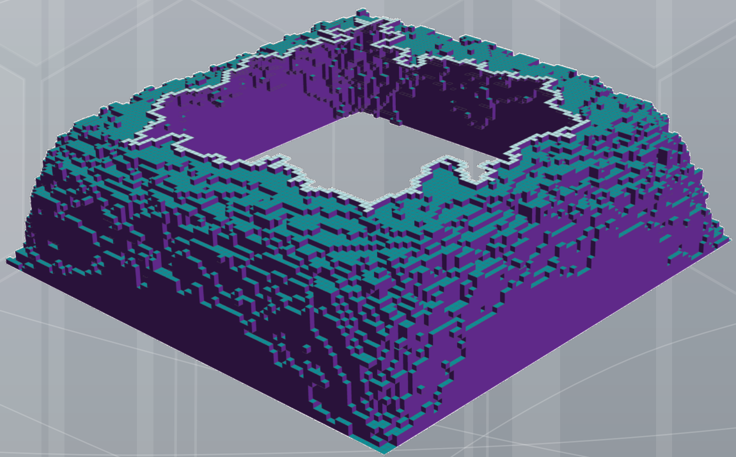
AMS Fall Eastern Sectional

Oct 2020

Rigidity and LD of 3D Ising interfaces: an approximate Domain Markov property



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based on joint works with
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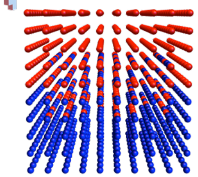
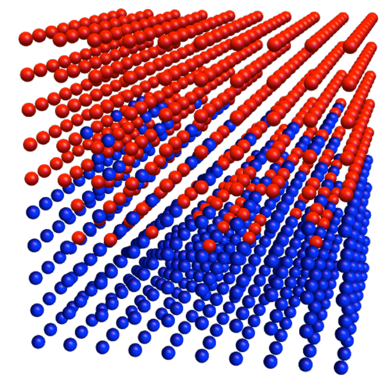
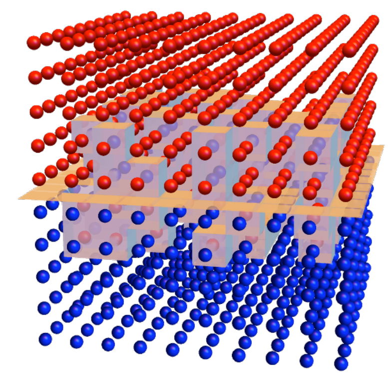
3D Ising interfaces

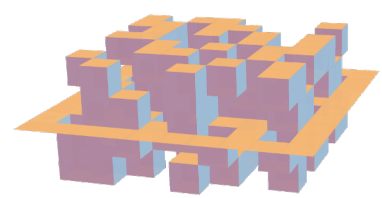


- Consider surfaces generated as follows:
  - 3D cylinder $\Lambda = [\![-n,n]\!]^2 \times (\mathbb{Z} + \frac{1}{2})$
  - σ is a 2-coloring of the vertices:
    - boundary vertices: $\begin{cases} - & \textit{upper half-space} \\ + & \textit{lower half-space} \end{cases}$
    - internal vertices: arbitrarily (*for now*).
  - Draw a **dual-face** $(u,v)^*$ if $\sigma_u \neq \sigma_v$.



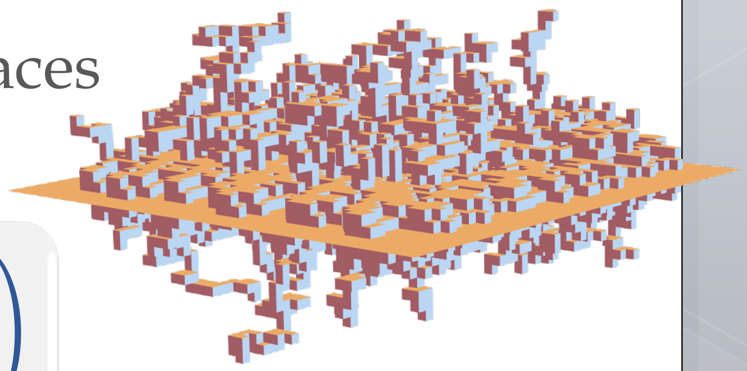
- **Interface**: (max) *-connected component $\mathcal{I}$ of dual-faces separating the boundary.





3D Ising interfaces (ctd.)

- Goal: understand random interfaces sampled via the distribution:



$$\mu(\mathcal{I}) \propto \exp\left(-\beta|\mathcal{I}| + \sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})\right)$$

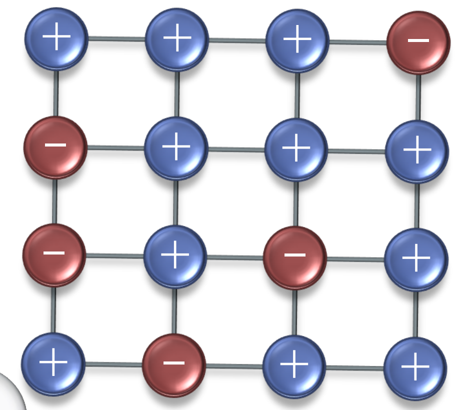
  - $\beta > 0$: inverse temperature (large, fixed).
  - $\mathbf{g}(\cdot,\cdot)$: some complicated function, yet satisfying

1) $\mathbf{g} \leq K_{\mathbf{0}}$
2) $|\mathbf{g}(f,\mathcal{I}) - \mathbf{g}(f',\mathcal{I}')| \leq e^{-c_0 \mathbf{r}}$ if $B_{\mathbf{r}}(f,\mathcal{I}) \cong B_{\mathbf{r}}(f',\mathcal{I}')$

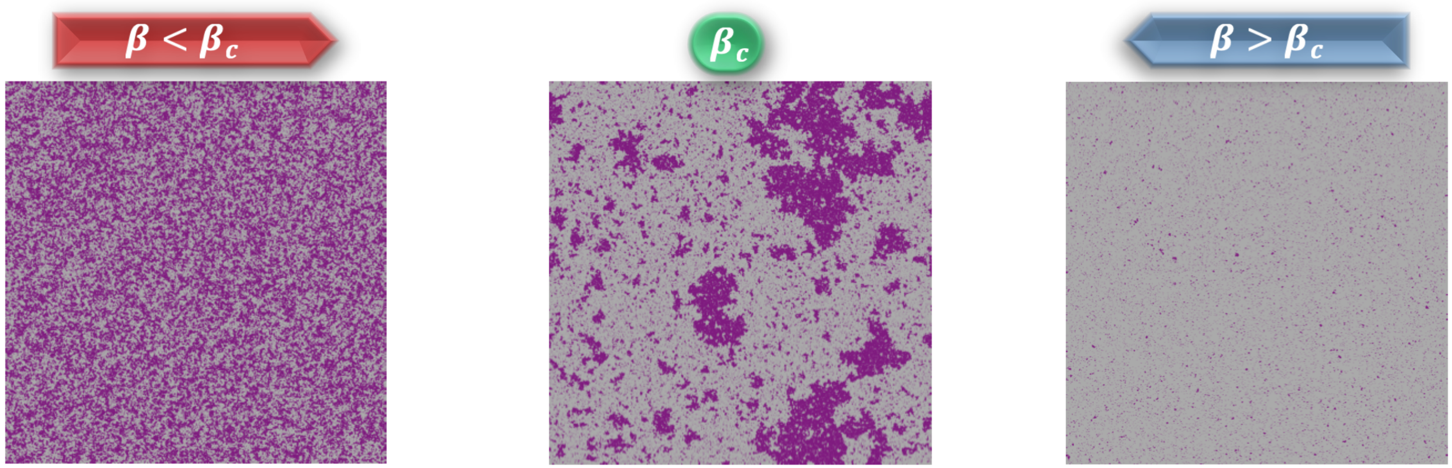
for **absolute** constants c_0, K_0.

Definition: the classical Ising model

- Underlying geometry: finite $\Lambda \subset \mathbb{Z}^d$.
- Set of possible configurations: $\Omega = \{\pm 1\}^\Lambda$
- Probability of a configuration $\sigma \in \Omega$ given by the *Gibbs distribution*:



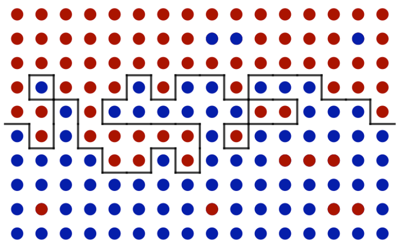
$$\mu_\Lambda(\sigma) \propto \exp\left(-\beta \sum_{x\sim y} \mathbf{1}_{\{\sigma_x \neq \sigma_y\}}\right)$$



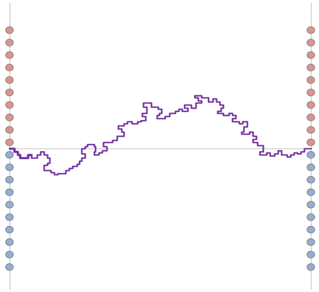
$\beta = 0.75$ $\beta = 0.88$ $\beta = 1$

2D Ising interfaces

- $\mu_\Lambda^\mp$: Ising model on 2D cylinder $\Lambda = [-n, n] \times (\mathbb{Z} + \frac{1}{2})$
  - Boundary conditions: $\begin{cases} - & \textit{upper half}\text{–plane} \\ + & \textit{lower half}\text{–plane} \end{cases}$
  - Draw a dual-**edge** $(u, v)^*$ if $\sigma_u \neq \sigma_v$.



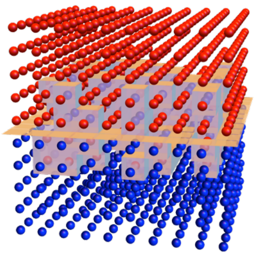
- **Interface**: connected component $\mathcal{I}$ of dual-edges that separates the the boundary components.



- Known [Higuchi '79], [Greenberg, Ioffe '95], [Dobrushin, Hryniv '97], [Hryniv '98], [Dobrushin, Kotecký, Shlosman '92] :
  - Interface has a scaling limit: $\frac{\mathcal{I}(x/n)}{\sqrt{c_\beta n}} \to$ Brownian bridge
  - Maximum M_n is $O_P(\sqrt{n})$, and $M_n - \mathbb{E}[M_n]$ is also $O_P(\sqrt{n})$.

3D Ising interfaces

- $\mu_\Lambda^{\mp}$: Ising model on 3D cylinder $\Lambda = [-n,n]^2 \times (\mathbb{Z} + \frac{1}{2})$
  - Boundary conditions: $\begin{cases} \ominus \ \textit{upper half}-\text{plane} \\ \oplus \ \textit{lower half}-\text{plane} \end{cases}$
  - Draw a dual-face $(u,v)^*$ if $\sigma_u \neq \sigma_v$.



- **Interface**: maximal *-connected component $\mathcal{I}$ of dual-faces that separates the boundary components.
- [Minlos, Sinai '67],[Dobrushin '72]: *(cluster expansion)*

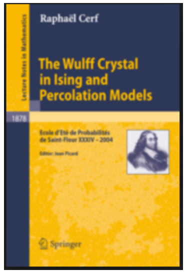
$$\mu_\Lambda^{\mp}(\mathcal{I}) \propto e^{-\beta|\mathcal{I}| + \sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})} \quad \text{for large } \beta$$

(for a uniformly bounded "local" $\mathbf{g}$ as described above)

  - Via this: Dobrushin showed the interface, unlike 2D, is **rigid**.

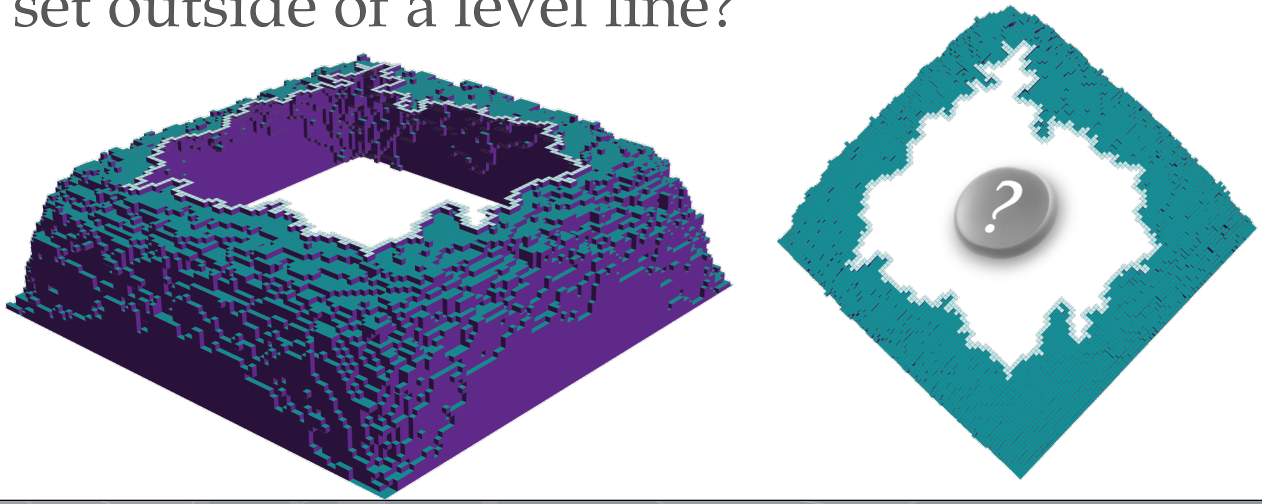
Related work on 3D Ising interfaces

- Alternative simpler argument by [van Beijeren '75] for [Dobrushin '72]'s result on the rigidity of the 3D Ising interface.
- Rigidity argument extended to
  - Widom–Rowlinson model [Bricmont, Lebowitz, Pfister, Olivieri '79a], [Bricmont, Lebowitz, Pfister '79b, '79c]
  - Super-critical percolation / random cluster model conditioned to have interfaces [Gielis, Grimmett '02]
- Tilted interfaces: [Cerf, Kenyon '01] (zero temp, 111 interface), [Miracle Sole '95] (1-step interface), [Sheffield '03] ($|\nabla\varphi|^p$ models), **many** works on the conjectured behavior, related to the (non-)existence of non-translational invariant Gibbs measures
- Wulff shape, large deviations for the magnetization, surface tension [Pisztora '96], [Bodineau '96], [Cerf, Pisztora '00], [Bodineau '05], [Cerf '06]

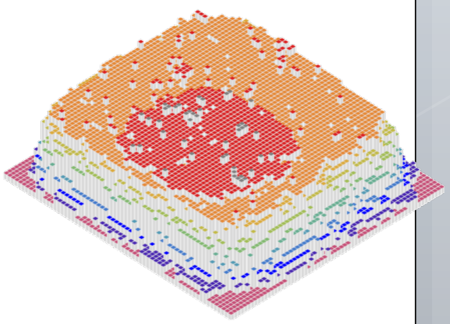


Plus/minus interface in 3D Ising

- Describing the 3D Ising interface $\mathcal{I}$:
  - Height fluctuations at the origin (or in the bulk)?
  - Correlation between height oscillations?
  - Maximum height: Asymptotics (LLN)? Tightness?
- Our focus: (Approx) Domain Markov Property for $\mathcal{I}$: What does the interface look like if we condition on its face set outside of a level line?



Motivation: entropic repulsion



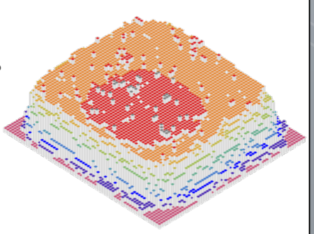
- Despite the penalizing energy, if the interface is constrained to be **nonnegative**, it should propel itself to height $h_n \gg 1$ to gain entropy.
- [Caputo, L., Martinelli, Sly, Toninelli '14, '16]: detailed picture for the (2+1)D **Solid-On-Solid** model which approximates low temperature 3D Ising; e.g., THEOREM [1:2:3 *for the maximum with entropic repulsion*]:

> Let M_n = max of the unconstrained surface.
> Let $\widehat{H}_n$ = height of origin and $\widehat{M}_n$ = maximum when restricting the surface to be ≥ 0. Then $\exists\ h_n \asymp \log n$ s.t.
> $$\widehat{H}_n / h_n \xrightarrow{\text{p}} 1, \qquad M_n / h_n \xrightarrow{\text{p}} 2, \qquad \widehat{M}_n / h_n \xrightarrow{\text{p}} 3$$

- *In 3D Ising: a DMP for level lines would give "half the proof"*

Domain Markov Property

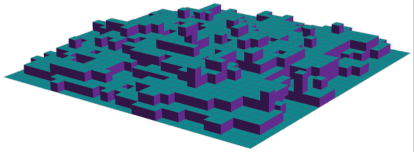
- DMP: $\forall$ *subset of sites* S*, the conditional law of the model given the values on* ∂S *gives the same model on* S *with these boundary conditions (BC), independently of* S^c*.*
- Holds for Ising spin configurations (MRF).
- Fundamental feature in many (2+1)D height function models (viewed as random surfaces) such as DGFF.
- Example: $|\nabla\varphi|^p$ model (Solid-On-Solid/Discrete Gaussian/…):
  - $\varphi : [\![-n, n]\!]^2 \to \mathbb{Z}$ height function.
  - $\pi_\Lambda^0(\varphi) \propto \exp\left(-\beta \sum_{x\sim y} |\varphi(x) - \varphi(y)|^p\right)$ with BC 0.
  - Conditioning on $\varphi(\partial S) \equiv h$ gives π_S^h (BC h).



(No) DMP for Ising interfaces

- In SOS (etc.): conditioning on an h-level line shifts by h
- Fails to hold for Ising model **interfaces:** the finite components ("bubbles") of the Ising model, hidden from $\mathcal{I}$, carry the interaction through ∂S.
- 2D Ising interfaces : Ornstein–Zernike theory for approximate DMP at high temperature (low: duality).
- 3D Ising interfaces: no analog at low temperature (interfaces are surfaces rather than curves).
- *We will show an approximate DMP for the height oscillations of $\mathcal{I}$ when ∂S is a level line.*

Plus/minus interface in 3D Ising



- *Rigidity of the interface* ([Dobrushin '72]):

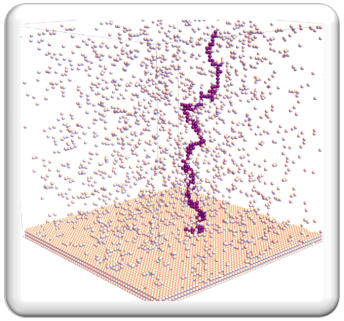
  There exists $\beta_0 > 0$ such that $\forall \beta > \beta_0$ and $\forall x_1, x_2, k$,

  $$\mu_\Lambda^{\mp}(\mathcal{I} \cap (x_1, x_2)\times[k,\infty) \neq \emptyset) \leq \exp\left(-\tfrac{1}{3}\beta k\right)$$

- Consequently: M_n = maximum height of $\mathcal{I}$ satisfies
  - $\mu_\Lambda^{\mp}\left(M_n \leq C_\beta \log n\right) \to 1$ for any $C_\beta > 6/\beta$.
- Recently: [Gheissari, L. '19a, '19b]
  - Identified the correct exponential rate above:

    $$\exists \lim_{k\to\infty} -\frac{1}{k}\log \mu_\Lambda^{\mp}(\mathcal{I} \cap (x_1, x_2)\times[k,\infty) \neq \emptyset) =: \alpha$$
  - Led to a LLN for the maximum M_n.
  - Subsequently (via more subtle analysis): tightness.

LLN and tightness for the maximum



- M_n = maximum of the interface $\mathcal{I}$ in 3D Ising; [Dobrushin '72]: $M_n = O_{\mathrm{P}}(\log n)$.
- <u>THEOREM</u>: ([Gheissari, L. '19a, '19b])

There exists β_0 such that for all $\beta > \beta_0$,

1. $M_n / \log n \to 2/\alpha$ in probability for
$$\alpha(\beta) = \lim_{h\to\infty} -\frac{1}{h} \log \mu^{\mp}_{\mathbb{Z}^3}\left((0,0,0) \overset{+}{\underset{\mathbb{R}^2\times[0,h]}{\longleftrightarrow}} (\mathbb{R}^2 \times \{h\})\right)$$
2. $M_n - \mathbb{E}M_n = O_{\mathrm{P}}(1)$, and $\mathbb{E}M_n = m_n^* + O(1)$ for an explicit deterministic sequence (m_n^*).
3. $\exists C, c$ such that $\forall\, k \in \mathbb{Z}$,
$$e^{-Ce^{-(4\beta+C)k}} \le \mu_n^{\mp}(M_n < m_n^* - k) \le e^{-ce^{-(4\beta-C)k}}$$

LLN

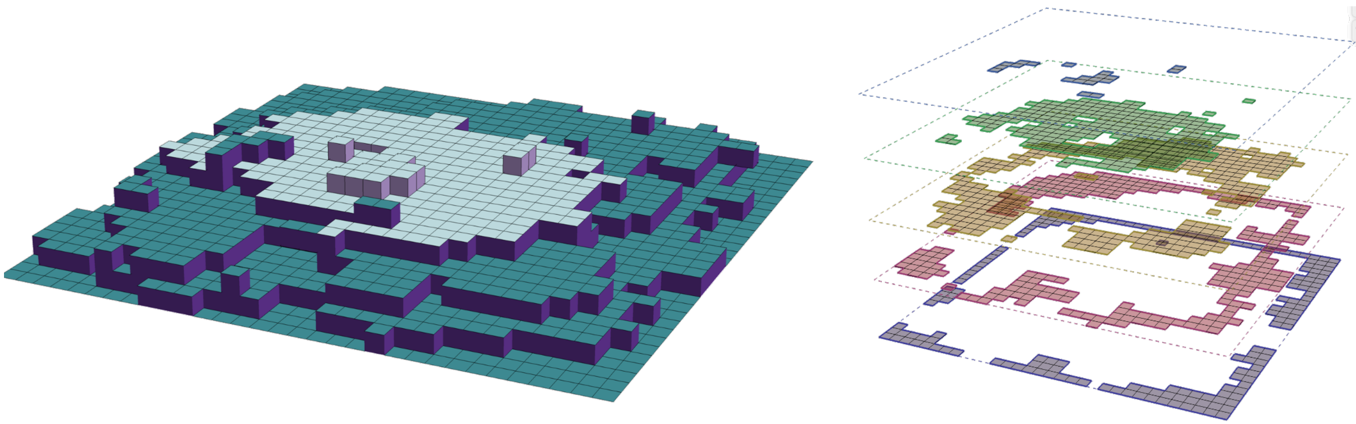
Tightness

Gumbel tails

* *existence of the limit α is nontrivial*

Level lines in 3D Ising interface

- By [Dobrushin '72] : w.h.p., 0.99 of faces $x \in [-n, n]^2$ have exactly 1 horizontal face of $\mathcal{I}$ in $x \times \mathbb{Z}$; gives rise to **level sets**.
- What if we condition on $\gamma = \partial S$ being a **level line**?



- Existing estimates fail (e.g., TV decorrelation [Dobrushin '72] [Bricmont, Lebowitz, Pfister '79], or viewing it as a tilt of $\mu^{\mp}_{S \times \mathbb{Z}}$ via cluster expansion/Pirogov–Sinai ([Holicky, Zahradnik '93]):
  - The R–N derivative is $e^{c|\gamma|}$ (destroys the bound).

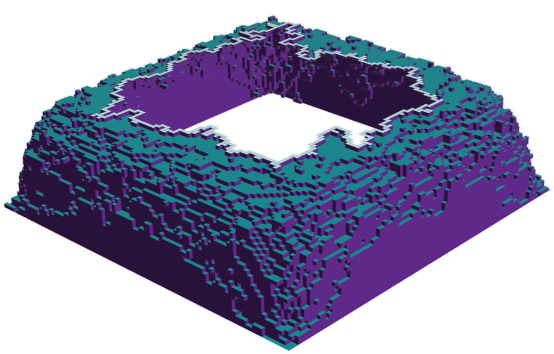
Approximate DMP for maximum

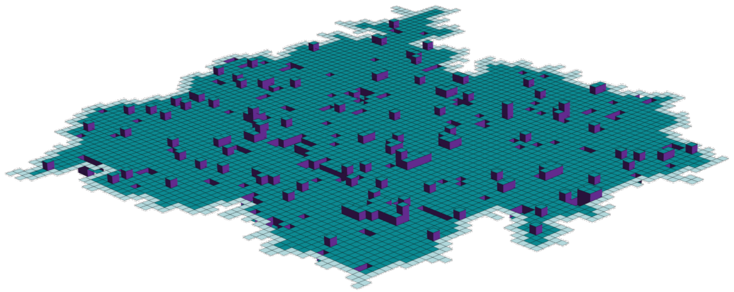
- THEOREM: ([Gheissari, L. '20+])

There exists β_0 such that for all $\beta > \beta_0$, there exist C, c such that $\forall\, k \in \mathbb{Z}$, if γ_n has interior S with area $s > |\gamma_n|^{1.1}$,

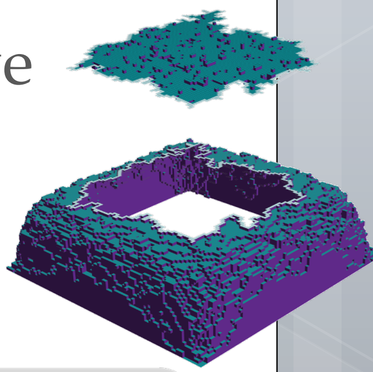
$$e^{-Ce^{-(4\beta+C)k}} \le \mu_n^{\mp}\left(M_S - h < m_S^* - k \mid \mathcal{F}_\gamma\right) \le e^{-ce^{-(4\beta-C)k}}$$

where $\mathcal{F}_\gamma = \{\mathcal{I} \cap (S^c \times \mathbb{Z}); \gamma_n$ is a height$-h$ level line$\}$.





Prerequisite: rigidity in a level line



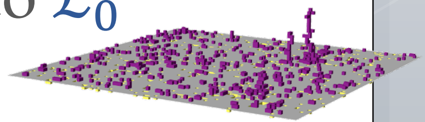
- To address the maximum in a level line, first we would need an analog of Dobrushin's rigidity and exponential tails within the level line.
- <u>THEOREM</u>: ([Gheissari, L. '20+])

> There exists β_0 such that for all $\beta > \beta_0$, if γ_n is a closed simple curve with interior S, then for $\forall(x_1, x_2) \in S$ and h, k,
>
> $$\mu_\Lambda^\mp\left(\mathcal{I} \cap (x_1, x_2)\times[h+k,\infty) \neq \emptyset \mid \mathcal{F}_\gamma\right) \leq \exp(-(4\beta - C)k)$$
>
> where $\mathcal{F}_\gamma = \{\mathcal{I} \cap (S^c\times\mathbb{Z}); \gamma_n \text{ is a height}-h \text{ level line}\}$.

  - *Crucially: no restriction on* (x_1, x_2) *to be far from* ∂S.

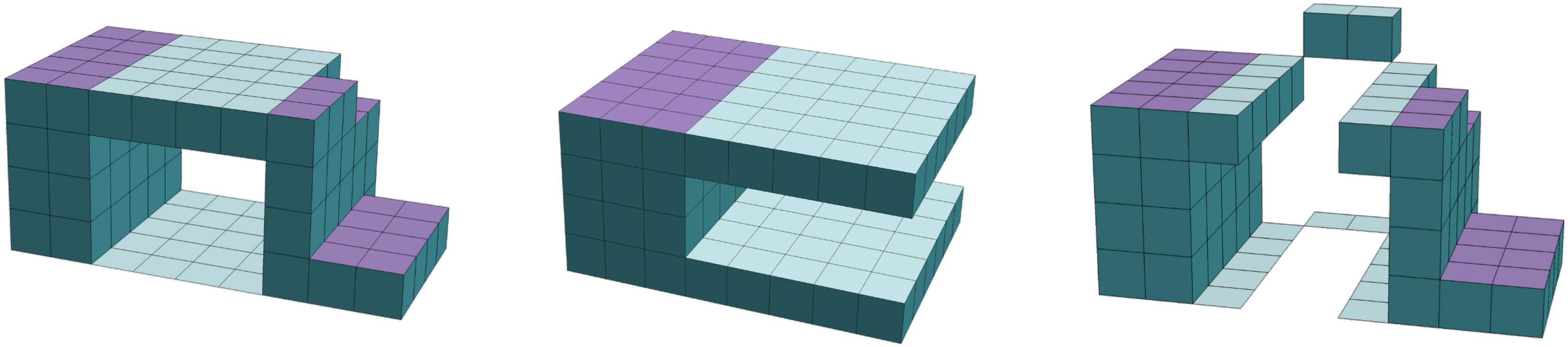
Reviewing Dobrushin's argument

- Notation: $\mathcal{L}_0 = \mathbb{R}^2 \times \{0\}$; π = projection onto $\mathcal{L}_0$
- <u>DEFINITION</u>: [**ceiling** and **walls**]



1. *Ceiling face* : a horizontal face $f \in \mathcal{I}$ such that
$$\pi(f') \neq \pi(f) \quad \forall f' \neq f .$$
Ceiling $\mathcal{C}$: connected component of ceiling faces.

2. *Wall face* : all other faces.
Wall $\mathcal{W}$: connected component of wall faces.



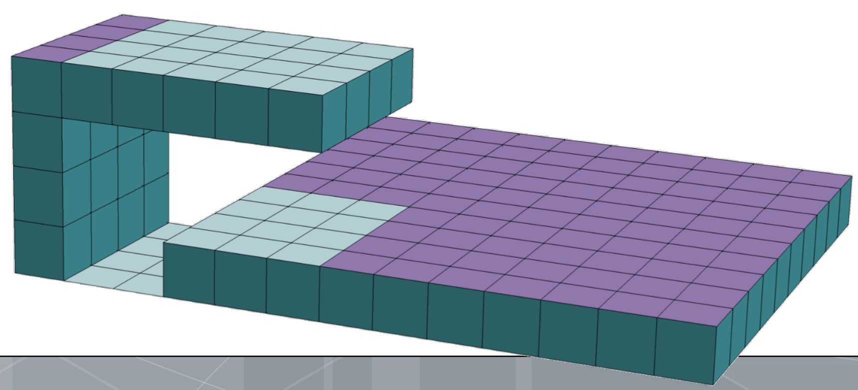
Reviewing Dobrushin's argument

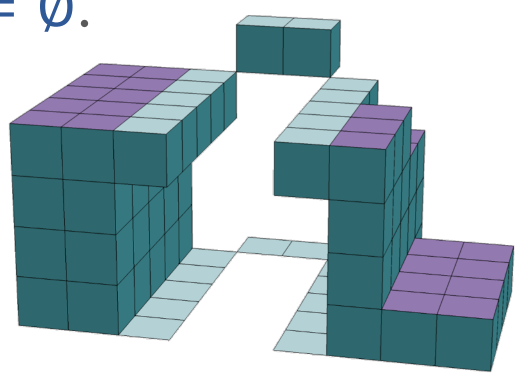
- DEFINITION: [**ceiling** and **walls**]

1. *Ceiling face* : a horizontal face $f \in \mathcal{I}$ with $\pi(f') \neq \pi(f) \;\; \forall f' \neq f$.
   Ceiling $\mathcal{C}$: connected component of ceiling faces.
2. *Wall face* : all other faces.
   Wall $\mathcal{W}$: connected component of wall faces.

- FACTS:

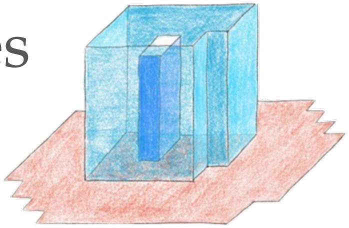
1. $\forall$ ceiling $\mathcal{C}$ has a single height.
2. $\forall$ wall $\mathcal{W}$: $\pi(\mathcal{W})$ is connected.
3. $\forall$ walls $\mathcal{W} \neq \mathcal{W}'$: $\pi(\mathcal{W}) \cap \pi(\mathcal{W}') = \emptyset$.

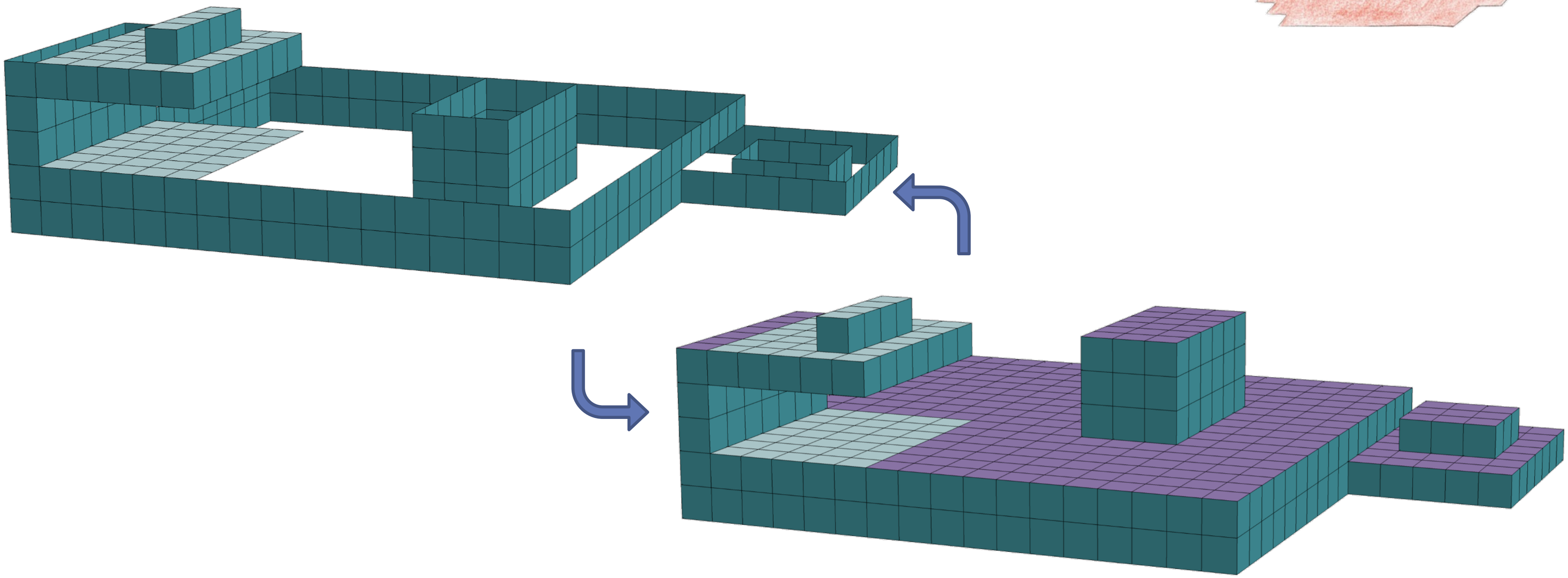




Reviewing Dobrushin's argument

- A wall $\mathcal{W}$ is **standard** if $\exists\ \mathcal{I}$ whose only wall is $\mathcal{W}$.
- <u>FACT</u>: 1:1 correspondence between interfaces and *admissible** collections of standard walls.

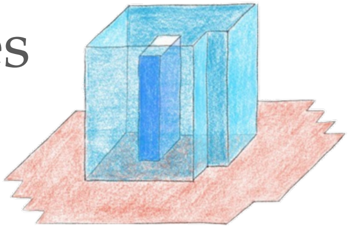




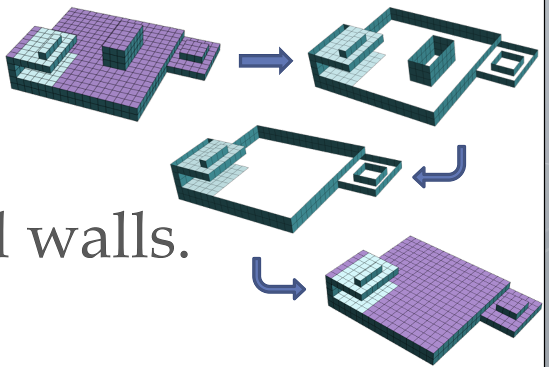
** admissible: walls are disjoint components and so are their projections*

Dobrushin's rigidity argument

- A wall $\mathcal{W}$ is **standard** if $\exists\, \mathcal{I}$ whose only wall is $\mathcal{W}$.
- <u>FACT</u>**:** 1: 1 correspondence between interfaces and *admissible* collections of standard walls.



- Basic idea: given $x \in \mathcal{L}_0$, construct a map Φ:
  - "*standardize*" every wall $\mathcal{W}$ in $\mathcal{I}$;
  - delete the wall $\mathcal{W}_x$ of x;
  - "*reconstruct*" $\mathcal{I}'$ from other standard walls.



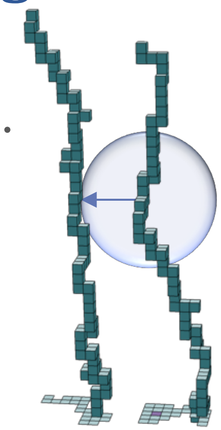
- Goal: establish for this map Φ:

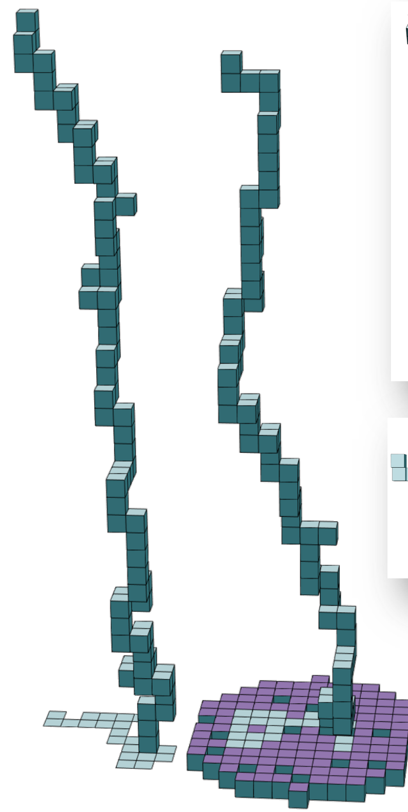
1. (*Energy bound*) $\dfrac{\mu(\mathcal{I})}{\mu(\Phi(\mathcal{I}))} \leq e^{-c\beta|\mathcal{W}_x|}$
2. (*Multiplicity bound*) $\#\{\mathcal{I} \in \Phi^{-1}(\mathcal{I}') : |\mathcal{W}_x| = \ell\} \leq e^{c\ell}$

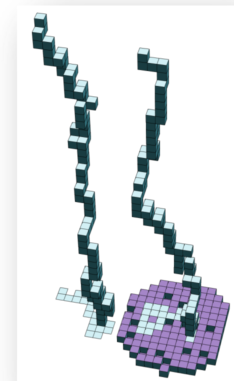
Dobrushin's rigidity argument

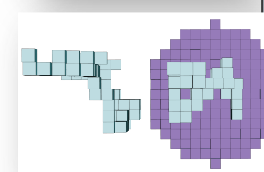
recall $\mu_\Lambda^\mp(\mathcal{I}) \propto e^{-\beta|\mathcal{I}|+\sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})}$

- Basic idea: delete the wall $\mathcal{W}_x$ of x.
- *Energy bound* ($\frac{\mu(\mathcal{I})}{\mu(\Phi(\mathcal{I}))} \leq e^{-c\beta|\mathcal{W}_x|}$) :
  - Gain $\beta|\mathcal{W}_x|$ from $\beta(|\mathcal{I}| - |\Phi(\mathcal{I})|)$
  - Problem: effect on **non-deleted** faces that moved due to $\mathbf{g}$…
    - The effect of $\mathbf{g}$ is **local** (decays exp. in distance).
    - BUT: **tall** nearby walls can pick up a cost that cancels our $\beta|\mathcal{W}_x|$ gain.







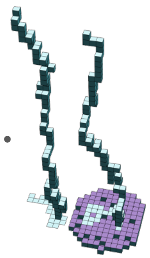


- Solution: also delete **tall** walls that are **close** to $\mathcal{W}_x$.

Dobrushin's rigidity argument

recall $\mu_\Lambda^\mp(\mathcal{J}) \propto e^{-\beta|\mathcal{J}| + \sum_{f\in\mathcal{J}} \mathbf{g}(f,\mathcal{J})}$

- *Energy bound* ($\frac{\mu(\mathcal{J})}{\mu(\Phi(\mathcal{J}))} \leq e^{-c\beta|\mathcal{W}_x|}$) :
  - Gain $\beta|\mathcal{W}_x|$ from $\beta(|\mathcal{J}| - |\Phi(\mathcal{J})|)$, but must handle **g**…
  - … must also **delete tall** walls that are **close**.



- *Multiplicity bound* ($\#\{\mathcal{J} \in \Phi^{-1}(\mathcal{J}') : |\mathcal{W}_x| = \ell\} \leq e^{c\ell}$) :
  - Problem: accounting for the **extra** walls we deleted…

- **Dobrushin's criterion**: **groups of walls**: for $x, y \in \mathcal{L}_0$,
  $\mathcal{W}_x \sim \mathcal{W}_y \quad \Longleftrightarrow \quad d(x,y)^2 \leq \max\{|\pi^{-1}(x)|, |\pi^{-1}(y)|\}$.
  (a "tall" $\mathcal{W}_x$ (many faces above x) is easier to group with)
- The map Φ deletes the entire **group of walls** of $\mathcal{W}_x$: analysis becomes 2D (but too crude for detailed questions).

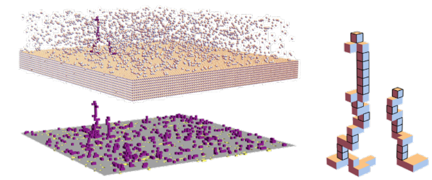
Proof ideas: rigidity in a ceiling

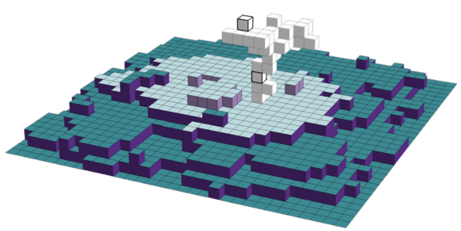
- Dobrushin's argument is robust: extended to various models (Widom–Rowlinson/random cluster/…)

  BUT: his **group-of-walls criterion** is more delicate, used verbatim in all these extensions.
- This criterion will not respect a level line boundary; moreover, it might *delete the level line wall* as part of a group-of-walls (moving us out of our space of permitted interfaces), breaking the Peierls argument.
- New idea: a one-sided criterion: **wall clusters**
  (*say $\mathcal{W}_1$ is* ***closely nested*** *in a $\mathcal{W}_2$ if there is a ceiling $\mathcal{C}$ of $\mathcal{W}_2$ nesting $\mathcal{W}_1$ and $d(\partial\dot{\mathcal{C}}, \mathcal{W}_1) \leq |\mathcal{W}_1|$; the* ***wall cluster*** *is obtained by repeatedly adding closely nested walls.*)

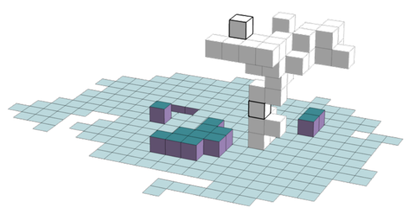
Proof ideas: rate function in a ceiling

- If **Step I** was showing rigidity within a level line, then **Step II** would be to obtain the correct rate function α.
- KEY: couple the conditional law of $\mu_\Lambda^\mp$ on **pillars** (local height oscillation of the interface about a point) to the unconditional distribution in $\mu_{\mathbb{Z}^3}^\mp$.





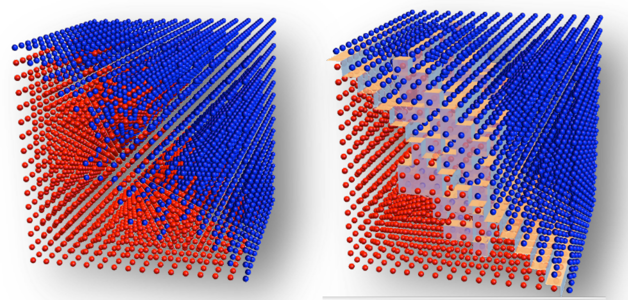
- To do so: we construct a family of *isolated pillars* which may be **swapped** in a pair of interfaces (a $2 \to 2$ map) for coupling $\mu_{\mathbb{Z}^3}^\mp$ to the conditional $\mu_\Lambda^\mp$.



- Showing such pillars are typical requires much of the machinery of the shape theorem used for tightness.

Some open problems

- Understand the LD rate $\alpha(\beta) = \lim_{h\to\infty} -\frac{1}{h}\log \mu^{\mp}_{\mathbb{Z}^3}\left((0,0,0) \overset{+}{\underset{\mathbb{R}^2\times[0,h]}{\longleftrightarrow}} (\mathbb{R}^2\times\{h\})\right)$:
  - Is it equal to $\lim_{h\to\infty} -\frac{1}{h}\log \mu^{-}_{\mathbb{Z}^3}\left((0,0,0) \overset{+}{\longleftrightarrow} (\mathbb{R}^2\times\{h\})\right)$ (pure phase) ?
  - Is $\alpha < 4\beta$? (“*Ising is rougher than SOS*”) [Known: $\alpha \in 4\beta \pm C$].
- Extensions to other (including non-monotone) models, e.g. 3D Potts? (new results did not use on FKG).
- Interfaces under **tilted BC**?



Thank you!