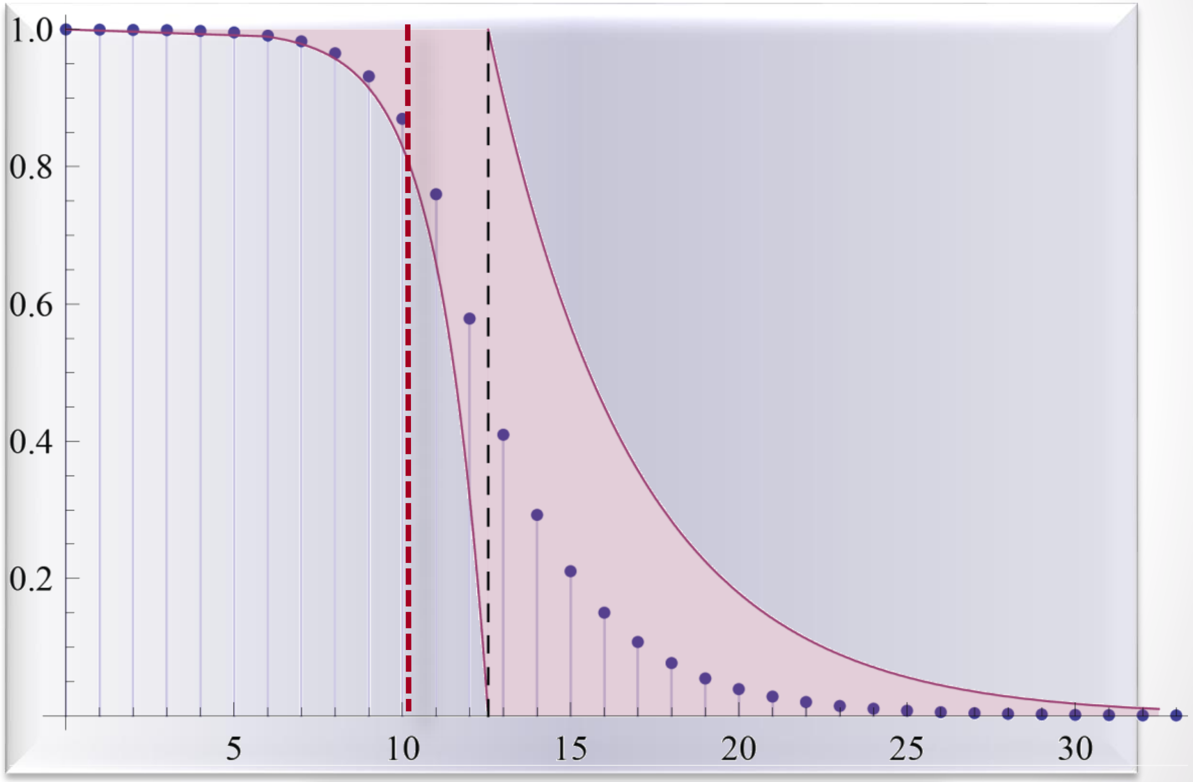
Simulations of RWs on $\mathcal{G}(n,d)$

SRW 6-regular graph on 5000 vertices



$O(\sqrt{\log n})$ cutoff window

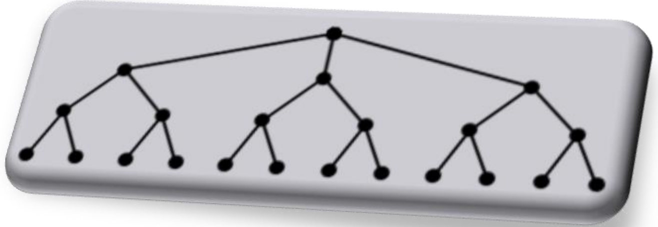
NBRW 3-regular graph on 2000 vertices



$O(1)$ cutoff window

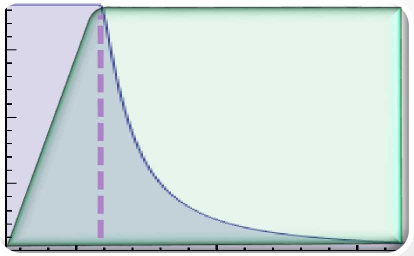
Insight: cutoff for SRW & NBRW

- Consider a d-regular tree, rooted at the starting point of the RW (mixes upon hitting leaves).



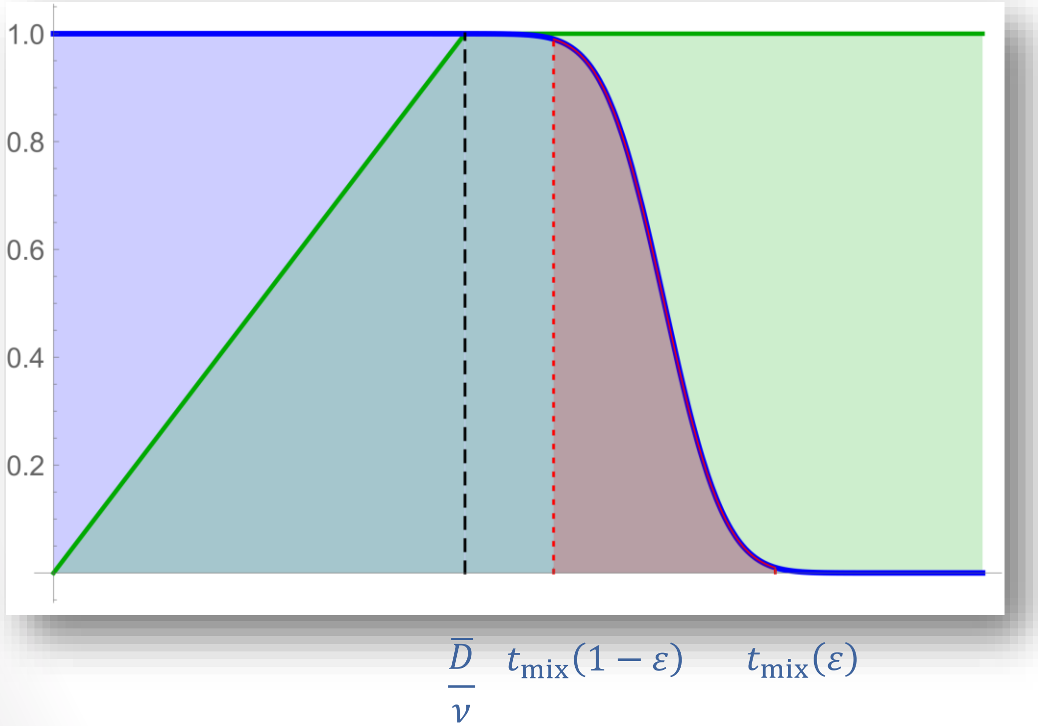
- Height of NBRW vs. SRW:
  - NBRW cannot backtrack up the tree
    $\Rightarrow$ hits bottom after precisely $\log_{d-1} n$ steps.
  - SRW $\equiv$ biased 1D RW with speed $\nu = {}^{d-2}/_{d}$
    $\Rightarrow$ hits bottom after $\frac{d}{d-2}\log_{d-1} n + O_{\mathrm{P}}(\sqrt{\log n})$ steps.
- In both cases: cutoff once the entropy of $P^t(v_0,\cdot)$ reaches $\log n$, which occurs at $t = \frac{1}{\nu}\underbrace{\frac{1}{\log(d-1)}\log n}_{\overline{D}}$.

$\overline{D}$ (average distance)

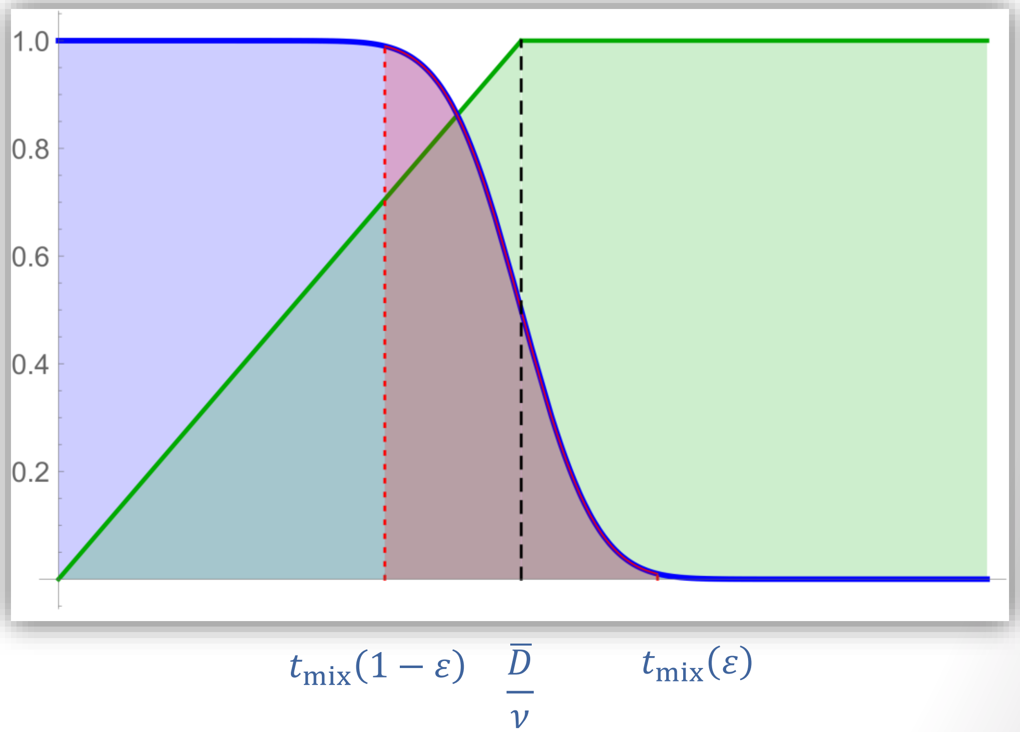


Mixing vs. the distance from the origin

- Mixing on irregular graphs is delayed beyond the stabilization of the distance, since the rate at which entropy drops further involves the dimension $\mathbf{d}$:

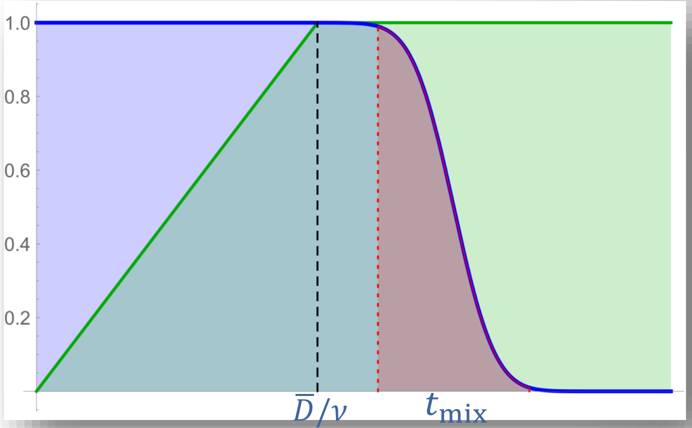


Irregular graph



Regular graph

New results: RW on the giant



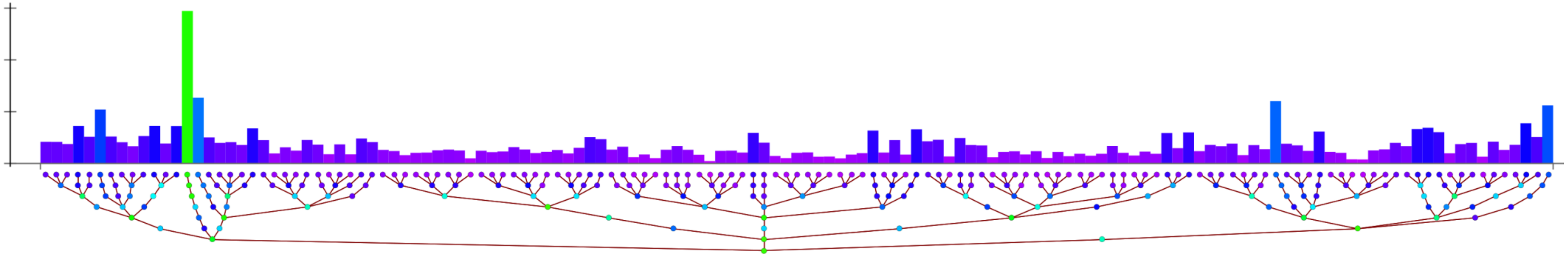
- Setup:
  - $\mathcal{C}_1 =$ largest component of $\mathcal{G}(n, p = \lambda/n)$ [$\lambda > 1$ fixed].
  - $\nu =$ speed of RW on a Po(λ)-GW tree.
  - $\mathbf{d} =$ dimension of harmonic measure Po(λ)-GW tree $\overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} \log \frac{1}{\theta(\xi_t)}$ where $(\xi_t) =$ LERW and $\theta(x) =$ probability it visits x.

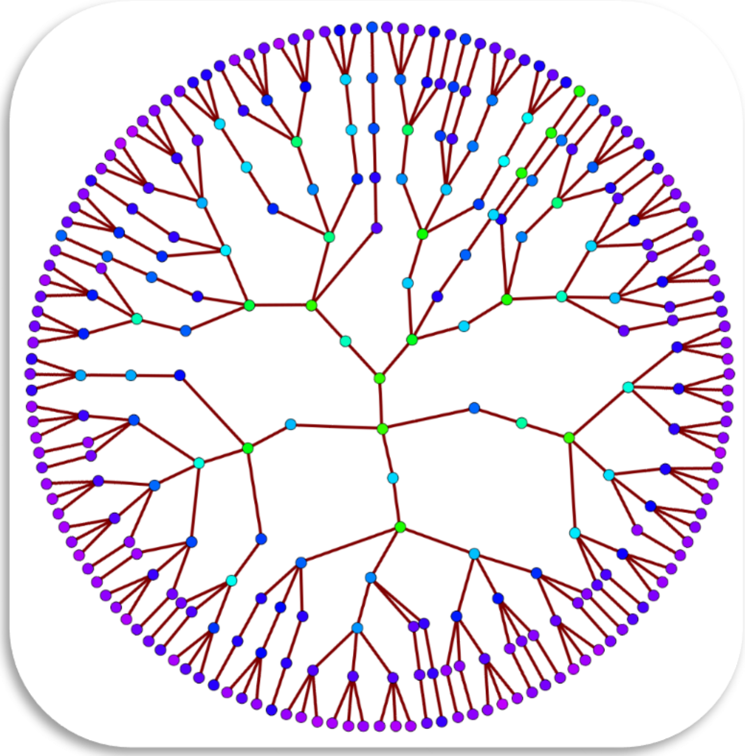
- **<u>Theorem</u>** [Berestycki, L., Peres, Sly]:

> RW from a uniform vertex $v_1 \in \mathcal{C}_1$ w.h.p. satisfies
> $$t_{\mathrm{mix}}^{(v_1)}(\varepsilon) = \nu^{-1}\mathbf{d}^{-1} \log n \pm (\log n)^{1/2+o(1)}$$

- Cutoff from a typical starting point!

Dimension of harmonic measure





$$\mathbf{d} \overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} \log \frac{1}{\theta(\xi_t)}$$

where (ξ_t) = LERW
and $\theta(x)$ = probability it visits x.

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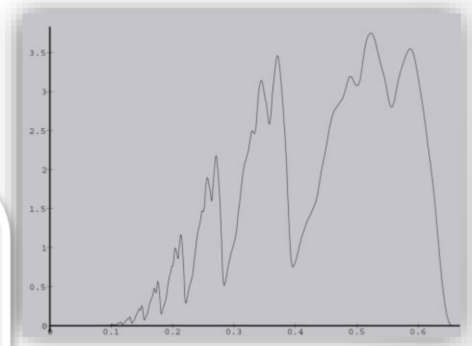
Dimension of harmonic measure

- For a.e. GW-tree: $\mathbf{d} \overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} \log \frac{1}{\theta(\xi_t)}$
  where $(\xi_t) =$ LERW and $\theta(x) =$ probability it visits x.

- Can be written as an integral w.r.t. to the measure on effective conductance in the GW-tree.
- Pioneering work [Lyons, Pemantle, Peres '94] showed that $d < \log \mathbb{E}Z$ for a.e. GW-tree !

Density of the C_{eff} distribution for $Z \sim \begin{cases} 1 & 1/3 \\ 2 & 1/3 \\ 3 & 1/3 \end{cases}$



[$\nu\mathbf{d} = \int_{s=0}^{\infty}\int_{t=0}^{\infty} \frac{\log(1+s)}{1+s^{-1}+t^{-1}} d\mu(t)\mu(s)$ with μ = dist. of $C_{\text{eff}}(\rho,\infty)$.]

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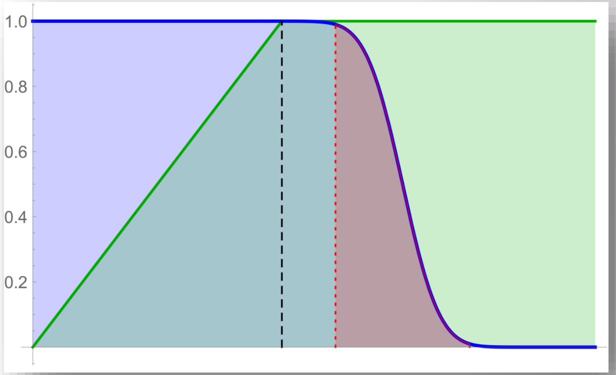
RW on random graphs with given degrees



- Random graph with given degrees ≥ 3 (*e.g.*, half 3 half 4): similarly, dimension reduction due to irregularity of degrees…

- **THEOREM** [Berestycki, L., Peres, Sly]:

Let G be a uniformly chosen graph with degree frequencies (p_{k}) s.t. Z with $\mathbb{P}(Z=k) \propto k\, p_k$ satisfies $\mathbb{E}Z = O(1)$, $2 \leq Z \leq e^{(\log n)^{1/2-\delta}}$.
Then **RW** from a uniform vertex of $v_1 \in G$ w.h.p. satisfies

$$t_{\mathrm{mix}}^{(v_1)}(\varepsilon) = \nu^{-1}\mathbf{d}^{-1}\log n \pm O\left(\sqrt{\log n}\right)$$

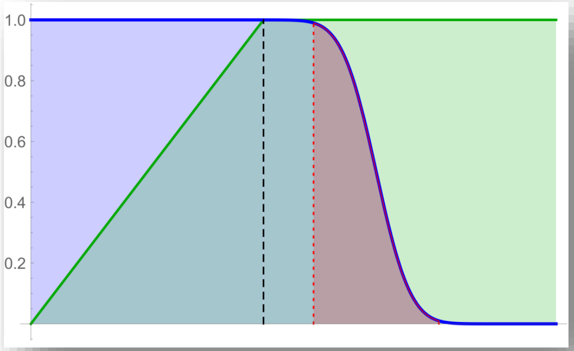
and the same statement holds for **NBRW** (from typical/worst v_1).

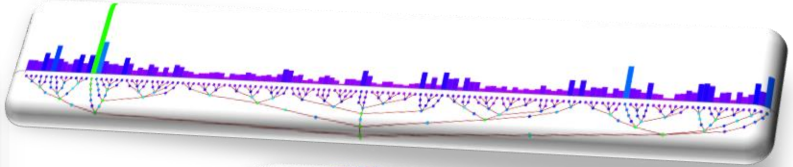
Proof ingredients for $\mathcal{G}(n,p)$

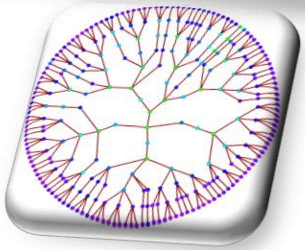
- The correct cutoff window requires sharp fluctuation estimates on $\log\theta(\xi_t)$ for $\theta =$ harmonic measure.
  - Build on arguments of [Lyons, Pemantle, Peres ’95, ‘96] and [Dembo, Gantert, Peres, Zeitouni ‘02].
- Exploit fact (using the structure theorem for $\mathcal{C}_1$) that bottlenecks are rare/spread-out to help expansion.
- Additional difficulties: delays from hanging trees, coupling the walk on the tree to that on the graph, …
- Proof extends to random graphs with given degrees.
  - **NBRW** directly analyzed by an adaptation of the random regular graph proof (sharp cutoff window).

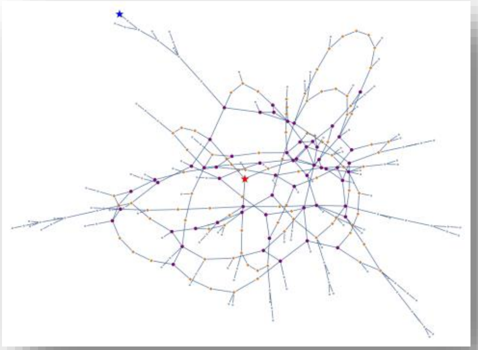
Open problems

- *What is the dimension* $\mathbf{d}$ *of harmonic measure on a* $\mathrm{Po}(\lambda)$*-GW-tree?*
- *Does* RW *exhibit cutoff on* ***every*** *family of* ***transitive*** 3*-regular expanders?* [conjectured to be true by Y. Peres]
- *Does* RW *exhibit cutoff on* ***any*** *family of* ***transitive*** 3*-regular expanders?* (explicit / probabilistic)









Thank you