Glauber Dynamics for Spin Systems at High & Critical Temperatures

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Ising model

- Underlying geometry: finite graph $G=(V,E)$.
- Set of possible configurations: $\Omega=\{\pm1\}^V$ (assignments of +/- *spins* to the vertices).
- Probability of a configuration $\sigma\in\Omega$ is given by the *Gibbs distribution* (no external field):

$$\mu(\sigma) = \frac{1}{Z(\beta)} \exp \, \beta \sum_{xy\in E} \sigma(x)\sigma(y)$$

- *Ferromagnetic* with inverse-temperature β : as $\beta\uparrow$ the measure μ favors configurations with aligned neighboring spins.

Heat-bath Glauber dynamics

- MC sampler for the Gibbs distribution:
  - Update each $u \in V$ via an independent Poisson(1) clock, replacing it by a new spin $\sim \mu$ conditioned on $V \setminus \{u\}$.
- Ergodic reversible MC with stationary measure μ.
- How fast is the convergence to equilibrium?
  - Measuring convergence in $L^2(\mu)$: Spectral gap :

    $$\text{gap} = 1 - \lambda$$

    (where λ = largest nontrivial eigenvalue of the kernel H).
  - Measuring convergence in L^1 : Mixing time :

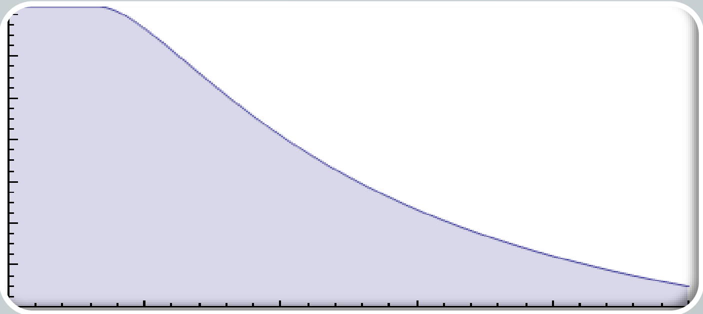
    $$t_{\text{mix}}(\varepsilon) = \inf\left\{ t : \max_{\sigma} \left\| H_t(\sigma, \cdot) - \mu \right\|_{\text{TV}} \le \varepsilon \right\}$$

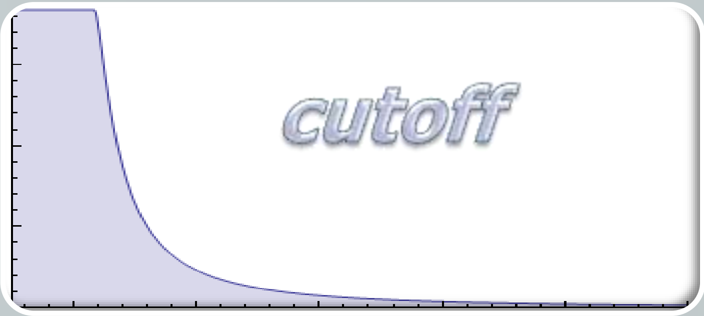
General (believed) picture for Glauber dynamics

- Setting: Ising model on the lattice $(\mathbb{Z}/n\mathbb{Z})^d$.
  Belief: For some critical inverse-temperature β_c :
- Low temperature ($\beta > \beta_c$) :
  gap^{-1} and t_{mix} are ***exponential*** in the surface area.
- Critical temperature ($\beta = \beta_c$) :
  gap^{-1} and t_{mix} are ***polynomial*** in the surface area.
- High temperatures:
  1. ***Rapid*** mixing: gap^{-1}$=O(1)$ and $t_{\text{mix}} \asymp \log n$
  2. Mixing occurs abruptly, i.e., there is ***cutoff***.

The Cutoff Phenomenon

- Describes a sharp transition in the convergence of finite ergodic Markov chains to stationarity.



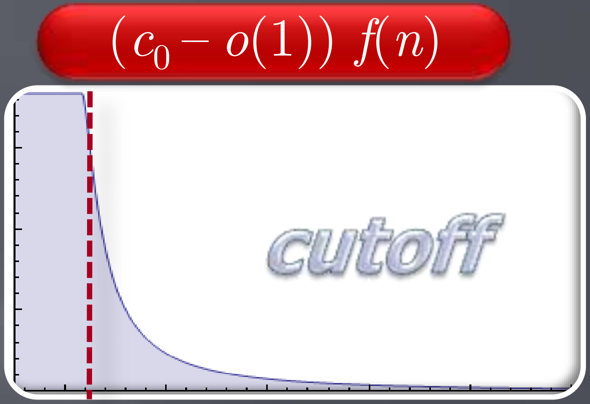


Steady convergence
it takes a while to reach distance ½ from π, then a while longer to reach distance ¼, etc.

Abrupt convergence
the distance from π quickly drops from 1 to 0

The importance of cutoff

- Suppose we run Glauber dynamics for the Ising Model satisfying $t_{\text{mix}} \asymp f(n)$ for some $f(n)$.
- Cutoff ⇔ $\exists$ some $c_0 > 0$ so that:
  - Must run the chain for at least $\sim c_0 \cdot f(n)$ steps to even reach distance (1 - ε) from μ.
  - Running it any longer than that is essentially redundant.



- Proofs usually require (and thus provide) a deep understanding of the chain (its reasons for mixing).
- Many natural chains are *believed* to have cutoff, yet proving cutoff can be extremely challenging.

Cutoff: formal definition

- Recall: $t_{\text{mix}}(\varepsilon) = \inf\{t : \max_{\sigma} \|H_t(\sigma,\cdot) - \mu\|_{\text{TV}} \le \varepsilon\}$,

$$\|\mu - \nu\|_{\text{TV}} = \sup_{A \subset \Omega}[\mu(A) - \nu(A)].$$

- The chain has *cutoff* if the following holds:

$$\lim_{n\to\infty} \frac{t_{\text{mix}}(\varepsilon)}{t_{\text{mix}}(1-\varepsilon)} = 1 \quad \text{for any } 0 < \varepsilon < 1.$$

- That is, for any $0 < \alpha, \beta < 1$ we have

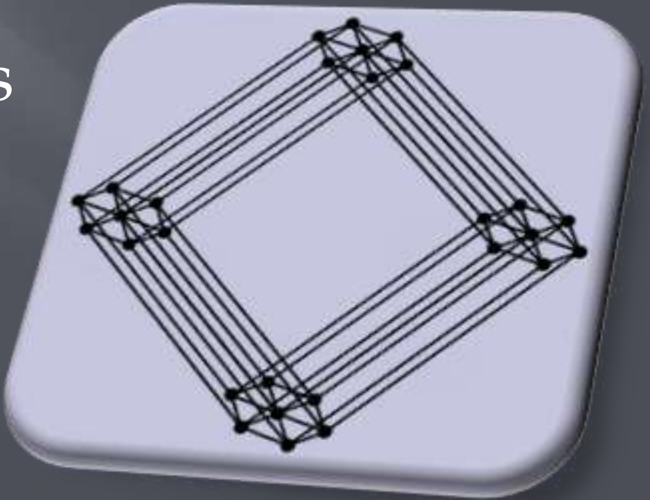
$$t_{\text{mix}}(\alpha) = (1+o(1))\, t_{\text{mix}}(\beta).$$

Cutoff example: Glauber dynamics for Ising at $\beta = 0$

- No interactions: Uniform $\{\pm 1\}$ updates on the n sites [the discrete-time analogue of the chain is the lazy random walk on the hypercube]
- The magnetization is a Markov Chain (analogous to the classical "Ehrenfest's Urn" birth & death chain).
- The Coupon Collector approach:

  $t_{\text{mix}}(\varepsilon) \leq \log n + c_\varepsilon$, whereas

  $t_{\text{mix}}(1-\varepsilon) \geq \frac{1}{2}\log n - c'_\varepsilon$.
- [Aldous '83]: lower bound is tight: ½ $\log n + O(1)$ time suffices!



Picture for Glauber dynamics on the square lattice $(\mathbb{Z}/n\mathbb{Z})^2$

- Known:
  - $\beta > \beta_c$ Low temperature:
    $\text{gap}^{-1} \geq \exp(c\,n)$ and $t_{\text{mix}} \geq \exp(c\,n)$ for some $c > 0$.
  - $\beta < \beta_c$ High temperatures:
    $\text{gap}^{-1} = O(1)$ and $t_{\text{mix}} \asymp \log n$
- *Open*: Polynomial gap^{-1} , t_{mix} at the critical $\beta = \beta_c$?
  - Static critical μ already highly involved...
- *Open*: Cutoff at high temperatures?
  - Cutoff unknown even in 1-dimension (Q. of Peres)…
  - To this date, no proof of cutoff for any chain whose stationary distribution is not completely understood...

Ising on other geometries

1. **Complete graph** (Curie-Weiss model):
   - Believed to predict the behavior of the Ising model on high-dimensional tori.
   - Size of the system plays the role of surface-area.

2. **Regular tree** (Bethe lattice):
   - Canonical example of a non-amenable graph (with boundary proportional to its volume).
   - Height of tree plays the role of surface-area.

The Curie Weiss model: (mean field Ising model)

- Rescaling β : $\mu(\sigma) \propto \exp\left((\beta/n) \sum_{x<y} \sigma(x)\sigma(y) \right)$
- [Griffiths, Weng, Langer '66], [Ellis '87]),… :
  For $\beta > 1$: gap^{-1}, t_{mix} exponential in n.
- [Aizenman, Holley '84], [Bubley, Dyer '97]),… :
  For $\beta < 1$: $\text{gap}^{-1} = O(1)$, $t_{\text{mix}} \asymp \log n$.
- [Levin, Luczak, Peres '07], [Ding, L., Peres '09]:
  For $\beta = 1$: gap^{-1}, $t_{\text{mix}} \asymp n^{1/2}$
  For $\beta < 1$: ***cutoff*** : $t_{\text{mix}} = \frac{1}{2(1-\beta)} \log n + O(1)$
  Critical window around β_c is of order $1/\sqrt{n}$
- Picture completely ***verified***.

The Curie Weiss model: complete picture

- Theorem [Ding, L., Peres '09]:

$\beta = 1 - \delta$ with $\delta^2 n \to \infty$:

Cutoff at $\frac{1}{2\delta}\log(\delta^2 n)$ with a window of $1/\delta$.

$\text{gap}^{-1} = (1+o(1))/\delta$.

$\beta = 1 \pm \delta$ with $\delta = O(n^{-1/2})$:

gap^{-1}, $t_{\text{mix}} \asymp n^{1/2}$ and there is no cutoff.

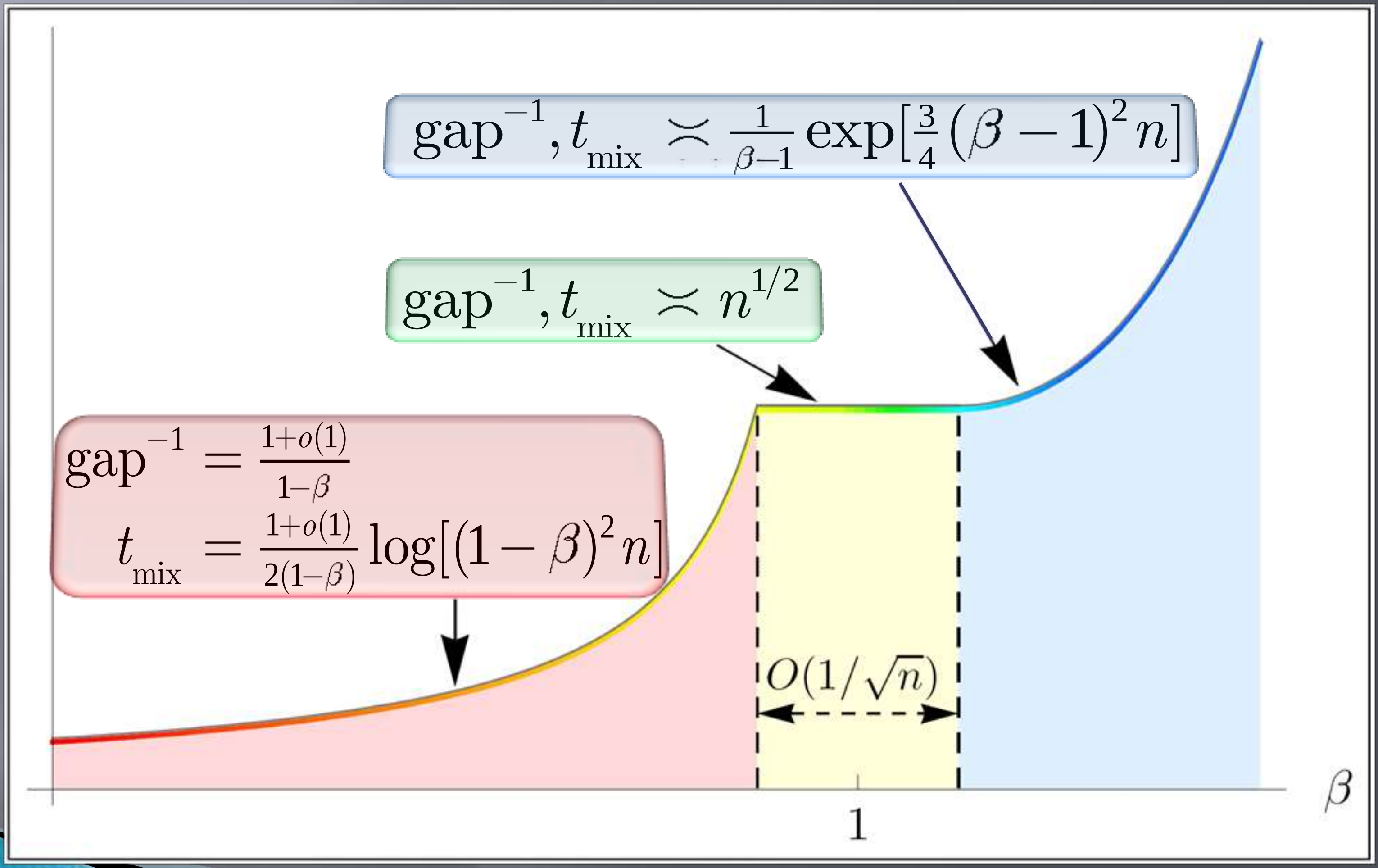
$\beta = 1 + \delta$ with $\delta^2 n \to \infty$:

gap^{-1} , $t_{\text{mix}} \asymp \frac{1}{\delta}\exp[\frac{n}{2}\int_0^{\zeta}\log(\frac{1+g(x)}{1-g(x)})dx]$, where

ζ is the unique positive root of $g(x) = \dfrac{\tanh(\beta x) - x}{1 - x\tanh(\beta x)}$.

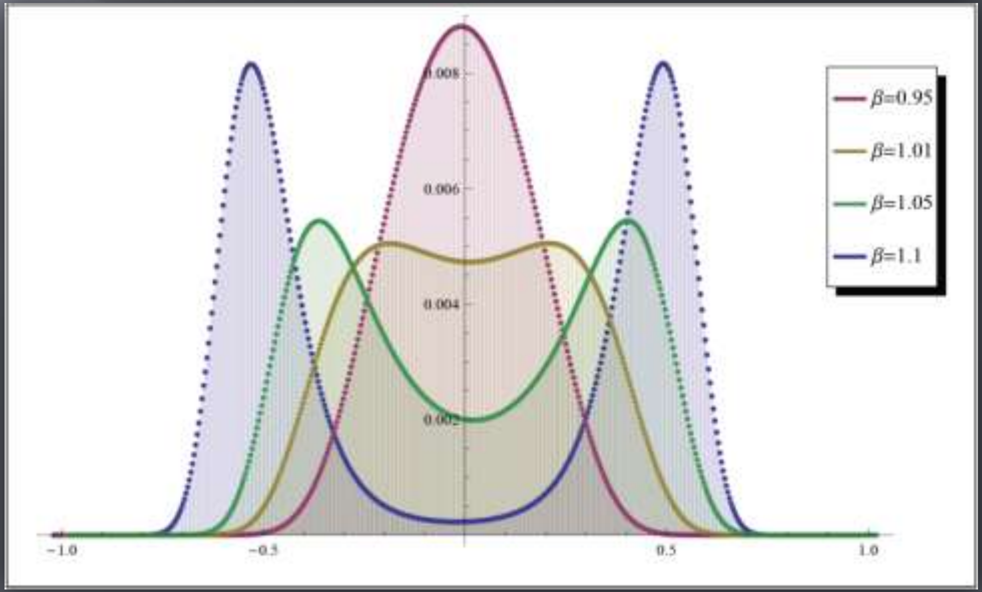
There is no cutoff.

Curie-Weiss model:
(mean-field Ising model)
Scaling window in the gap/mixing-time evolution



Mean field Ising model (ctd.)

- Key element in the analysis:
  By the complete symmetry, the magnetization is in fact a birth-and-death Markov Chain.
- Its mixing governs the mixing of the full dynamics.
- Proof involves a delicate analysis of certain hitting times to establish the precise point of reaching the "center of mass".



Stationary distribution of the magnetization chain for the dynamics on $n = 500$ vertices.

Same picture for the q-state Potts model ?

- Potts model: $\Omega = [q]^V$ (assigning *colors* to the sites)

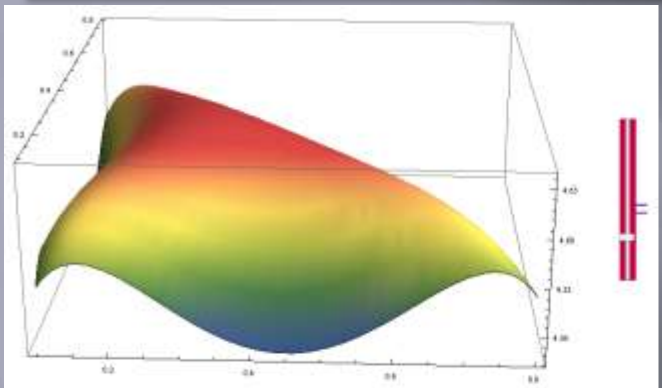
$$\mu(\sigma) = Z^{-1}(\beta)\exp\left(-\beta\sum_{xy\in E}\mathbf{1}_{\{\sigma(x)\neq\sigma(y)\}}\right)$$

- [Ellis '87]: Mean-Field Potts has a phase transition in the structure of the μ around $\beta_c = 2\frac{q-1}{q-2}\log(q-1)$:
  - *Disordered phase* for $\beta < \beta_c$: Each of the q spins appears roughly the same # of times.
  - *Ordered phase* for $\beta > \beta_c$: One of the spins dominates.
- [Gore, Jerrum '96]: Mixing is exponentially slow at $\beta = \beta_c$ even for (faster) Swendsen-Wang dynamics…
- No power-law of gap^{-1}, t_{mix} at criticality !?!

Mean-field Potts model

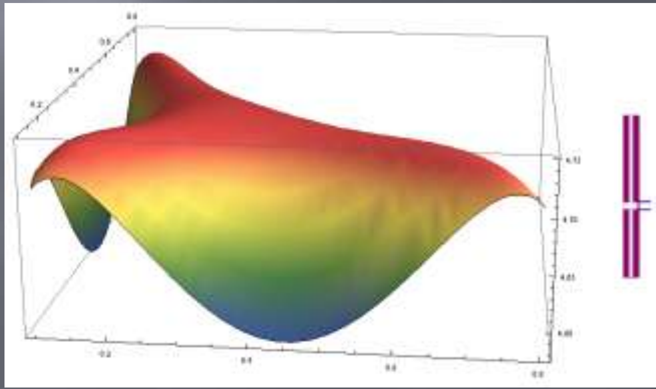
- [Cuff, Ding, L., Louidor, Peres, Sly]: confirm the picture w.r.t. a new critical point $\beta_{\mathrm{m}} < \beta_{\mathrm{c}}$!
  - The smallest β such that $g(x) = \dfrac{e^{\beta x}}{e^{\beta x} + (q-1)e^{\beta(1-x)/(q-1)}} - x$ has a non-trivial root in $(\frac{1}{q}, 1]$.
- Analogous behavior to Ising around β_{m}, e.g.:
  - Rapid mixing ($\log n$) with <u>cutoff</u> at $\beta < \beta_{\mathrm{m}}$.
  - Power law at criticality: $\mathrm{gap}^{-1},\ t_{\mathrm{mix}} \asymp n^{1/3}$ for $\beta = \beta_{\mathrm{m}}$.
  - Critical window has order $n^{-2/3}$.
  - Exp. mixing, yet fast "<u>essential mixing</u>" in $(\beta_{\mathrm{m}}, \beta_{\mathrm{c}})$.

Mean-field Potts model, *q*=3



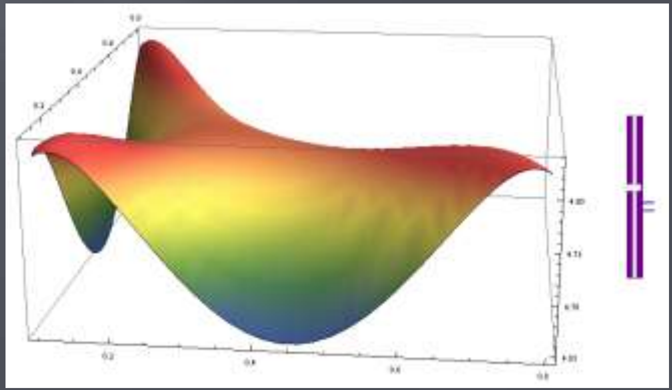
$\beta < \beta_m \approx 2.746$: Rapid mixing with cutoff

$\beta = \beta_m$: Power law mixing ($n^{1/3}$)



Between (β_{m}, β_c):
Exponential mixing, yet fast "essential mixing" with cutoff.

$\beta \geq \beta_c \approx 2.773$: Exponential mixing



Ising on the Bethe lattice

- Underlying graph = b-ary tree of height h .
- μ has a constructive representation (free boundary):
  - Assign a uniform spin to the root
  - Scan the tree top to bottom: Every site inherits the spin of its parent with probability $\frac{1}{2}(1+\tanh\beta)$ and mutates o/w.
- Two critical temperatures:
  1. $\beta_1 = \operatorname{arctanh}(1/b)$:Uniqueness threshold (does the effect of a plus boundary on the root vanish as $h \to \infty$).
  2. $\beta_c = \operatorname{arctanh}(1/\sqrt{b})$: Purity threshold (does the effect of a "typical" boundary on the root vanish as $h \to \infty$).
- β_c coincides with critical spin-glass parameter.

Glauber for Ising on trees

- β_c is the critical mixing parameter :
- [Kenyon, Mossel, Peres ’01], [Berger, Kenyon, Mossel, Peres ’05]: Under free boundary:
  - For $\beta < \beta_c$: $\text{gap}^{-1} = O(1)$, $t_{\text{mix}} \asymp h$
  - For $\beta > \beta_c$: $\log(\text{gap}^{-1}) \asymp h$
  - At criticality: No upper bound; shown that gap^{-1} is at least linear in h and conjectured that this is tight.
- [Martinelli, Sinclair, Weitz ’04]:
  - Under all-plus BC: $\text{gap}^{-1} = O(1)$, $t_{\text{mix}} \asymp h$ <u>for all $\beta > 0$</u> .
- What is the behavior at the critical β_c ?

Mixing for Ising on critical trees

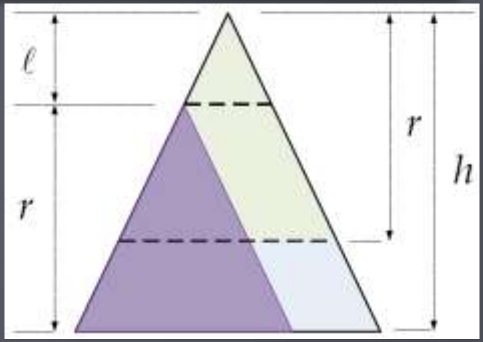
- Theorem [Ding, L., Peres '09]:

Fix $b \geq 2$ and let $\beta_c = \operatorname{arctanh}(b^{-1/2})$ be the critical inverse-temperature for the Ising model on the b-ary tree of height h.
There exist $C,\ c_1,\ c_2 > 0$ independent of b such that :

1. For any boundary condition τ :
$$\text{gap}^{-1}\ ,\ t_{\text{mix}} = O(\ h^C\ )\ .$$
2. In the free boundary case:
$$\text{gap}^{-1} \geq c_1\, h^2\ ,\ t_{\text{mix}} \geq c_2\, h^3\ .$$

- (first geometry other than the complete graph where power-law mixing at criticality is verified)

Mixing on critical trees (ctd.)
Main ingredients in the proof



- Use block dynamics [Martinelli '97] to form a recursion on the tree:
  $$\inf_{\tau}\{\text{gap}_h^{\tau}\} \geq \tfrac{1}{2}\inf_{\tau}\{\text{gap}_{\mathcal{B}}^{\tau}\} \cdot \inf_{\tau}\{\text{gap}_r^{\tau}\}$$
- Reduces to estimating a quantity that, in the free boundary case, corresponds to a reconstruction-type result of [Pemantle, Peres '05] ("propagate a spin at the root down to level ℓ, then reconstruct it back").
- In our case: we have τ, an arbitrary boundary condition (greatly complicates the proof)...
- Still open: Is there cutoff for $\beta \leq \beta_c$?

Recent progress: Ising on lattices

- Theorem [L., Sly]:

Let $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ be the critical inverse-temperature for the Ising model on $\mathbb{Z}^2$. Then the continuous-time Glauber dynamics for the Ising model on $(\mathbb{Z}/n\mathbb{Z})^2$ with periodic boundary conditions at $0 \le \beta < \beta_c$ has cutoff at $(1/\lambda)\ \log\ n$, where λ is the spectral gap of the dynamics on the infinite volume lattice.

- Analogous result holds for *any* dimension $d \ge 1$. [e.g., for $d = 1$ there is cutoff at $\frac{1}{2(1-\tanh(2\beta))}\log n$ for any temperature].

Recent progress on lattices (ctd.)

- Main result hinges on an L^1-L^2 reduction, enabling the application of log-Sobolev inequalities.
- Generic method that gives further results on:
  - Arbitrary external field and non-uniform interactions.
  - Boundary conditions (including free, all-plus, mixed).
  - Other spin system models:
    - Anti-ferromagnetic Ising
    - Gas Hard-core
    - Potts (ferromagnetic /anti-ferromagnetic)
    - Coloring
    - Spin-glass
  - Other lattices (e.g., triangular, graph products).

THANK YOU.