Cutoff for the Ising model on the Lattice

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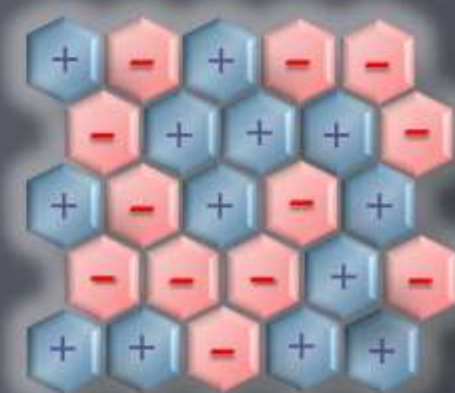
Ising model

- Underlying geometry: finite graph $G=(V,E)$.
- Set of possible configurations:

$$\Omega=\{\pm 1\}^V$$



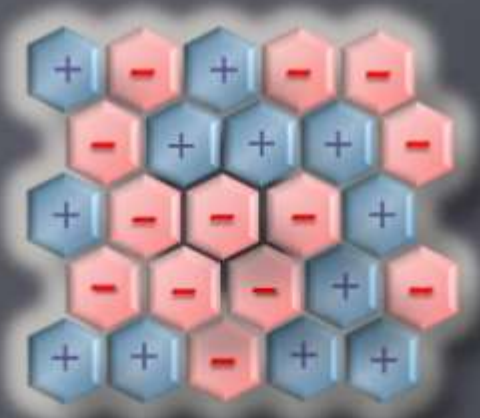
- Probability of a configuration $\sigma\in\Omega$ given by the *Gibbs distribution*

$$\mu(\sigma)=\frac{1}{Z(\beta)}\exp\left(\beta\sum_{xy\in E}\sigma(x)\sigma(y)\right)$$ [no external field]

- *Ferromagnetic* $\leftrightsquigarrow$ inverse-temperature $\beta\geq 0$.
- Phase transition as β varies (in some geometries).

Glauber dynamics for Ising

- One of the most commonly used MC samplers for the Gibbs distribution:
  - Update sites via *iid* Poisson(1) clocks
  - Each update replaces a spin at $u \in V$ by a new one $\sim \mu$ conditioned on $V \setminus \{u\}$ (heat-bath version).



- Ergodic reversible MC with stationary measure μ.
- Introduced by Glauber in 1963. Other versions of the dynamics include e.g. Metropolis.
- *How fast does it converge to equilibrium?*

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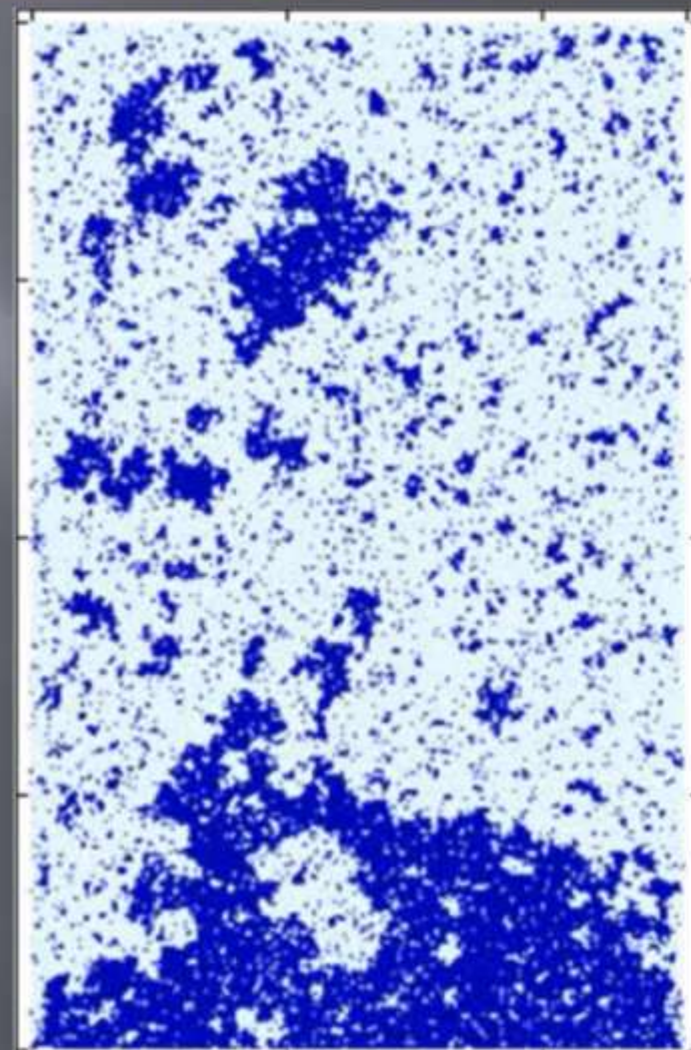
Example: Glauber dynamics for critical Ising on the square lattice

- 256 x 400 square lattice w. boundary conditions:
  - $(+)$ at bottom
  - $(-)$ elsewhere.
- Frame after 2^{20} steps, i.e. $\sim$10 updates per site.



Rate of convergence to equilibrium

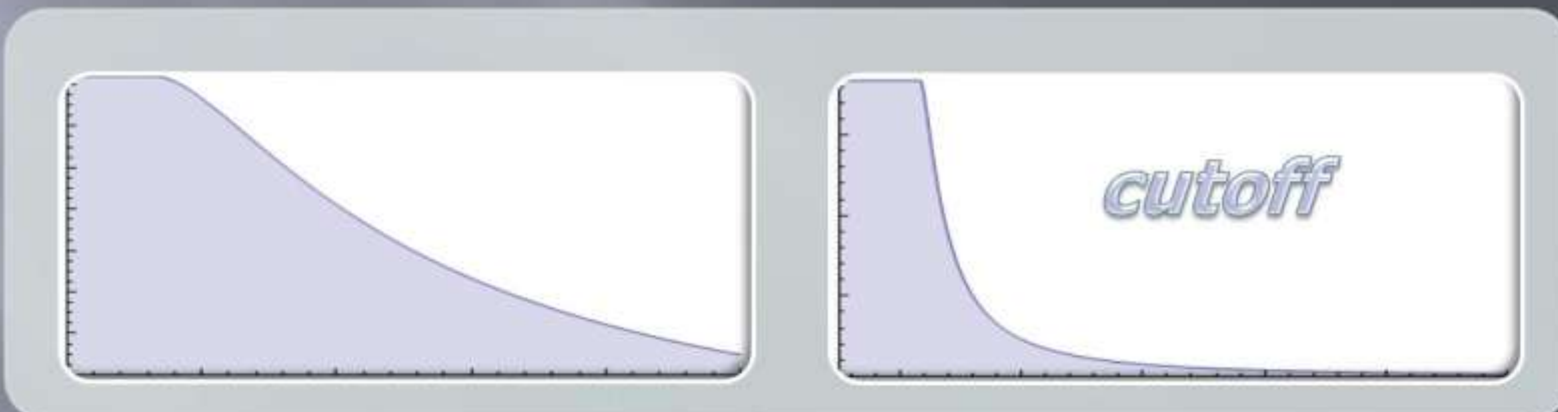
- Mixing time : standard measure of convergence:
  - The L^1 (total-variation) mixing time within ε is
    $$t_{\mathrm{mix}}(\varepsilon) = \inf\left\{t : \max_{\sigma} \left\| H_t(\sigma,\cdot) - \mu \right\|_{\mathrm{TV}} \leq \varepsilon\right\}$$
    where H is the heat-kernel.
  - "Mixing time" usually taken as $t_{\mathrm{mix}}(1/4)$ by convention.
- Spectral gap : governs convergence in $L^2(\mu)$:
  gap $=$ smallest positive eigenvalue of the kernel H.

General (believed) picture for Glauber dynamics

- Setting: Ising model on the lattice $(\mathbb{Z}/n\mathbb{Z})^d$.
  Belief: For some critical inverse-temperature β_c :
- Low temperature: $(\beta > \beta_c)$
  gap^{-1} and t_{mix} are ***exponential*** in the surface area.
- Critical temperature: $(\beta = \beta_c)$
  gap^{-1} and t_{mix} are ***polynomial*** in the surface area.
- High temperature: $(\beta < \beta_c)$
  1. ***Rapid*** mixing: $\text{gap}^{-1} = O(1)$ and $t_{\text{mix}} \asymp \log n$
  2. Mixing occurs abruptly, i.e., there is ***cutoff***.

The Cutoff Phenomenon

- Describes a sharp transition in the convergence of finite ergodic Markov chains to stationarity.



Steady convergence

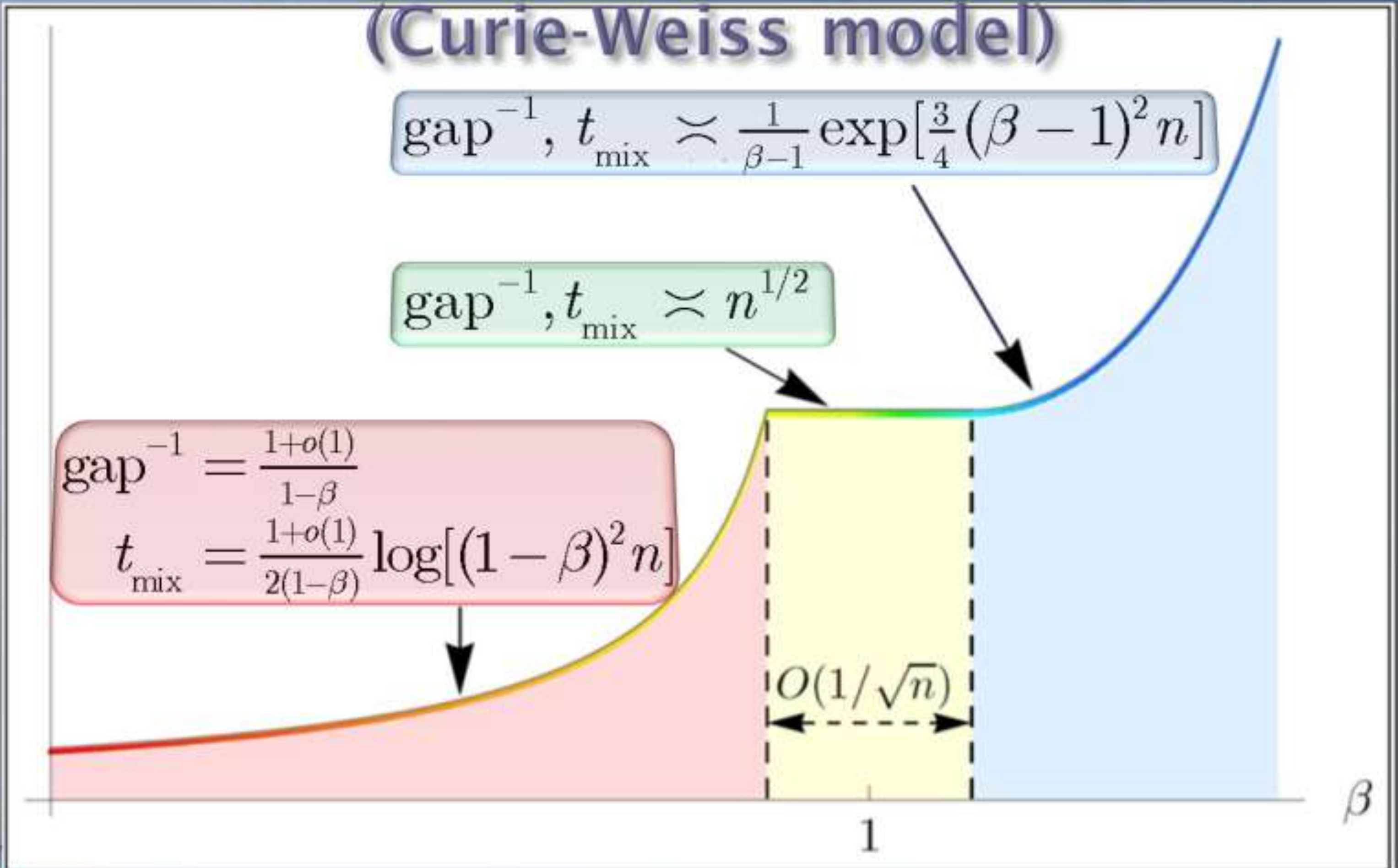
it takes a while to reach distance ½ from stationarity then a while longer to reach distance ¼, etc.

Abrupt convergence

distance from equilibrium quickly drops from 1 to 0

Gap/mixing-time evolution for Ising on the complete graph

(Curie-Weiss model)



Above picture established in [Ding, L., Peres '09].

Mixing time for Ising on lattices: High temperature regime

- Mixing time of Ising on the lattice at high temp. was established in a series of seminal papers:
  - [Aizenman, Holley '84]
  - [Dobrushin, Shlosman '87]
  - [Holley, Stroock '87, '89]
  - [Holley '91]
  - [Stroock, Zegarlinski '92a, '92b, '92c]
  - [Zegarlinski '90, '92]
  - [Lu, Yau '93]
  - [Martinelli, Olivieri '94a, '94b]
  - [Martinelli, Olivieri, Schonmann '94]
- $\Rightarrow$ Bounded log-Sobolev constant and $O(\log n)$ mixing.
- In two dimensions this is known for all $\beta < \beta_c$.

Mixing on the square lattice

- High temperature: $O(1)$ log-Sobolev constant and $O(\log n)$ mixing for all $\beta < \beta_c = \frac{1}{2}\log(1+\sqrt{2})$.
  - ? Dynamics conjectured to exhibit *cutoff* [Peres'04].

- Low temperature: for $\beta > \beta_c$ both gap^{-1} and the mixing time are $\exp[(c(\beta)+o(1))n]$.

  [Schonmann '87], [Chayes, Chayes, Schonmann'87], [Martinelli '94], [Cesi, Guadagni, Martinelli, Schonmann'96].

- Critical temperature: No known sub-exponential ? upper bounds at $\beta = \beta_c$ for mixing or gap^{-1} …

Cutoff: formal definition

- A family of chains (X_t^n) is said to have *cutoff* if:

$$\lim_{n\to\infty} \frac{t_{\mathrm{mix}}(\varepsilon)}{t_{\mathrm{mix}}(1-\varepsilon)} = 1 \quad \forall\ 0 < \varepsilon < 1.$$

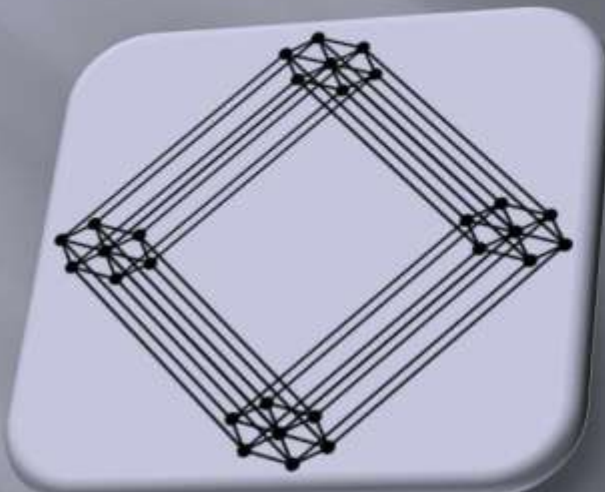
i.e., $t_{\mathrm{mix}}(\alpha) = (1+o(1))\, t_{\mathrm{mix}}(\beta)$ for any $0 < \alpha\,,\, \beta < 1$.

- A sequence (w_n) is called a *cutoff window* if

$$w_n = o(t_{\mathrm{mix}}(\tfrac{1}{4}))\ ,$$
$$t_{\mathrm{mix}}(\varepsilon) - t_{\mathrm{mix}}(1-\varepsilon) = O_\varepsilon(w_n) \quad \forall\, 0 < \varepsilon < 1\ .$$

Basic examples

Lazy discrete-time simple random walk



On the hypercube $\{0,1\}^n$:

☑ Exhibits cutoff at

$\frac{1}{2}\, n \log n + O(n)$

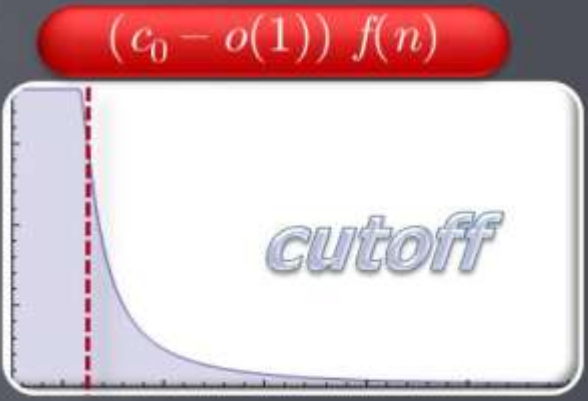
[Aldous '83]

On the n-cycle:

☒ No cutoff.

The importance of cutoff

- Suppose we run Glauber dynamics for the Ising Model satisfying $t_{\text{mix}} \asymp f(n)$ for some $f(n)$.
- Cutoff $\Leftrightarrow$ $\exists$ some $c_0 > 0$ so that:
  - Must run the chain for at least $\sim c_0 \cdot f(n)$ steps to even reach distance (1 - ε) from μ.
  - Running it any longer than that is essentially redundant.



- Proofs usually require (and thus provide) a deep understanding of the chain (its reasons for mixing).
- Many natural chains are *believed* to have cutoff, yet proving cutoff can be extremely challenging.

Cutoff History

- Random walks on graphs and groups:
  - Discovered:
    - Random transpositions on S_n [Diaconis, Shahshahani '81]
    - RW on the hypercube, Riffle-shuffle [Aldous '83]
  - Named "Cutoff Phenomenon" in the top-in-at-random shuffle analysis [Diaconis, Aldous '86]
  - RWs on finite groups [Saloff-Coste '04]
  - RWs on random regular graphs [L., Sly '10+]
- One-dimensional Markov chains:
  - Birth-and-Death chains
    [Diaconis, Saloff-Coste '06], [Ding, L., Peres '09]
- No proofs of cutoff except when stationary distribution is completely understood and has many symmetries.

Cutoff for the Glauber dynamics

- So far *only* spin-systems where cutoff was verified are Ising and Potts models on the *complete graph* [Levin, Luczak, Peres '10], [Ding, L., Peres '09], [Cuff, Ding, L., Louidor, Peres, Sly]  
- Conjectured to believe at high temperatures for:
  - ? Ising on the lattice, e.g. with periodic or free boundary.
  - ? Potts model on the lattice.
  - ? Gas Hard-core model on lattices.
  - ? Colorings of lattices.
  - ? Arbitrary boundary conditions / external field.
  - ? Anti-ferromagnetic Ising/Potts models, Spin-glass, Other lattices / amenable transitive graphs,…

Unknown even in 1 dimension (Q. of Peres)…

Cutoff for Ising on the lattice

- Theorem [L., Sly]:

Let $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ be the critical inverse-temperature for the Ising model on $\mathbb{Z}^2$. Then the continuous-time Glauber dynamics for the Ising model on $(\mathbb{Z}/n\mathbb{Z})^2$ with periodic boundary conditions at $0 \leq \beta < \beta_c$ has cutoff at $(1/\lambda_\infty)\log n$, where λ_∞ is the spectral gap of the dynamics on the infinite volume lattice.

- Analogous result holds for *any* dimension $d \geq 1$:
  - Cutoff at $(d/2\lambda_\infty)\log n$
  - E.g., cutoff at $[2(1-\tanh(2\beta))]^{-1}\log n$ for $d=1$.

Cutoff for Ising on the lattice

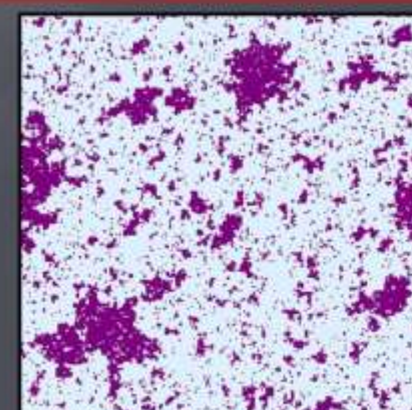
- Main result hinges on an L^1-L^2 reduction, enabling the application of log-Sobolev inequalities.
- Generic method gives further results on many other models conjectured to have cutoff:
  - ✓ Ising on the lattice, e.g. with periodic or free boundary.
  - ✓ Potts model on the lattice.
  - ✓ Gas Hard-core model on lattices.
  - ✓ Colorings of lattices.
  - ✓ Arbitrary boundary conditions / external field.
  - ✓ Anti-ferromagnetic Ising/Potts models, Spin-glass, Other lattices / amenable transitive graphs,…

Recent development: Critical Ising on square lattice

- Theorem [L., Sly]: Critical slowdown verified in $\mathbb{Z}^2$:

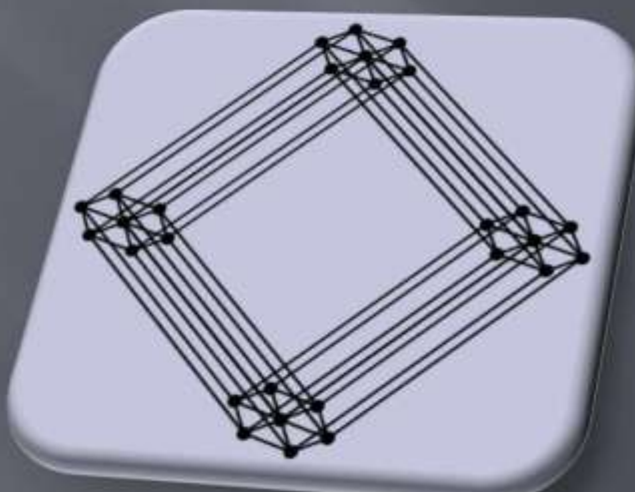
Consider the critical Ising model on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length n, i.e. at inverse-temperature $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$. Let $\mathrm{gap}_\Lambda^\tau$ denote the spectral-gap in the generator of the corresponding Glauber dynamics under an arbitrary fixed boundary condition τ. Then there exists an absolute $C > 0$ (independent of Λ, τ) such that $(\mathrm{gap}_\Lambda^\tau)^{-1} \leq n^C$.

More on this in the next Harvard probability seminar Thursday (Mar 11) 3:10pm, Science Center 232.



Proving Cutoff for Ising: Toy example: cutoff at $\beta = 0$

- No interactions:
  - Stationary distribution is uniform.
  - Spins evolve via independent cont.-time MCs.
- Equivalent to the lazy RW on the hypercube $\{0,1\}^n$.
- [Aldous '83]: Cutoff at $\frac{1}{2}\log n + O(1)$
  - Constant window
  - Twice faster than trivial upper bound.



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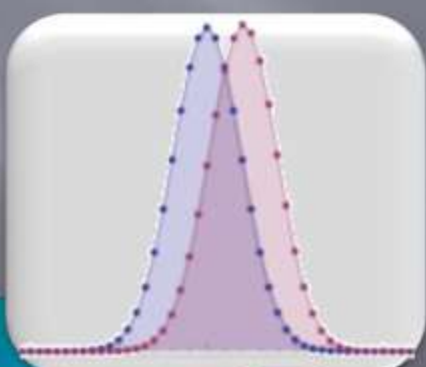
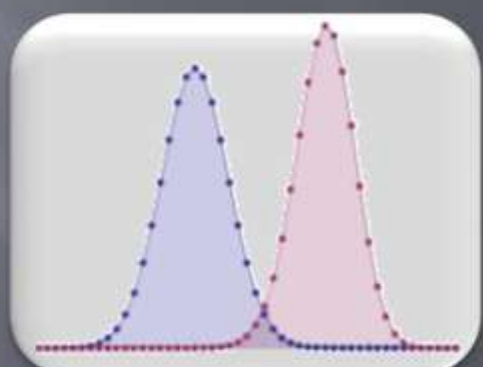
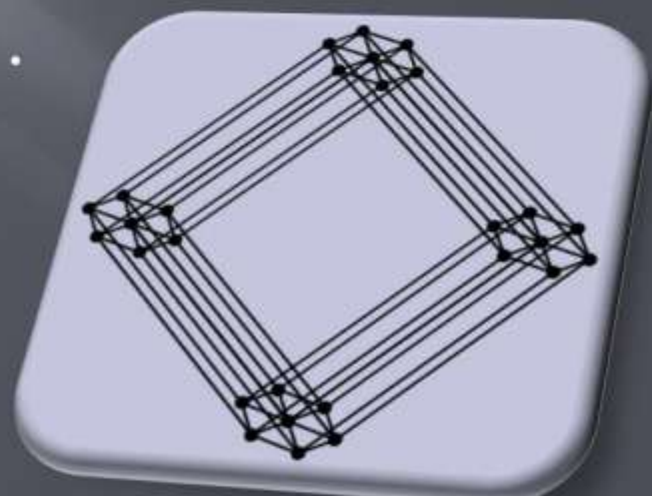
Proving Cutoff for Ising: Toy example: cutoff at $\beta = 0$ (ctd.)

- Magnetization is a birth-and-death chain:
  - By symmetry start at the all-plus state.
  - # of +'s at time t is $\sim \mathrm{Bin}(n, \frac{1}{2}(1+e^{-t}))$.
  - # of +'s under stationary measure $\sim \mathrm{Bin}(n, \frac{1}{2})$ which has Gaussian fluctuations of $O(\sqrt{n})$.
  - Mixing occurs when $\frac{1}{2}\, e^{-t} \asymp \sqrt{n}$.







L^1-L^2 reduction for product chains

- Setup: general family of ergodic product chains:

$$(X_{t(n)}) = \{X_{t(n)}^{i(n)} : i = 1,\dots,m(n)\}$$
$$\lim_{n\to\infty} \left\| \mathbb{P}(X_t^i \in \cdot) - \pi^i \right\|_\infty = 0$$

- Define: $$M \triangleq \sum_1^m \left\| \mathbb{P}(X_t^i \in \cdot) - \pi^i \right\|_{L^2(\pi^i)}^2$$

- The following then holds:

$$M \to 0 \quad \Rightarrow \quad \left\| \mathbb{P}(X_t \in \cdot) - \pi \right\|_{\mathrm{TV}} \to 0$$
$$M \to \infty \quad \Rightarrow \quad \left\| \mathbb{P}(X_t \in \cdot) - \pi \right\|_{\mathrm{TV}} \to 1$$

- For the hypercube $m = n$ and we want to drop the individual L^2 distances ($\asymp \mathrm{e}^{-t}$) below $1/\sqrt{n}$.

L^1-L^2 reduction for Ising

- Framework:
  - (X_t) : continuous-time Glauber dynamics for $\mathbb{Z}_n^d$
  - (X_t^*) : continuous-time Glauber dynamics on a smaller lattice: $\mathbb{Z}_r^d$ for $r = 3\log^3 n$.
  - B : smaller box within $\mathbb{Z}_r^d$ of side-length $2\log^3 n$.
- Define:

$$\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t^*(B) \in \cdot) - \mu_B^* \right\|_{L^2(\mu_B^*)}$$

measuring the L^2 convergence of the projection of (X_t^*) onto the box B.

L^1-L^2 reduction for Ising (ctd.)

- Recall: $\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t^*(B) \in \cdot) - \mu_B^* \right\|_{L^2(\mu_B^*)}$
- Theorem:

Let $s = s(n)$ and $t = t(n)$ satisfy

$$(10d / \alpha_s^*) \log\log n \le s < \log^{4/3} n \quad ,$$

$$(20d / \alpha_s^*) \log\log n \le t < \log^{4/3} n \quad ,$$

where α_s^* is the infimum over log-Sobolev constants.

$$(n / \log^5 n)^d \mathfrak{m}_t^2 \to 0 \;\Rightarrow \limsup_{n\to\infty} \max_{x_0} \left\| \mathbb{P}_{x_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} = 0$$

$$(n / \log^3 n)^d \mathfrak{m}_t^2 \to \infty \Rightarrow \liminf_{n\to\infty} \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t \in \cdot) - \mu \right\|_{\mathrm{TV}} = 1$$

- Translates L^1 mixing to L^2 mixing (to within a finer scale) on projections in smaller boxes.

Existence of cutoff

- Recall that $\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t^*(B) \in \cdot) - \mu_B^* \right\|_{L^2(\mu_B^*)}$
  and choose: $t^\star \triangleq \inf\left\{ t : \mathfrak{m}_t^2 \le \frac{\log^{3d+1}}{n^d} \right\}$.
- Log-Sobolev inequalities ensure that $t^\star = O(\log n)$.
- Take $s = (10d / \alpha_s^*) \log\log n$.
  - $\Rightarrow (n / \log^5 n)^d \, \mathfrak{m}_{t^\star}^2 = \log^{1-2d} n = o(1)$
  - Theorem implies L^1-distance of o(1) by time $t^\star + s$.
- Since $t^\star \asymp \log n \Rightarrow t^\star \ge (20d / \alpha_s^*) \log\log n$.
  - $\Rightarrow (n / \log^3 n)^d \, \mathfrak{m}_{t^\star}^2 = \log n \to \infty$
  - Theorem implies L^1-distance of 1-o(1) at time $t^\star$. ∎

Ideas from the proof: L^1-L^2 reduction & cutoff location

- Additional effort needed to establish cutoff location in terms of λ_∞ :
  - Express cutoff location in terms of the spectral-gaps on the smaller $\mathbb{Z}_r^d$ and show these converge to λ_∞ .
- Reduction is enabled by the following:
  - Information spreads at rate 1 while mixing is $\mathrm{O}(\log n)$: No time for information to spread…
  - Consider the (random) "update support": the smallest set of spins whose value at time t is needed in order to determine the state at time $t+s$.
  - Geometric properties of support $\Rightarrow$ product chain.

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Random support of update seq.

THANK YOU.