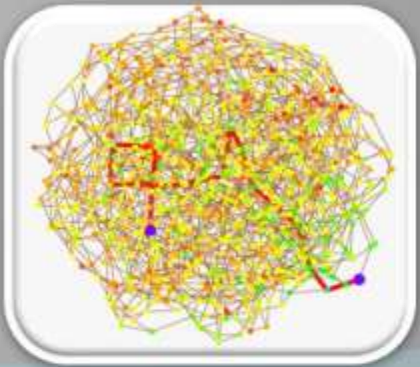
Anatomy of a Young Giant Component in the Random Graph



Eyal Lubetzky
Microsoft Research

Based on joint works with J. Ding, J.H. Kim and Y. Peres

Feb 2010

Univ. Roma TRE

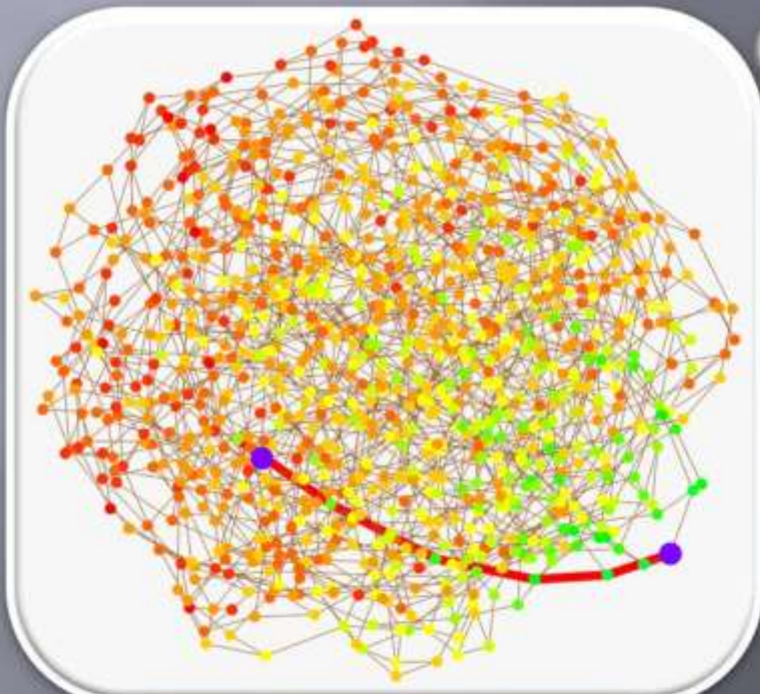
A random metric model:
random *iid* edge-weights on an expander

- A graph $\mathcal{K}$ is an *expander* if the Simple Random Walk on it has a spectral-gap bounded away from 0.
- Numerous applications for sparse expanders in Mathematics and Computer Science.

- ***Q***: Put *iid* $\mathrm{Exp}(1)$ weights on the edges of $\mathcal{K}$.
What does the new metric typically look like?
  - E.g.: diameter, typical distance (equivalent to First Passage Percolation), heaviest simple paths, …
- We will be interested in this question for $\mathcal{G}(n\,,3)$, a random uniform 3-regular graph on n vertices.

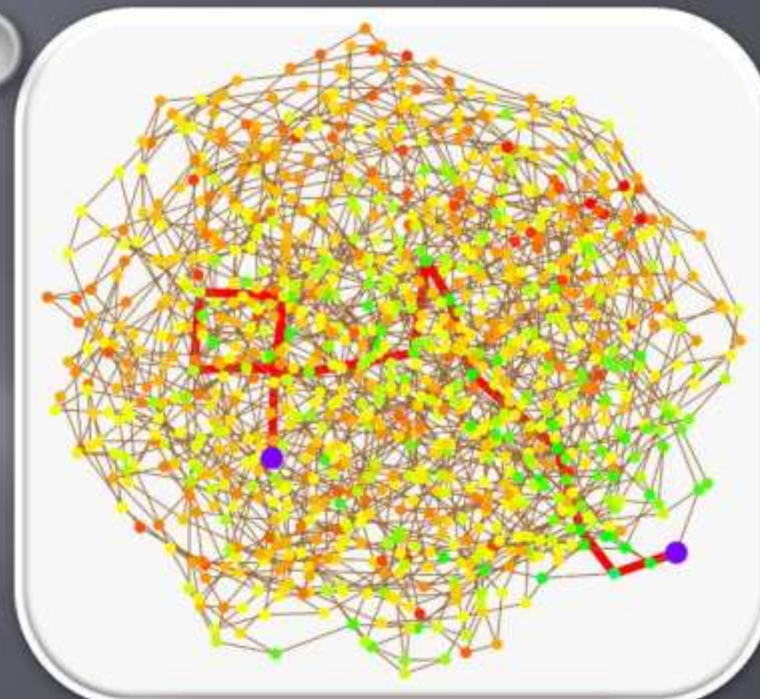
Example: $\mathrm{Exp}(1)$ weights on a random cubic graph

$\mathcal{G}(1000{,}3)$



Shortest path:
6 hops, weight 6.59

Typical length	$\sim \log_2 n$
Typical weight	$\sim \log_2 n$
Worst length	$\sim \log_2 n$



Lightest path:
14 hops, weight 5.36

Typical length	$\sim 2\log n \approx 1.4 \log_2 n$
Typical weight	$\sim \log n \approx 0.7 \log_2 n$
Worst weight	$\sim \frac{5}{3} \log n \approx 1.2 \log_2 n$

Erdős-Rényi random graphs

- Setting: $\mathcal{G}(n,p)$ around the critical point $p = 1/n$.
- "Double jump" phenomenon for order of $|\mathcal{C}_1|$:
  [Erdős-Rényi (1960's)], [Bollobás '84] , [Łuczak '90]
  - $\log n$ for $p = c/n$, $c < 1$ fixed.
  - n for $p = c/n$, $c > 1$ fixed.
  - $n^{2/3}$ at criticality, and throughout the *critical window:* $p = (1 \pm \varepsilon)/n$ for $\varepsilon = O(n^{-1/3})$.
- Emerging from the critical window:
  ($\varepsilon^3 n \to \infty$ and $\varepsilon \to 0$) : $|\mathcal{C}_1| \sim 2\varepsilon n$
  giant component is gradually formed…

Understanding $\mathcal{G}(n,p)$ beyond criticality

- Commonly studied geometric properties:
  - Component sizes (vertices, edges), e.g. $|\mathcal{C}_1|$.
  - Distances: typical, maximal (diameter).
  - Expansion and isoperimetric profile.
  - Long simple paths and cycles.
  - …
- Parameters interplaying with graph geometry:
  - Cover time of the random walk.
  - Mixing time of the random walk.
  - …

Emerging from the window

- $\mathrm{diam}(\mathcal{C}_1)$:
  - $\Theta(n^{1/3})$ at critical window [Nachmias, Peres '08].
  - $\Theta(\log n)$ for $p = (1+\varepsilon)/n$ with $\varepsilon > 0$ fixed. Asymptotics obtained by [Fernholz, Ramachandran '07] and by [Bollobás, Janson, Riordan '07].
- Mixing time of the random walk on $\mathcal{C}_1$:
  - $\Theta(n)$ at critical window [Nachmias, Peres '08].
  - $\Theta(\log^2 n)$ for $p = (1+\varepsilon)/n$ with $\varepsilon > 0$ fixed. [Fountoulakis, Reed '08] and independently by [Benjamini, Kozma, Wormald].
- What does the transition look like when $\varepsilon \to 0$?

Eyal Lubetzky, Microsoft Research
Feb 2010

Emerging from the window

- Prior to this work:

	Critical window $p = (1 \pm \varepsilon)/n$ $\varepsilon = O(n^{-1/3})$	Mildly supercritical $p = (1+\varepsilon)/n$ $\varepsilon^3 n \to \infty$, $\varepsilon \to 0$	**Strictly Supercritical** $p = (1+\varepsilon)/n$ $\varepsilon > 0$ fixed
$\mathrm{diam}(\mathcal{C}_1)$	$\Theta(n^{1/3})$ [NP'08] *not concentrated*		$\Theta(\log n)$ [CL'01], [FR'07], [BJR'07] *precise asymptotics*
Mixing time of random walk on $\mathcal{C}_1$	$\Theta(n)$ [NP'08] *tree-like behavior*		$\Theta(\log^2 n)$ [FR'08], [BKW] *weak-expansion*

Eyal Lubetzky, Microsoft Research
Feb 2010

Emerging from the window

- Prior to this work:

	Critical window $p=(1\pm\varepsilon)/n$ $\varepsilon = O(n^{-1/3}))$	Mildly supercritical $p=(1+\varepsilon)/n$ $\varepsilon^3 n \to \infty$, $\varepsilon \to 0$	**Strictly Supercritical** $p=(1+\varepsilon)/n$ $\varepsilon > 0$ fixed
$\mathrm{diam}(\mathcal{C}_1)$	$\Theta(n^{1/3})$ [NP'08] *not concentrated*	$\Theta(\ (1/\varepsilon)\ \log(\varepsilon^3 n)\ )$ [ŁS], [RW] *precise asymptotics except for very slow* $\varepsilon^3 n$	$\Theta(\log\ n)$ [CL'01], [FR'07], [BJR'07] *precise asymptotics*
Mixing time of random walk on $\mathcal{C}_1$	$\Theta(n)$ [NP'08] *tree-like behavior*	**?** *Hard to interpolate between the regimes*	$\Theta(\log^2\ n)$ [FR'08], [BKW] *weak-expansion*

1. Q: *Diameter* asymptotics *throughout intermediate regime?*
2. Q: *Order of the mixing time in that regime?*

Emerging from the window

- New results:

	Critical window $p=(1\pm\varepsilon)/n$ $\varepsilon = O(n^{-1/3}))$	Mildly supercritical $p=(1+\varepsilon)/n$ $\varepsilon^3 n \to \infty$, $\varepsilon \to 0$	**Strictly Supercritical** $p=(1+\varepsilon)/n$ $\varepsilon > 0$ fixed
$\mathrm{diam}(\mathcal{C}_1)$	$\Theta(n^{1/3})$ [NP'08] *not concentrated*	$\Theta(\ (1/\varepsilon)\ \log(\varepsilon^3 n)\ )$ [ŁS], [RW] *precise asymptotics except for very slow $\varepsilon^3 n$* $(3+o(1)(1/\varepsilon)\ \log(\varepsilon^3 n)$	$\Theta(\log\ n)$ [CL'01], [FR'07], [BJR'07] *precise asymptotics*
Mixing time of random walk on $\mathcal{C}_1$	$\Theta(n)$ [NP'08] *tree-like behavior*	**?** *Hard to interpolate between the regimes* $\Theta(\ (1/\varepsilon^3)\ \log^2(\varepsilon^3 n)\ )$	$\Theta(\log^2\ n)$ [FR'08], [BKW] *weak-expansion*

1. Q: *Diameter asymptotics throughout intermediate regime?*
2. Q: *Order of the mixing time in that regime?*

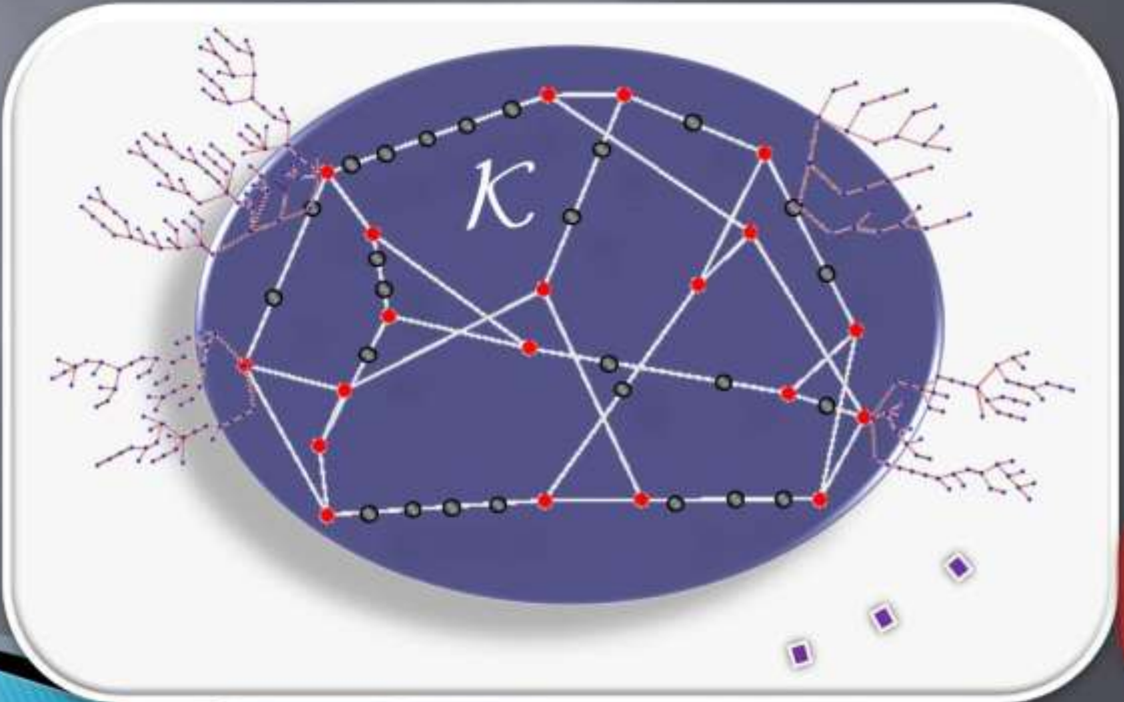
Characterizing the supercritical $\mathcal{C}_1$

- Notable decomposition theorems:
  - [Łuczak '91]: *Kernel* is a random multigraph on a certain degree sequence, which is almost entirely cubic.
  - [Pittel, Wormald '05]: Local CLT for $\mathcal{C}_1$ and 2-*core*.
  - [Benjamini, Kozma, Wormald]: Supercritical $\mathcal{C}_1$ as an expander "decorated" by trees of up to logarithmic size.
- Not precise enough; e.g., for mixing time order.
- New result: complete characterization of $\mathcal{C}_1$.
- Instead of decomposing $\mathcal{C}_1$, we <u>construct</u> it:
  We define a simple model contiguous to it.

Eyal Lubetzky, Microsoft Research
Feb 2010

Back to the giant component: Constructing the young giant

1. kernel : $\mathcal{K} \sim \mathcal{G}(N,3)$ with $N \sim (4/3)\,\varepsilon^3 n$
2. 2-core : edges $\rightarrowtail$ paths of lengths *iid* $\mathrm{Geom}(\varepsilon)$.
3. $\mathcal{C}_1$: attach $\mathrm{Poisson}(1-\varepsilon)$-Galton-Watson trees.



$\mathcal{C}_1$ in $\mathcal{G}(n,p)$
$p = (1+\varepsilon)/n$
$\varepsilon^3 n \to \infty$, $\varepsilon \to 0$

Structure of the young giant

- Theorem 1 [Ding, Kim, L., Peres]:

Let $\mathcal{C}_1$ *be the largest component of the random graph* $\mathcal{G}(n,p)$ *for* $p = \frac{1+\varepsilon}{n}$, *where* $\varepsilon^3 n \to \infty$ *and* $\varepsilon = o(n^{-1/4})$.
Then $\mathcal{C}_1$ *is contiguous to the model* $\tilde{\mathcal{C}}_1$ *defined as follows:*

1. *Let* $Z \sim \mathcal{N}(\frac{2}{3}\varepsilon^3 n, \varepsilon^3 n)$, *and let* $\mathcal{K}$ *be a random 3-regular (multi)graph on* $N = 2\lfloor Z \rfloor$ *vertices.*
2. *Replace edges of* $\mathcal{K}$ *by paths of lengths iid* $\mathrm{Geom}(\varepsilon)$.
3. *Attach iid* $\mathrm{Poisson}(1-\varepsilon)$-Galton-Watson *trees to vertices.*

That is, $\mathbb{P}(\tilde{\mathcal{C}}_1 \in \mathcal{A}) \to 0$ *implies* $\mathbb{P}(\mathcal{C}_1 \in \mathcal{A}) \to 0$ *for any* $\mathcal{A}$.

Reading off key features

- Steps 1,2,3 $\leftrightsquigarrow$ kernel, 2-core and $\mathcal{C}_1$ resp.
- Some examples:
  - Size of 2-core $\sim 2\,\varepsilon^2 n$
  - Size of kernel $\sim (4/3)\,\varepsilon^3 n$ vertices
  - Max length of a 2-path in 2-core $\sim (1/\varepsilon)\log(\varepsilon^3 n)$
  - $\exists$ simple cycle in 2-core of length $\sim (4/3)\ \varepsilon^2 n$
- Distances in the 2-core $\leftrightsquigarrow$ First Passage Percolation.
  - E.g., typical distance in the 2-core obtained from a result of [Bhamidi, Hooghiemstra, van der Hofstad].

General structure result

- <u>Theorem 2</u> [Ding, Kim, L., Peres]:

Let $\mathcal{C}_1$ *be the largest component of the random graph* $\mathcal{G}(n,p)$ *for* $p = \frac{1+\varepsilon}{n}$, *where* $\varepsilon^3 n \to \infty$ *and* $\varepsilon = o(1)$. *Let* $\mu < 1$ *be the conjugate of* ε, *i.e.* $\mu\, \mathrm{e}^{-\mu} = (1+\varepsilon)\mathrm{e}^{-(1+\varepsilon)}$.

Then $\mathcal{C}_1$ *is contiguous to the model* $\tilde{\mathcal{C}}_1$ *defined as follows:*

1. *Let* $\Lambda \sim \mathcal{N}(1+\varepsilon-\mu, (\varepsilon n)^{-1})$, *and assign independent* $\mathrm{Poisson}(\Lambda)$ *variables* $\{D_u : u \in [n]\}$ *to the vertices. Let* $\mathcal{K}$ *be a random multigraph on degrees* ≥ 3 *(cond. the sum is even).*
2. *Replace edges of* $\mathcal{K}$ *by paths of lengths iid* $\mathrm{Geom}(1-\mu)$.
3. *Attach iid* $\mathrm{Poisson}(\mu)$-Galton-Watson *trees to vertices.*

The diameter of $\mathcal{C}_1$

- Theorem 3 [Ding, Kim, L., Peres]:

Let $\mathcal{C}_1$ be the largest component of the random graph $\mathcal{G}(n,p)$ for $p = \frac{1+\varepsilon}{n}$, where $\varepsilon^3 n \to \infty$ and $\varepsilon = o(1)$. Let $\mathcal{C}_1^{(2)}$ be the 2-core of $\mathcal{C}_1$ and $\mathcal{K}$ be its kernel. Then w.h.p.,

$$\operatorname{diam}(\mathcal{C}_1) = (3 + o(1))(1/\varepsilon)\log(\varepsilon^3 n),$$
$$\operatorname{diam}(\mathcal{C}_1^{(2)}) = (2 + o(1))(1/\varepsilon)\log(\varepsilon^3 n),$$
$$\max_{u,v\in\mathcal{K}} \operatorname{dist}_{\mathcal{C}_1^{(2)}}(u,v) = (\tfrac{5}{3} + o(1))(1/\varepsilon)\log(\varepsilon^3 n).$$

- Following this result, [Riordan, Wormald] extended their estimate of $\operatorname{diam}(\mathcal{C}_1)$ to all of the above regime.

Mixing time on $\mathcal{C}_1$

- Theorem 4 [Ding, L., Peres]:

  Let $\mathcal{C}_1$ be the largest component of the random graph $\mathcal{G}(n,p)$ for $p = \frac{1+\varepsilon}{n}$, where $\lambda = \varepsilon^3 n \to \infty$ and $\lambda = o(n)$. The mixing time of the lazy random walk on $\mathcal{C}_1$ is w.h.p. of order $(n/\lambda)\log^2 \lambda$.

- Note the transition from $\Theta(n)$ in critical window and $\Theta(\log^2 n)$ for the strictly supercritical case.

THANK YOU.