Princeton

Apr 2023

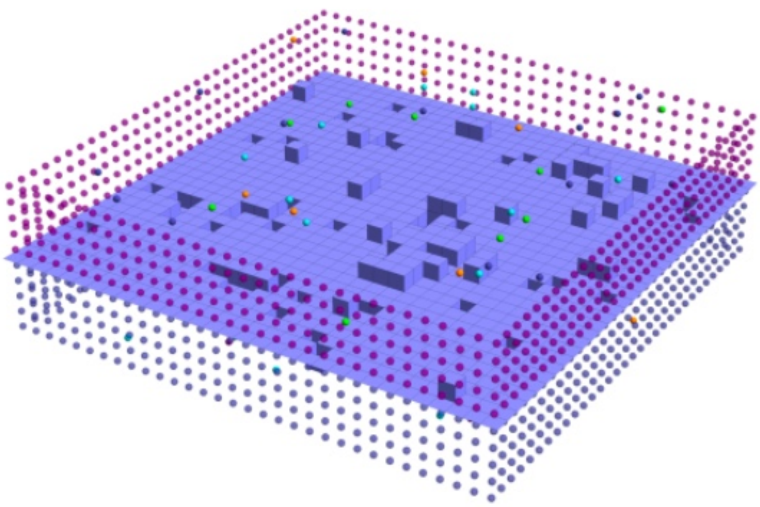
Extrema of Potts interfaces

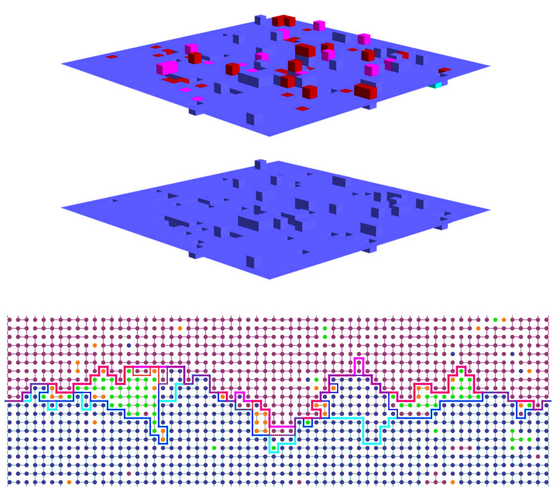
Eyal Lubetzky

Courant Institute (NYU)

Joint work with

Joseph Chen (NYU)



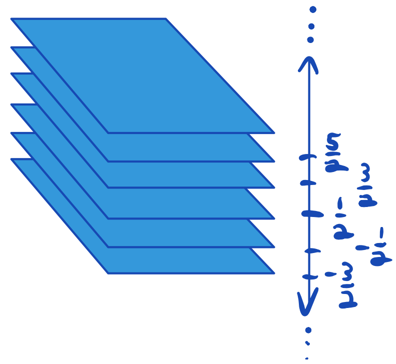


* The q-state POTTS MODEL: $(q \geq 2$ integer$)$

- Underlying geometry:

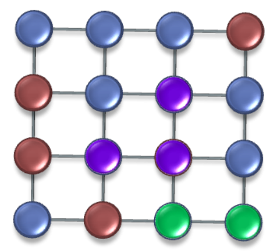
$$\Lambda_n \subset \mathbb{Z}^3.$$

Main focus: $\Lambda_n = [\![-n,n]\!]^2 \times (\mathbb{Z}+\tfrac{1}{2})$



- Set of configurations:

$$\{1,2,\dots,q\}^{\Lambda_n}$$



Probability of a configuration σ:

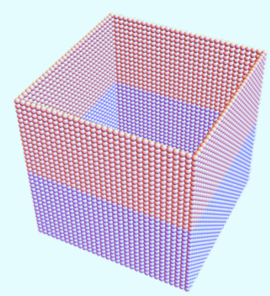
$$\boxed{\pi_n(\sigma) = \frac{1}{Z_p(n,\beta)} \exp\Big[-\beta \sum_{x\sim y} \mathbb{1}_{\{\sigma(x)\neq\sigma(y)\}}\Big]}$$

for $\beta > 0$ the inverse-temperature.

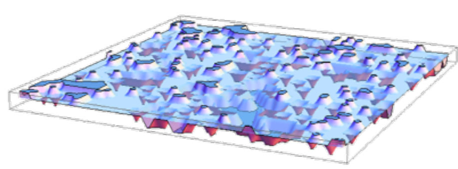
- Boundary conditions (b.c.): fixed coloring of Λ_n^c

- Main focus:

  - b.c. B in lower half space $\mathbb{H}_-$
  - b.c. R in upper half space $\mathbb{H}_+$
  - low temperature: $\beta > \beta_0$ fixed, large.

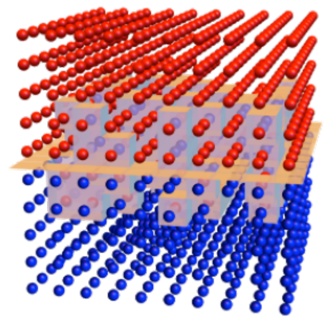


Q: What can we say about the random interfaces between the B and R of the boundary?

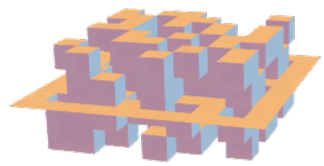


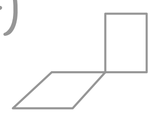
* Background: Ising model (the case $q=2$):

- Spins are either B or R.



- Dobrushin, in the early 1970's, studied the Interface:
  - Take $F_\sigma = \left\{ \begin{array}{l} f = (x,y)^* \text{ (dual) for} \\ x \sim y \text{ s.t. } \sigma(x) \neq \sigma(y) \end{array} \right\}$
  - $\mathcal{I}$ is the 0-connected component of faces in F_σ touching the boundary.



[f, f' are 0-adjacent (or $*$-adjacent) if $f \cap f' \neq \emptyset$ (even via a corner ]

* **Theorem** (Dobrushin, '72, '73) RIGIDITY OF THE INTERFACE

  In 3D Ising on Λ_n at $\beta > \beta_0$, for $\forall\, x = (x_1, x_2), h$

  $$\mathbb{P}\left((x_1, x_2, h) \in \mathcal{I} \right) \leq e^{-\frac{1}{3}\beta h}.$$

w.h.p.: $\mathcal{I}$ flat at height 0 above $0.99\, n^2$ faces $x \in [\![-n, n]\!]^2$.

* Corollaries

① $\exists$ non-translation invariant $\mathbb{Z}^3$ Gibbs measures.

② Max & Min height of $\mathcal{I}$ are $\leq \frac{10}{\beta} \log n$ w.h.p.

* Some of the follow-up works on LOW TEMP ISING:

- [van Beijeren '75] :
  alternative simpler argument for rigidity.
- [Bricmont, Lebowitz, Pfister, Olivieri '79a, '79b, '79c]
  extension of the rigidity argument to the Widom-Rowlinson model.
- [Gielis, Grimmett '02] :
  extension of rigidity argument to super-critical percolation / Random Cluster model conditioned to have interfaces.

- many other works on the Wulff shape, LD for magnetization, surface tension
  [Pisztora '96] , [Bodineau '96] , [Cerf, Pisztora '00]
  [Bodineau '05] , [Cerf '06] , ...

* Recent progress : Ising model (the case $q=2$) :

* Gheissari and L. ('21, '22) identified the correct exponential rate :

$$\boxed{\mathbb{P}\left((x_1,x_2,h)\in\mathcal{I}\right)=e^{-(\alpha+o(1))h} \quad \text{as } h\to\infty.}$$

for an explicit $\alpha\in 4\beta\pm C$.

* Led to the following result on Max / Min of $\mathcal{I}$:

Theorem: (Gheissari, L., '21, '22):

Tightness of MAX (MIN) of $\mathcal{I}$

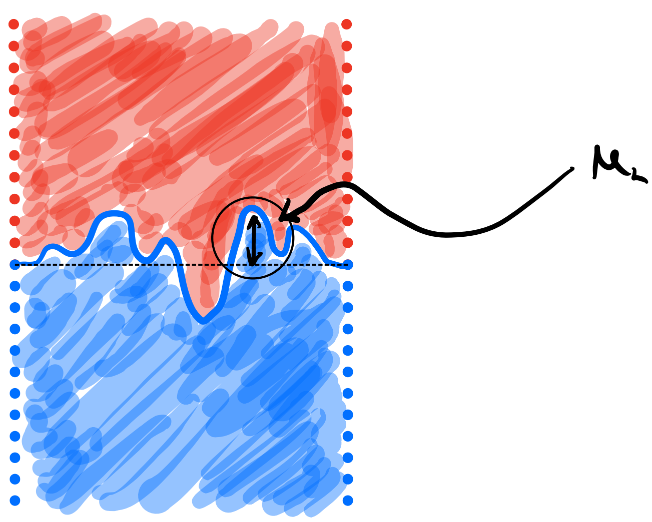
M_n the max height of $\mathcal{I}$ satisfies

$$M_n-\mathbb{E}M_n=O_p(1)$$

and

$$\mathbb{E}M_n=\left(\tfrac{2}{\alpha}+o(1)\right)\log n.$$

the interface $\mathcal{I}$ a.s. separates the two b.c. phases (finite bubbles aren't drawn)



Q: Analogue in Potts ??

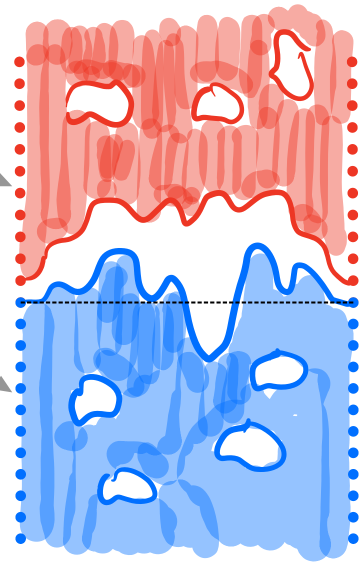
* Interfaces in the Potts model :

$\exists$ (a.s.) unique inf RED component :

V_{RED}

$\exists$ (a.s.) unique inf BLUE component :

V_{BLUE}



* Equiv. : say $V_{RED} = \{ v : \exists$ path of red vertices connecting it to $\partial\Lambda_n^+ \}$

$:= \partial\Lambda_n \cap \mathbb{H}_+$

$V_{BLUE} = \{ v : \exists$ path of blue vertices connecting it to $\partial\Lambda_n^- \}$

* Augment the components :

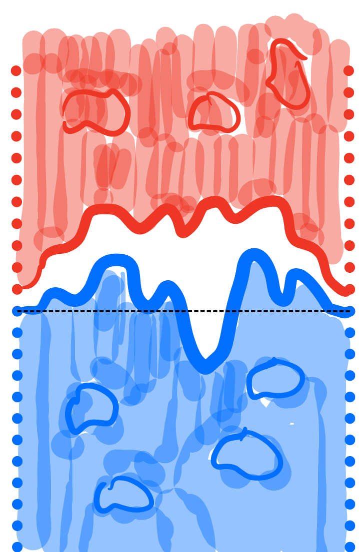
$$\hat{V}_{RED} = V_{RED} \cup \{ \text{finite components of } V_{RED}^c \}$$

$$\hat{V}_{BLUE} = V_{BLUE} \cup \{ \text{finite components of } V_{BLUE}^c \}$$

* Interface :

$\hat{V}_{RED}$

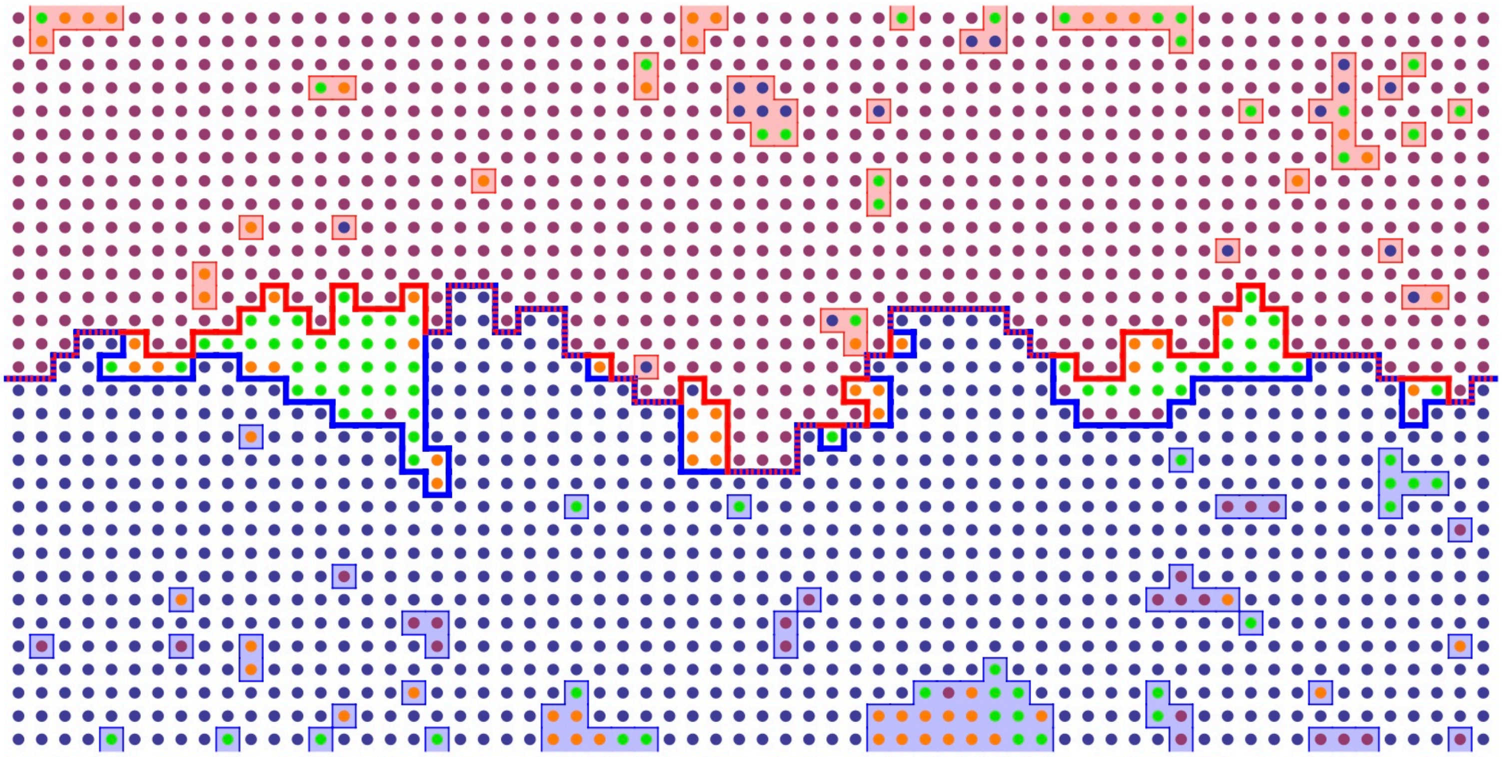
$\hat{V}_{BLUE}$

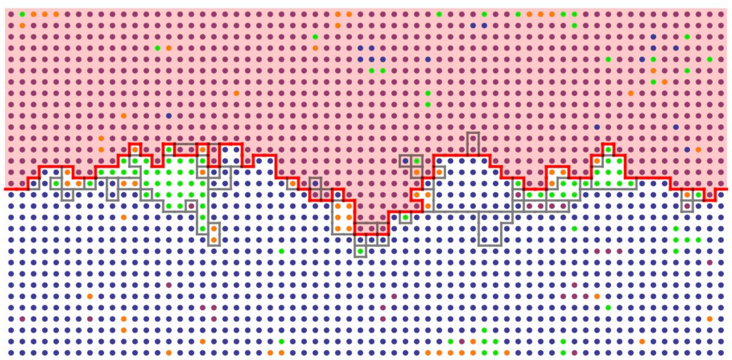


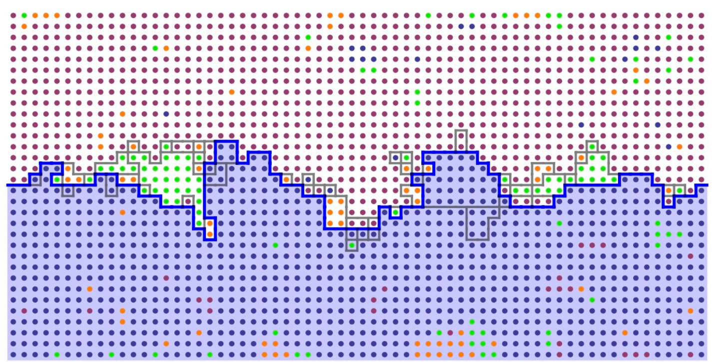
$$I_{BLUE} = \left\{ f = (u,v)^* \text{ for } \begin{matrix} u \in \hat{V}_{BLUE} \\ v \notin \hat{V}_{BLUE} \end{matrix} \right\}$$

analogously :

$$I_{RED} = \left\{ f = (u,v)^* \text{ for } \begin{matrix} u \in \hat{V}_{RED} \\ v \notin \hat{V}_{RED} \end{matrix} \right\}$$







$$I_{RED} = \left\{ f = (u,v)^{*} \quad \text{for} \quad \begin{array}{l} u \in \hat{V}_{RED} \\ v \notin \hat{V}_{RED} \end{array} \right\}$$

$$I_{BLUE} = \left\{ f = (u,v)^{*} \quad \text{for} \quad \begin{array}{l} u \in \hat{V}_{BLUE} \\ v \notin \hat{V}_{BLUE} \end{array} \right\}$$

* New results jointly with JOSEPH CHEN (NYU):

Theorem 1: [Chen, L.]

Consider q-state Potts on $\Lambda_n = [\![-n,n]\!]^2 \times (\mathbb{Z}+\frac{1}{2})$ w. Dobrushin b.c., $q \geq 2$ and $\beta > \beta_0$ (fixed).

Let M_n = MIN height of $\mathcal{I}_{\text{BLUE}}$

M'_n = MAX height of $\mathcal{I}_{\text{BLUE}}$

Then:

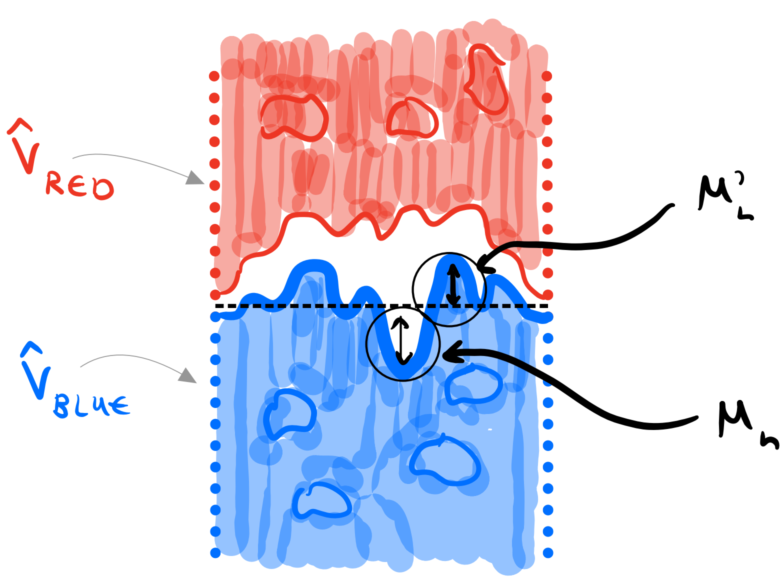
$$M_n - \mathbb{E}[M_n] = O_p(1)$$
$$M'_n - \mathbb{E}[M'_n] = O_p(1)$$

(Tightness)

Moreover, $\exists\ \gamma, \gamma' > 0$ s.t.

$$\mathbb{E} M_n = \left(\frac{2}{\gamma} + o(1)\right) \log n \quad , \quad \mathbb{E} M'_n = \left(\frac{2}{\gamma'} + o(1)\right) \log n$$

and $\gamma' > \gamma$ for $q \neq 2$. $(\gamma = \gamma'$ for $q = 2)$

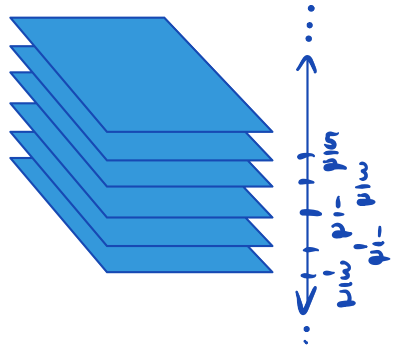


* **The Random Cluster model:** $(q>1,\ 0<p<1)$

- Underlying geometry:

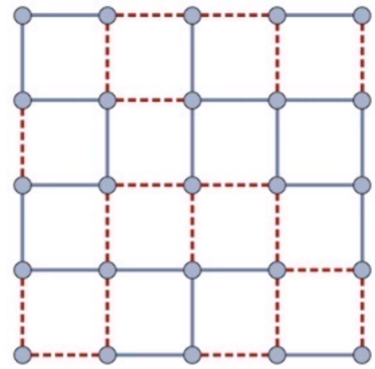
$$\Lambda_n \subset \mathbb{Z}^3 .$$

Main focus : $\Lambda_n = [\![-n,n]\!]^2 \times (\mathbb{Z}+\tfrac{1}{2})$



- Set of configurations:

$$\{\omega :\ \omega \subseteq E(\Lambda_n)\}$$



- Probability of a configuration ω :

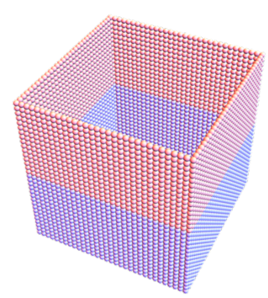
$$\boxed{\mu_n(\omega) = \frac{1}{Z_{RC}(n,p,q)}\, p^{|\omega|}\,(1-p)^{|E\setminus\omega|}\, q^{k(\omega)}}$$

open edges in ω # closed edges in ω # conn. components in ω

* Boundary conditions: a partition of the vertices of $\partial\Lambda_n$ to conn. comp (conn. "outside" of Λ_n).

- Main focus: all conn.

  - b.c. WIRED in lower half space $\mathbb{H}_-$
  - b.c. WIRED in upper half space $\mathbb{H}_+$
  - low temperature: $p > 1-\varepsilon_0$ fixed, large.



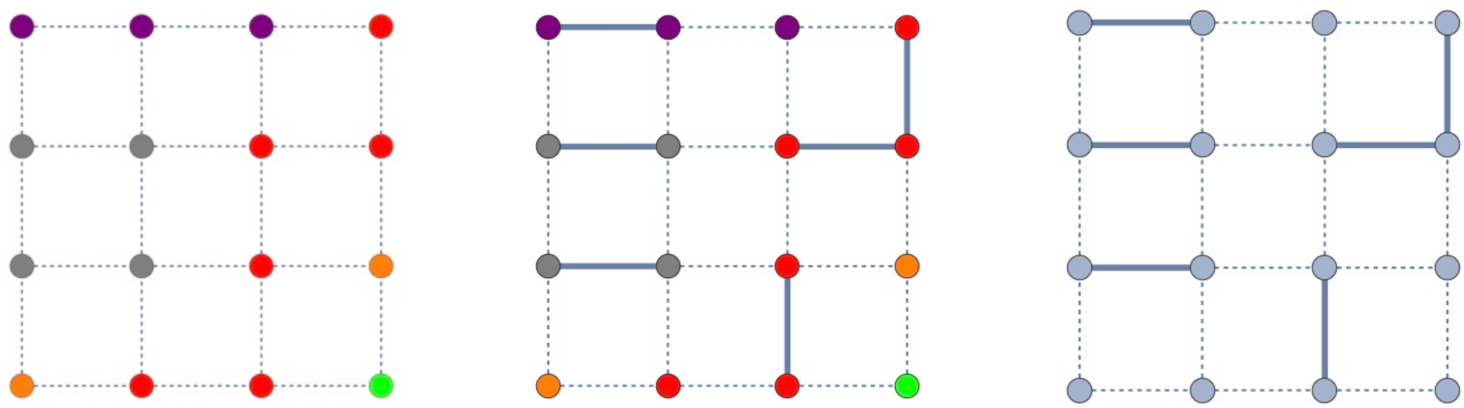
* [Edwards – Sokal '88] coupling $\phi = (\overset{\text{Potts}}{\pi}, \overset{\text{RC}}{\mu})$:

$$\boxed{\begin{aligned} &p := 1 - e^{-\beta} \\ &\phi_{p,q}(\sigma, \omega) = \frac{1}{Z_{ES}(n,p,q)}\, p^{|\omega|} (1-p)^{|E\setminus\omega|} \prod_{(x,y)\in\omega} \mathbb{1}_{\{\sigma(x)=\sigma(y)\}} \end{aligned}}$$

$\left[\pi(\sigma) \propto (1-p)^{\#\{x\sim y\,:\,\sigma(x)\neq\sigma(y)\}} \;;\; \mu(\omega) \propto p^{|\omega|}(1-p)^{|E\setminus\omega|} q^{k(\omega)}\right]$

\- moving between [Swendsen–Wang '87]

$$\sigma \xrightarrow[\text{color clusters}]{p\text{-percolation on}} (\sigma, \omega) \xleftarrow[\text{IID Uni}(\{1,\dots,q\})]{\text{color } \forall \text{ conn comp}} \omega$$

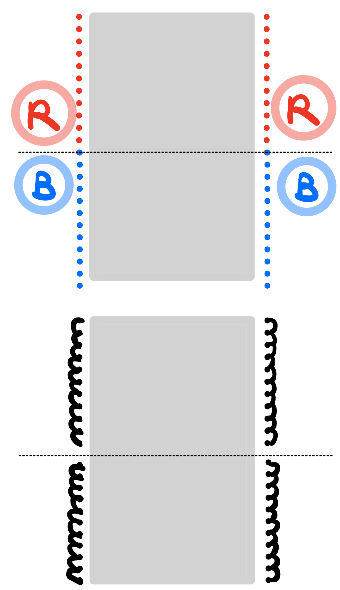


* σ in this coupling: Potts

corresponds to RC wired $\partial\Lambda_n^+$, wired $\partial\Lambda_n^-$

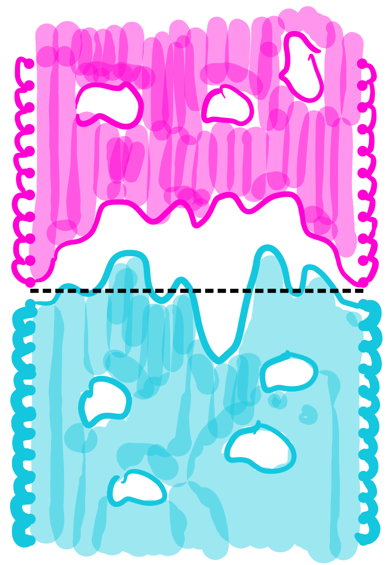
$$\boxed{\text{conditioned on } \nexists\ \partial\Lambda_n^+ \overset{\omega}{\longleftrightarrow} \partial\Lambda_n^-} =: \Omega_n$$

i.e., $\exists$ an interface I.



* Interfaces in the RC model:

$\exists$ (a.s.) unique inf Top component: V_{Top}

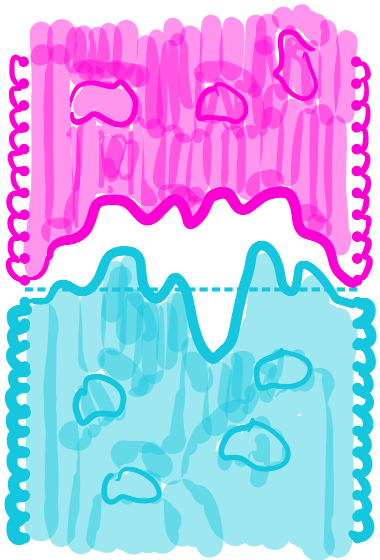


$\exists$ (a.s.) unique inf BoT component: V_{BoT}

* Augment the components:

$$\hat{V}_{Top} = V_{Top} \cup \{\text{finite components of } V_{Top}^c\}$$

$$\hat{V}_{BoT} = V_{BoT} \cup \{\text{finite components of } V_{BoT}^c\}$$



* Interfaces:

$$I_{Top} = \left\{ f = (u,v)^* \text{ for } \begin{matrix} u \in \hat{V}_{Top} \\ v \notin \hat{V}_{Top} \end{matrix} \right\}$$

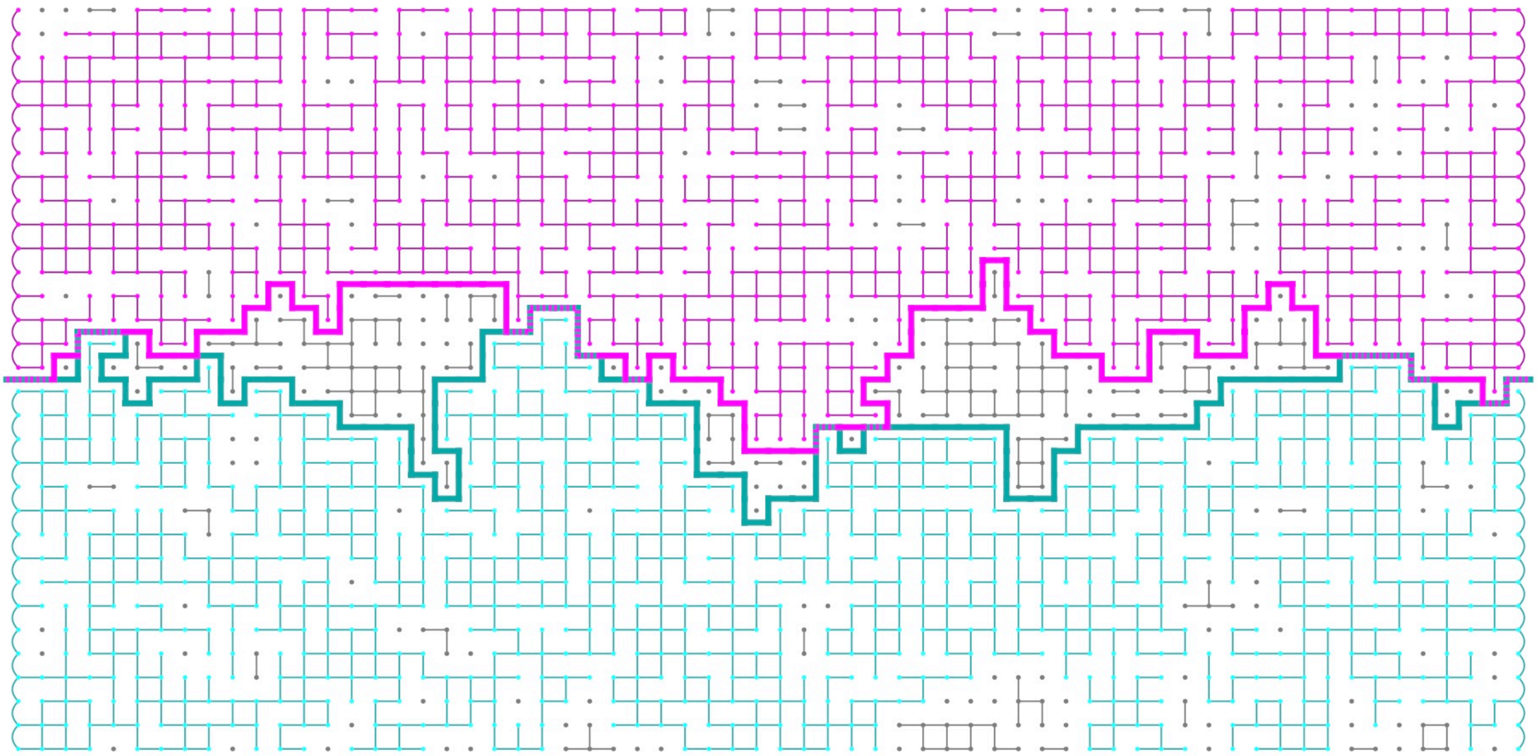
analogously:

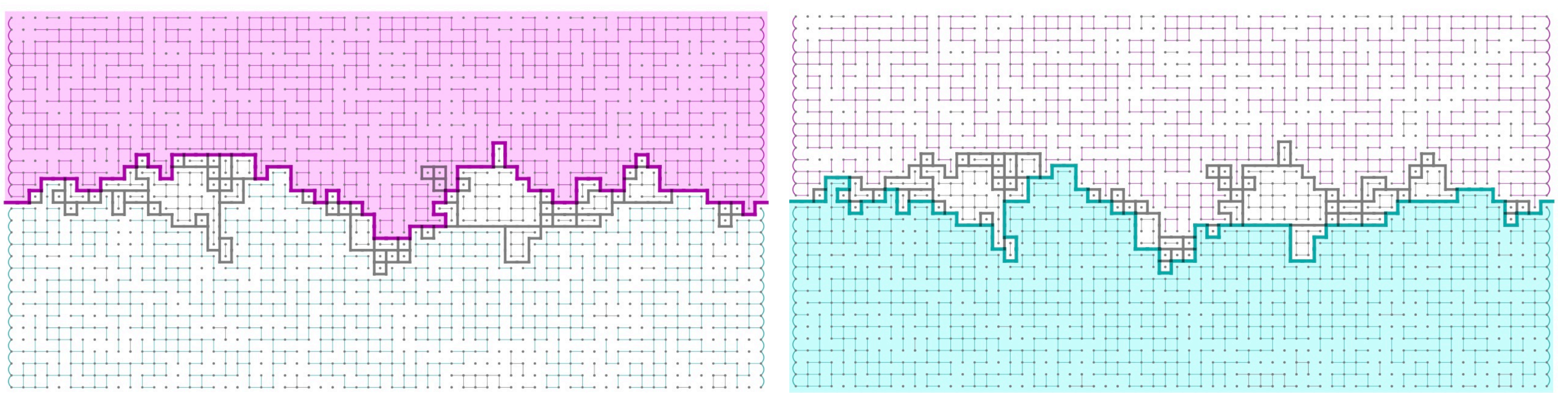
$$I_{BoT} = \left\{ f = (u,v)^* \text{ for } \begin{matrix} u \in \hat{V}_{BoT} \\ v \notin \hat{V}_{BoT} \end{matrix} \right\}$$

Last but not least:

$[f = e^* \text{ s.t. } e \notin \omega]$

$$I = \{ \text{1-connected comp of dual-closed faces touching the boundary} \}$$





$$I_{Top} = \left\{ f = (u,v)^* \quad \text{for} \quad \begin{matrix} u \in \hat{V}_{Top} \\ v \notin \hat{V}_{Top} \end{matrix} \right\}$$

$$I_{Bot} = \left\{ f = (u,v)^* \quad \text{for} \quad \begin{matrix} u \in \hat{V}_{Bot} \\ v \notin \hat{V}_{Bot} \end{matrix} \right\}$$

* New results on RC: $(p = 1 - e^{-\beta})$

Theorem 2: [Chen, L.]

Consider the RC model on $\Lambda_n = [\![-n, n]\!]^2 \times (\mathbb{Z} + \frac{1}{2})$ w. Dobrushin b.c., $q > 1$ and $\beta > \beta_0$ (fixed) condition on the existence of $\mathcal{I}$.

Let M_n = MIN height of $\mathcal{I}_{BOT}$

M'_n = MAX height of $\mathcal{I}_{BOT}$

Then:

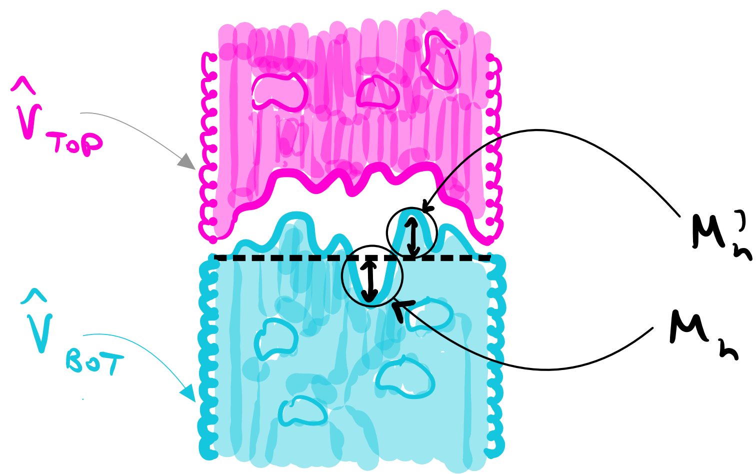
$$M_n - \mathbb{E}[M_n] = O_p(1)$$
$$M'_n - \mathbb{E}[M'_n] = O_p(1)$$

(Tightness)

Moreover, $\exists\ \alpha, \alpha' > 0$ s.t.

$$\mathbb{E} M_n = \left(\frac{2}{\alpha} + o(1)\right) \log n \quad , \quad \mathbb{E} M'_n = \left(\frac{2}{\alpha'} + o(1)\right) \log n$$

and $\alpha' > \alpha$.



* A Tale of Four Rates:

We can compare the rates as follows:

Theorem: [Chen, L.]

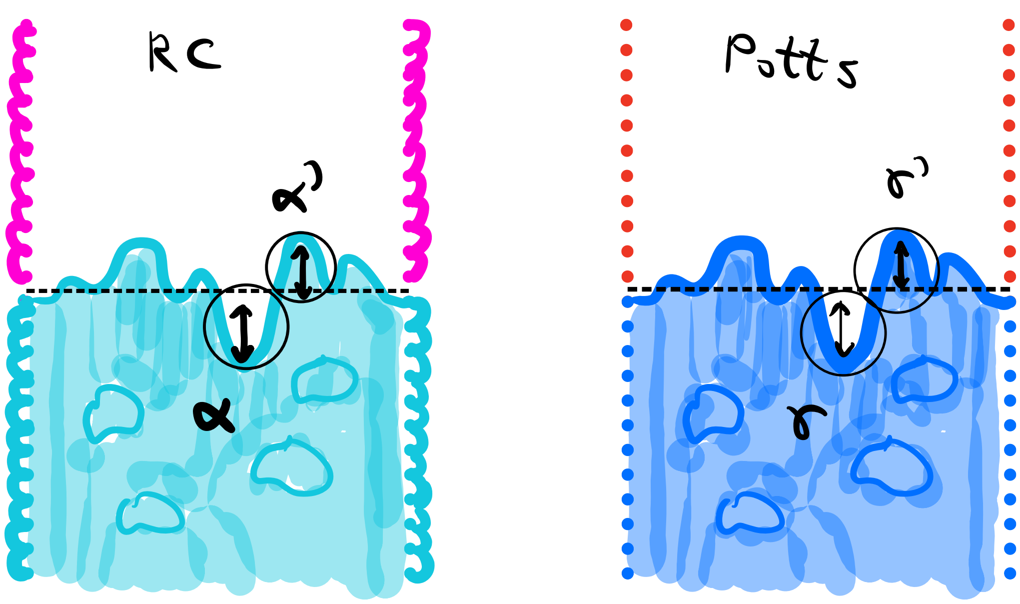
The rates from Thms. 1, 2 satisfy:

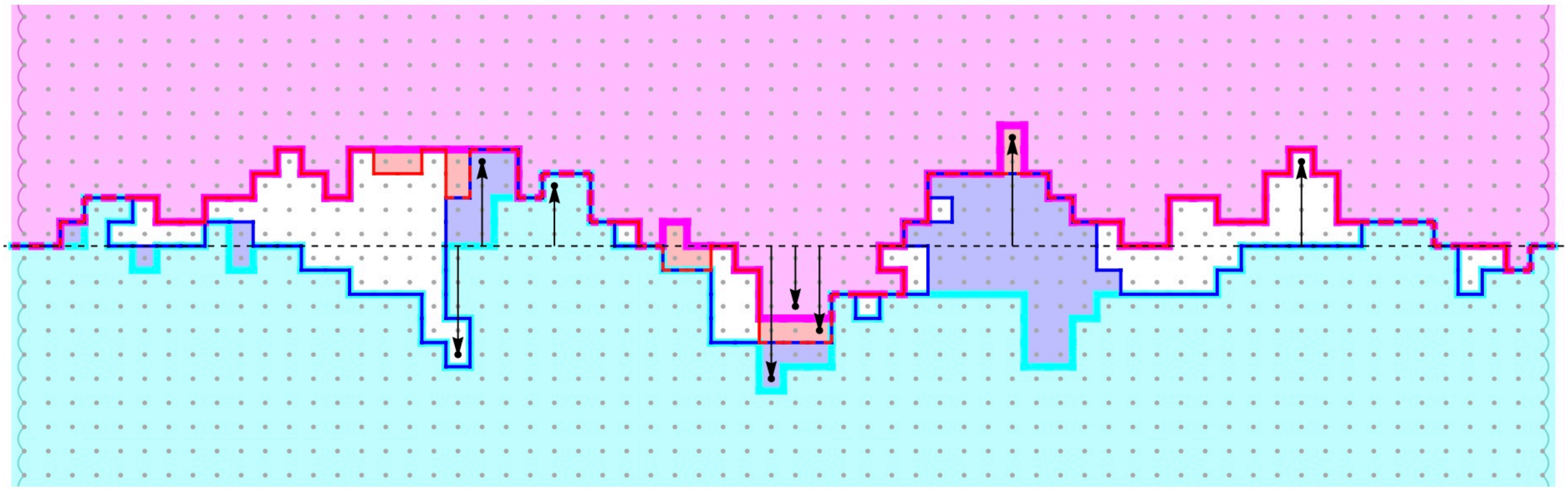
$$4\beta - C \le \alpha \le 4\beta$$

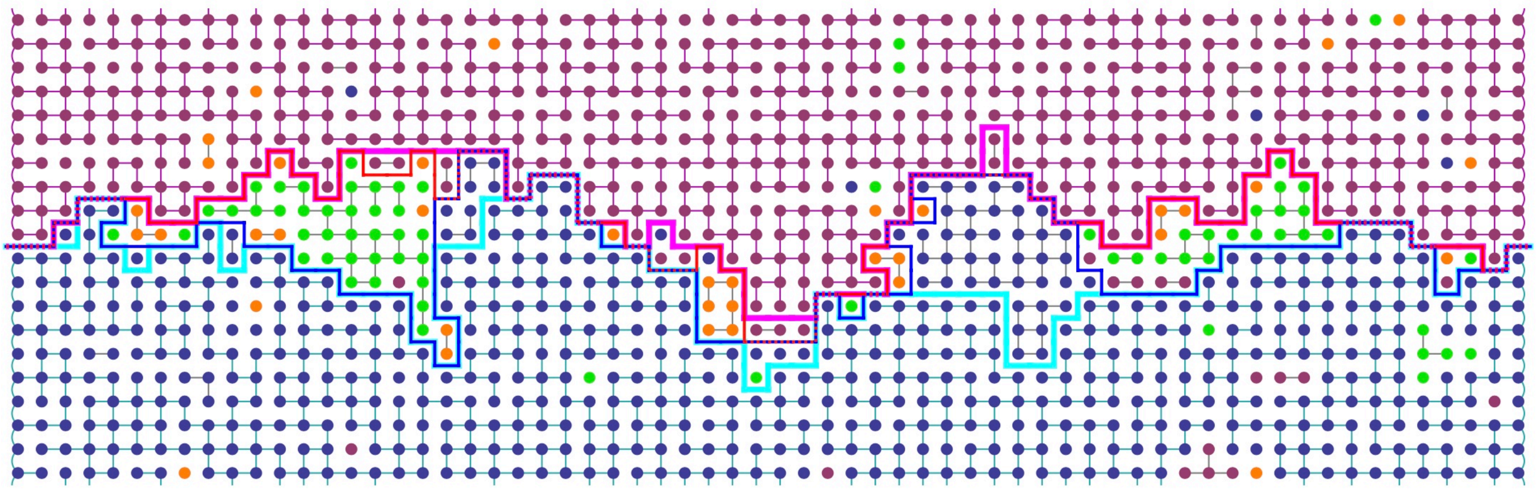
$$\gamma - \alpha = (1 \pm \varepsilon_\beta)\, e^{-\beta}$$

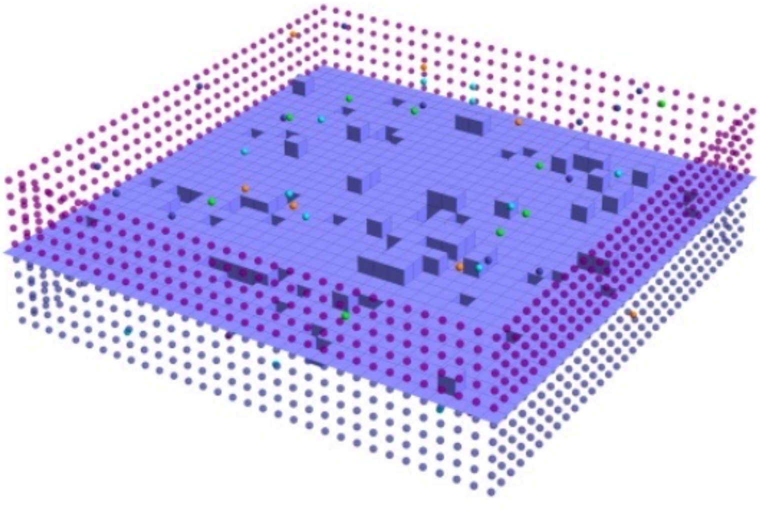
$$\gamma' - \alpha = (1 \pm \varepsilon_\beta)\,(q-1)\, e^{-\beta}$$

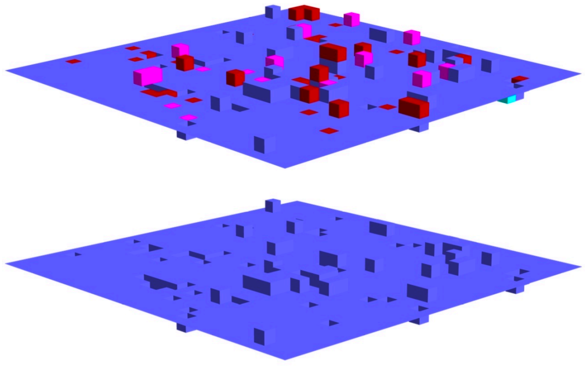
$$\alpha' - \alpha = (1 \pm \varepsilon_\beta)\, q\, e^{-\beta}$$











* How did the Ising pf work? (and why does it fail for Potts)

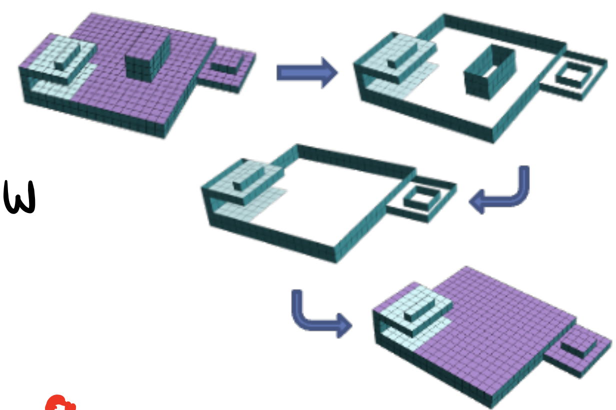
* Step I: Cluster expansion: [Minlos, Sinai '67] [Dobrushin '72]

$$\boxed{\mathbb{P}(\mathcal{I}) = \frac{1}{Z} e^{-\beta|\mathcal{I}| + \sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})}}$$

faces in $\mathcal{I}$

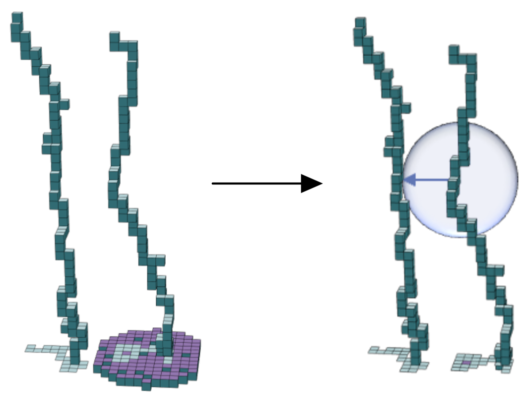
interaction func $\mathbf{g}$: uniformly bdd, local

* Step II: Dobrushin's rigidity framework:
  - classify conn sets of "excess" faces in $\mathcal{I}$ as WALLS
  - rest are CEILINGS.
  - to show rigidity: attempt to delete a wall W gaining $\beta|W|$



  - the tricky part: controlling $\mathbf{g}$.

E.g.: deleting W may shift other parts of $\mathcal{I}$ which accumulate interaction terms...



  - Must continue deleting "nearby" walls.
  - Dobrushin grouped walls together via size vs. distance to make this arg work.

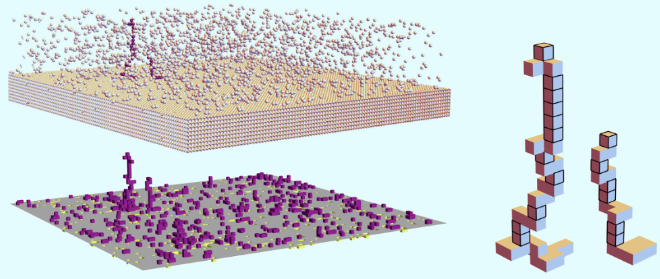
* Step (III): From WALLS to PILLARS:

- Dobrushin's deletion of complete groups-of-walls is too crude to recover LD rates:

Instead: [Gheissari, L.] looked at the

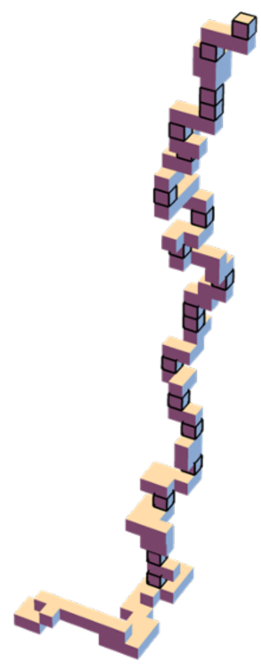
PILLAR $\mathcal{P}_x$:

the conn. comp in $\mathbb{H}_+$ of (+) spins containing x



- Conditional on the event
$$E_h^x = \{ \mathrm{ht}(\mathcal{P}_x) \geq h \}$$
it should behave as a (directed) RW in $\mathbb{Z}^3$: with regeneration pts.

- Break it into increments



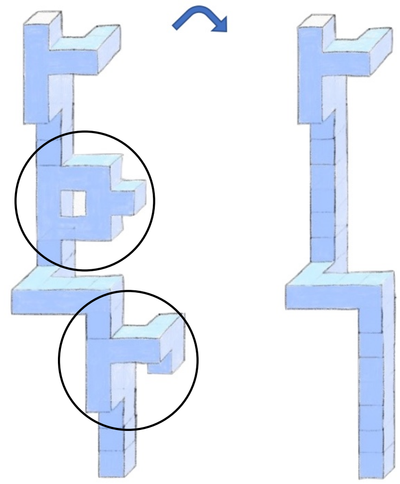
Goal:

(a) Show that a given increment tends to be "trivial": a cube (4 side faces)

(b) Including 1st (exceptional) increment

- How do we show (a) ?

By "straightening" $\mathcal{P}_x$:



* replacing i-th increment X_i by a trivial one.
* doing so for any j-th incr X_j whose size is too large compared to $\mathrm{dist}(X_i, X_j)$.

- How do we show (b) ?

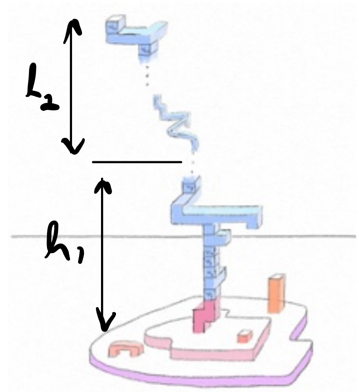
Complicated algorithm for modifying $\mathcal{I}$.

* Step (IV) : The LD rate α :

- $\mathcal{P}_x$ concerns a component of (+).

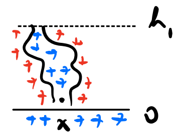
Can't we use FKG for SUPER - MULTIPLICATIVITY ?

No : due to the b.c. at height h_1 we are more negative.



- Instead : SUB - MULTIPLICATIVITY : (à la "BK-inequality")

Use $\boxed{\text{monotonicity}}$ and properties (a), (b).



* Random Cluster to the rescue?

- The toolkit to handle pillars is robust, but without the sub-mult argument: of no value...
- While $\mathbb{P}(\mathcal{I} \in \cdot)$ in Ising does not sat. FKG, the Ising dist on configurations does: monotonicity used in a crucial way.
- Standard remedy to Potts non-monotonicity: RC.

* [Gielis-Grimmett '02] extended the framework of Dobrushin to RC cond on an interface: call this measure

$$\boxed{\bar{\mu}_n = \mu_n(\cdot \mid \mathcal{D}_n)}$$

top b.c. $\partial\Lambda_n^+$ DISCONNECTED from bottom b.c. $\partial\Lambda_n^-$

* Still No montonicity because of the cond. on the (exponentially unlikely) event $\mathcal{D}_n$.
* However: at least the RC measure μ_n is monotone

* Cluster expansion and rigidity pf give us the foundations for studying $\mathcal{I}$ in $\bar{\mu}_n$

* The RC interface $\mathcal{I}$:

[$f = e^*$ s.t. $e \notin \omega$]

$\mathcal{I}$ = { 1-connected comp of dual-closed faces touching the boundary }

NOT the interface we'd want to study but the one [GG '02] developed tools for:

No longer just a surface

* BUT : many complications:

- Cluster Expansion:

$$\boxed{\bar{\mu}_n(\mathcal{I}) \propto (1-e^{-\beta})^{|\partial \mathcal{I}|} \, q^{k_{\mathcal{I}}} \, e^{-\beta|\mathcal{I}| + \sum_{f \in \mathcal{I}} g(f,\mathcal{I})}}$$

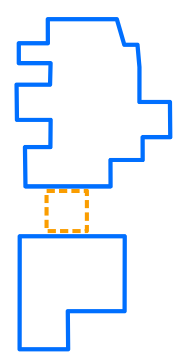
(dual open) faces 1-conn to $\mathcal{I}$ but not in $\mathcal{I}$

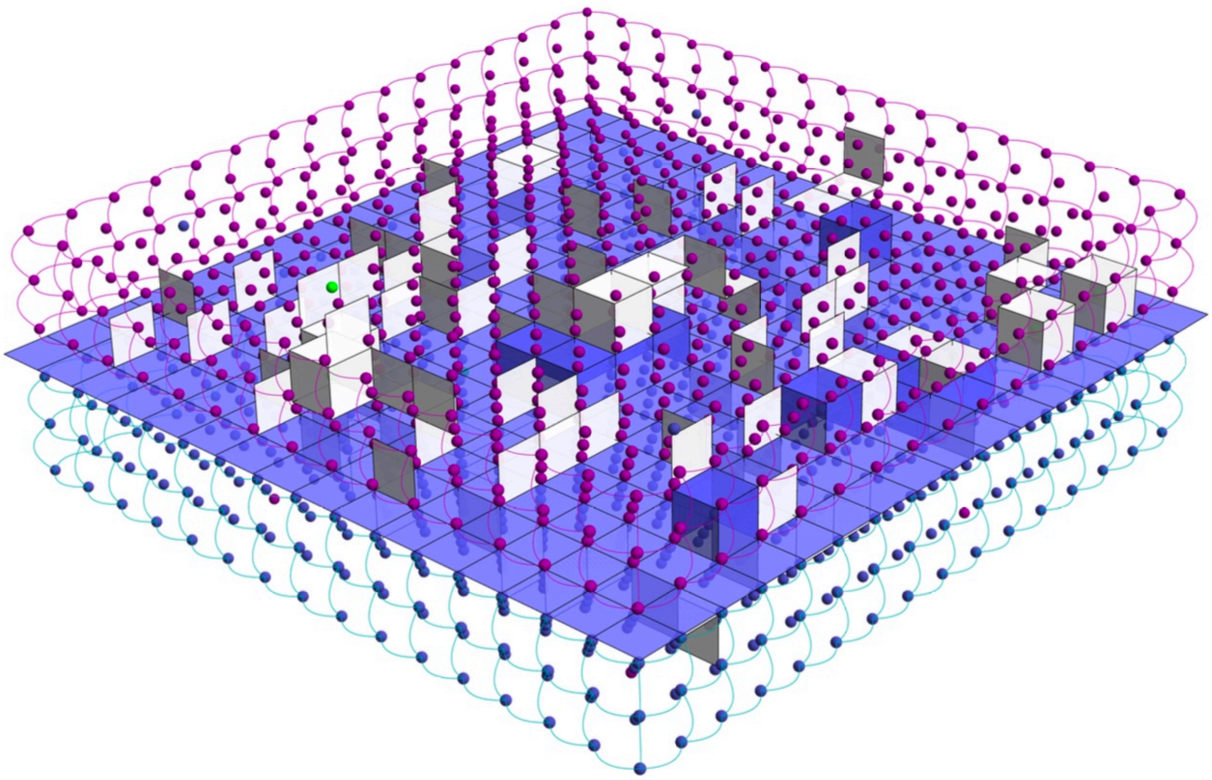
open clusters in the conf dual to $\mathcal{I}$

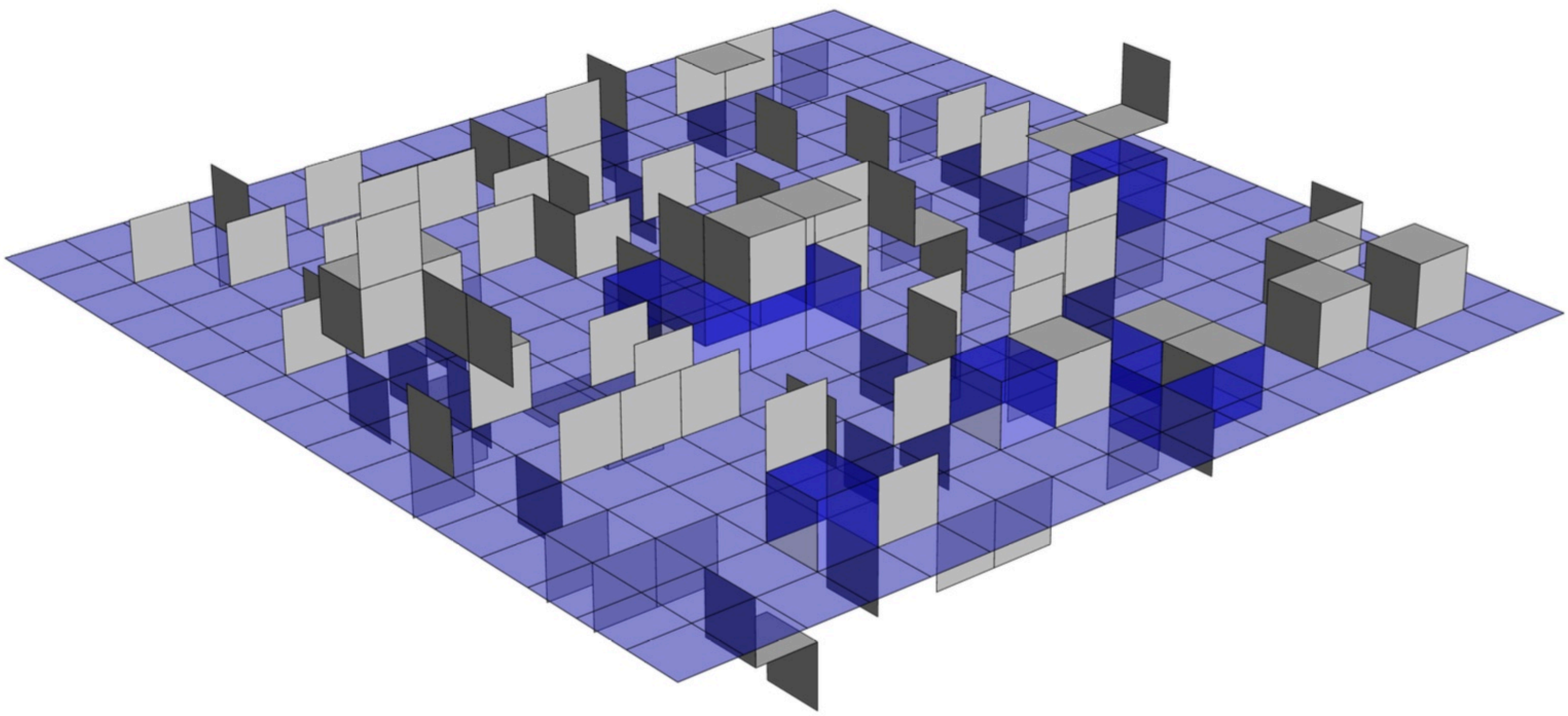
[In accordance with the RC $p^{\#\text{open}} (1-p)^{\#\text{closed}} q^{\#\text{conn comp}}$]

- Walls & Ceilings: done w.r.t. extending $\mathcal{I}$ into $\mathcal{I}^*$ via some open faces:

$$\boxed{\mathcal{I}^* := \mathcal{I} \cup \{ f \in \partial \mathcal{I} \text{ horizontal} \}}$$







* The RC PILLAR $\mathcal{P}_x$:

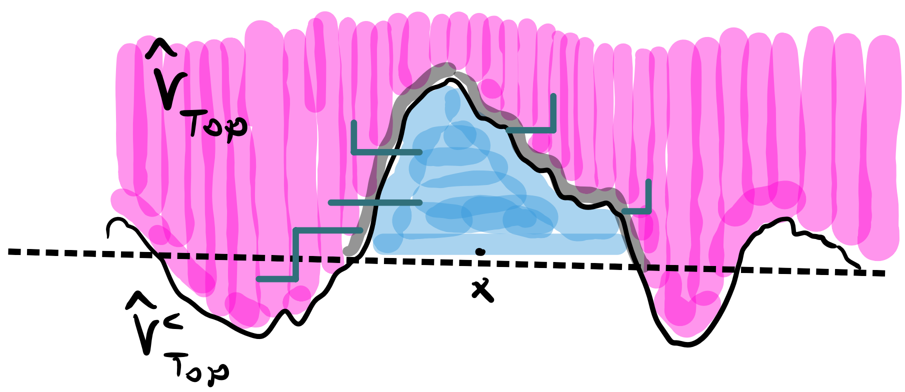
- Recall : Ising PILLAR = the Λ_n-conn. comp in $\mathbb{H}_+$ of $\oplus$ spins containing x

* RC PILLAR $\mathcal{P}_x$: the Λ_n-conn. comp in $\hat{V}^c_{Top} \cap \mathbb{H}_+$ containing x

Its faces def. by taking

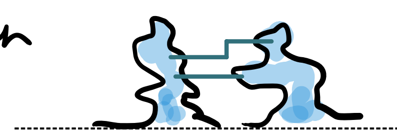
$$F = \{ f = (u,v)^* : u \in \mathcal{P}_x , v \in \mathbb{H}_+ \setminus \mathcal{P}_x \}$$

and adding to it any 1-conn. comp of faces $\mathcal{C}$ in $\mathcal{I} \setminus \mathcal{I}_{Top}$ s.t. $\mathcal{C} \cap \mathcal{P}_x \cap \mathbb{H}_+ \neq \emptyset$



- Added "hairs" necessary to deal with $\partial\mathcal{I}$ in the [GG'02] cluster expansion.

- But now separate pillars can touch each other ...



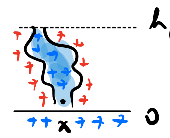
* Suppose we could control the pillar $\mathcal{P}_x$.

What about the SUB-MULTIPLICATIVITY argument?

- The goal: show

$$\bar{\mu}_n(A_{h_1+h_2}) \le \bar{\mu}_n(A_{h_1})\,\bar{\mu}_n(A_{h_2})$$

- In Ising: we exposed a $+$ component, by def surrounded by $-$'s.



- Here: much more delicate to def faces of $\mathcal{I}$ we expose to support a Domain Markov Property (starting from the open faces $\partial\mathcal{I}$)

- Last but not least: the missing bar:

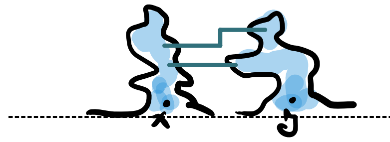
Even if this recipe gave

$$\bar{\mu}_n(A_{h_1+h_2}) \lesssim \bar{\mu}_n(A_{h_1})\,\mu_n(A_{h_2})$$

then the last term on the RHS is in a graph with different b.c. (no longer the $\bar{\mu}_n$ measure)

* <u>Some of the ideas to bypass these obstacles:</u>

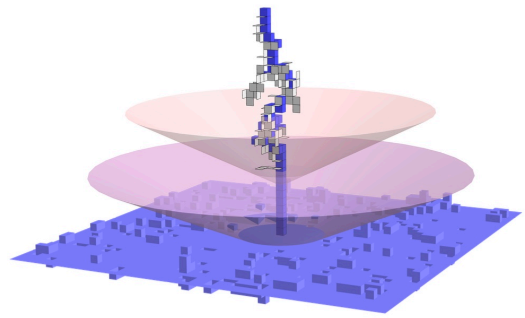
* $\mathcal{P}_x$ affects $\mathcal{P}_y$ via "hairs":



establish that "typically"

$\mathcal{P}_x \in$ cone

devoid of other walls



* Offset the new terms $(1-e^{-\beta})^{|\partial I|}\, q^{k_I}$

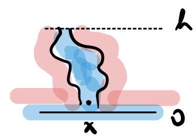
in the [GG '02] cluster expansion

via deleted faces in the "straightening" of $\mathcal{P}_x$.

* Approximate the event $E_h^x = \{ \mathrm{ht}(\mathcal{P}_x) \geq h \}$

by a suitable A_h^x that is

(i) amenable to exposing certain faces of I forming a b.c. on the graph above height h



(ii) not very sensitive to Ω_n at large h

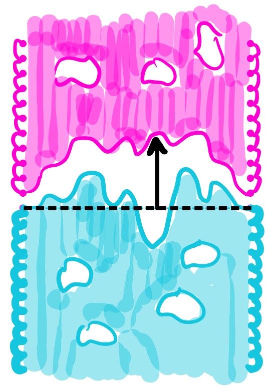
then add it to RHS by $\boxed{\text{monotonicity}}$:

$$\mu_n(A_{h_2}) \leq \bar{\mu}_n(A_{h_2})$$

Only works for a <u>DECREASING</u> A_h !

* <u>Carrying out the program:</u>

Max height of I_{Top}:



governed by:

$$\boxed{\alpha := \lim_{h\to\infty} -\frac{1}{h}\log \bar{\mu}_{\mathbb{Z}^2}\left(\mathrm{ht}(\mathcal{P}_x)\geq h\right)}$$

✓

* What about its Min height ?

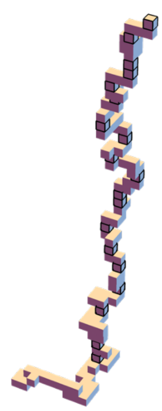
???

proof breaks down...

* What about Potts ?

<u>Promising approach:</u>

- Cond. on $\{\mathrm{ht}(\mathcal{P}_x)\geq h\}$ in the RC model, it behaves like a RW, in that its increments are asymp. stationary $\approx \nu$
  mixing
  measure on incr



- By the [ES] coupling, we need to consider the coloring of its interior.
- $\log \mathbb{P}\left(\;\; \mid \mathrm{ht}(\mathcal{P}_x)\geq h\right)$ will be approx a sum of h IID r.v.'s : $\log \mathbb{P}_\nu(\exists \{\;\} \in \mathcal{X})$

* The (retrospectively obvious) fault:

A typical $\mathcal{P}_x$ in $\bar{\mu}_n(\cdot \mid \mathrm{ht}(\mathcal{P}_x) \geq h)$ has the above structure.

But the Max of I_{BLUE} might (and will!) come from an atypical $\mathcal{P}_x$.

(Most increments should be trivial, still)

* Complicated optimization: shape of $\mathcal{P}_x$ wants to minimize surface area, but also give many options for BLUE paths climbing to h.

* Solution: show existence of the rate (rather than what its value is) by another SUB-MULTIPLICATIVITY argument:

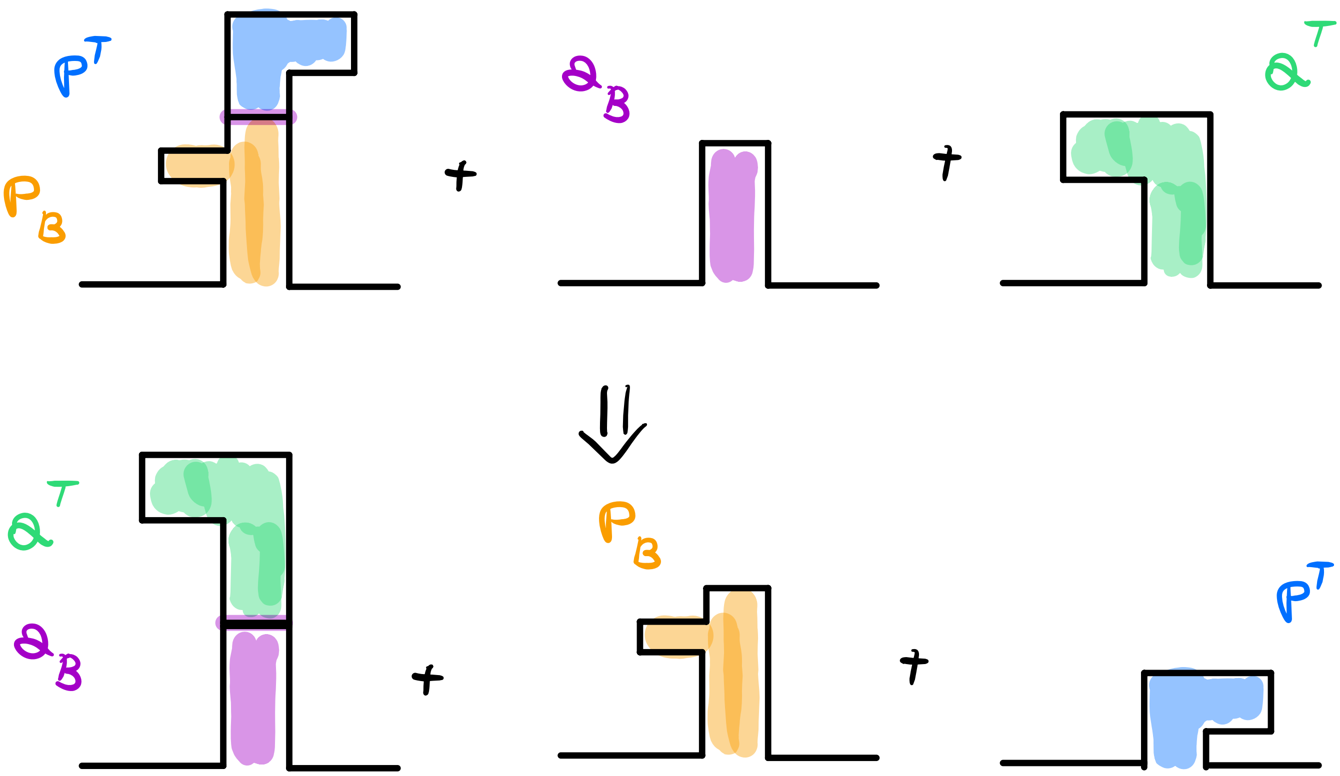
$$\phi_n\left(\mathrm{ht}(P_x^{\text{blue}}) \geq h_1 + h_2 \;\middle|\; \mathrm{ht}(P_x) \geq h_1 + h_2\right)$$

$$\leq C\, \phi_n\left(\mathrm{ht}(P_x^{\text{blue}}) \geq h_1 \;\middle|\; \mathrm{ht}(P_x) \geq h_1\right)$$

$$\cdot\, \phi_n\left(\mathrm{ht}(P_x^{\text{blue}}) \geq h_2 \;\middle|\; \mathrm{ht}(P_x) \geq h_2\right)$$

BUT HOW ??

* The 3-to-3 map:



cannot afford additive errors: we're cond on E_h^x ...

* want to show that:

$$\boxed{\nu(P_B \times P^T) \leq (1+\varepsilon)\, \nu_1(P_B)\, \nu_2(P^T)}$$

write

$$\nu(P_B \times P^T) - \nu_1(P_B)\, \nu_2(P^T)$$

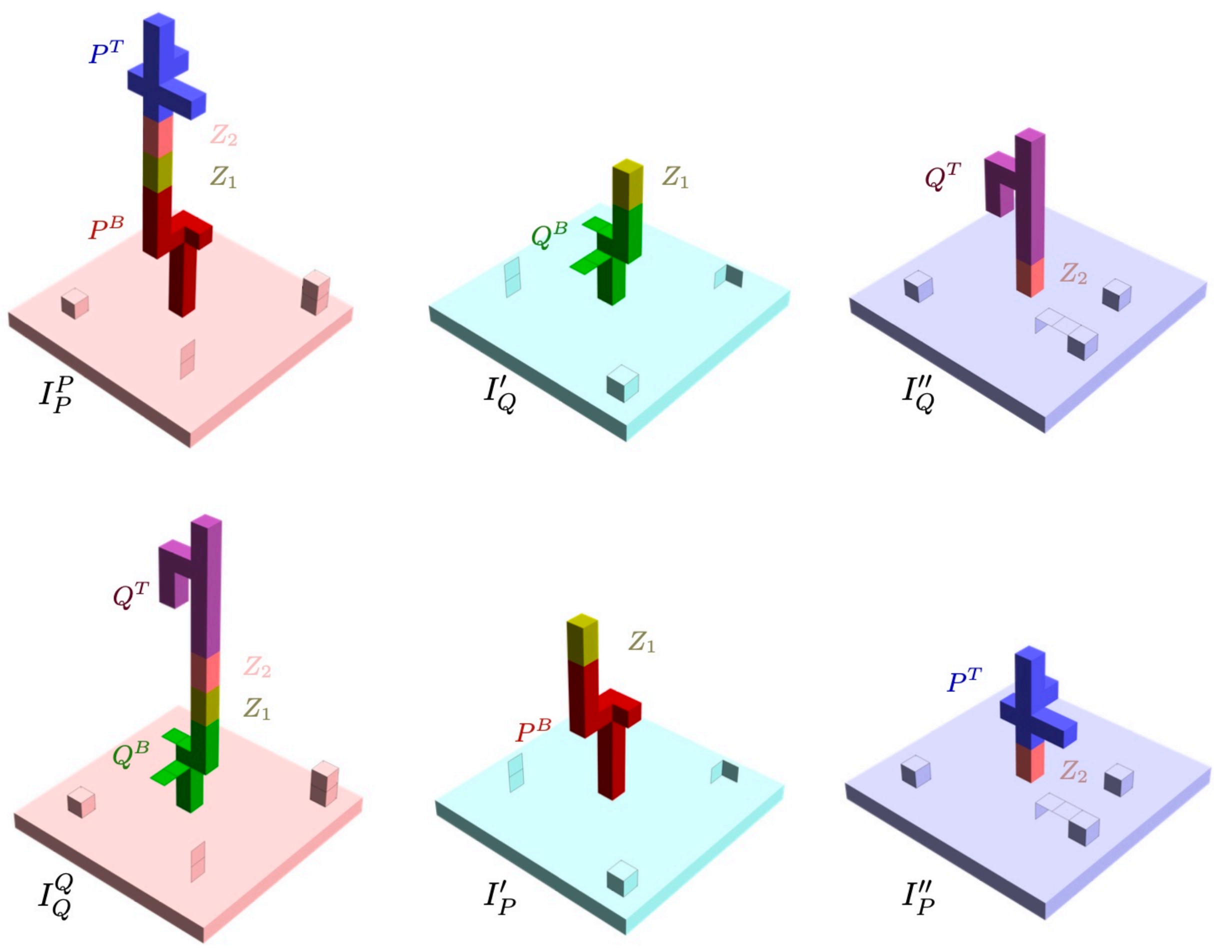
$$= \sum_{A, A_1, A_2} \left[\nu(P_B \times P^T, A) - \nu_1(P_B, A_1)\, \nu_2(P^T, A_2) \right]$$

$$= \sum_{\substack{A, A_1, A_2 \\ Q_B, Q^T}} \Big[\nu(P_B \times P^T, A)\, \nu_1(Q_B, A_1)\, \nu_2(Q^T, A_2) - \nu(Q_B \times Q^T, A)\, \nu_1(P_B, A_1)\, \nu_2(P^T, A_2) \Big]$$

$$= \sum_{\substack{A,\, A_1,\, A_2 \\ Q_B,\, Q^T}} \nu(Q_B \times Q^T, A)\, \nu_1(P_B, A_1)\, \nu_2(P^T, A_2) \cdot \underbrace{\left[\frac{\nu(P_B \times P^T, A)\, \nu_1(Q_B, A_1)\, \nu_2(Q^T, A_2)}{\nu(Q_B \times Q^T, A)\, \nu_1(P_B, A_1)\, \nu_2(P^T, A_2)} - 1 \right]}_{\varepsilon_B}$$

Control via cluster expansion with the 3-to-3 map

$$\leq \varepsilon\, \nu_1(P_B)\, \nu_2(P^T) \qquad \square$$



* Recovering the LD rates α', r', r :

With the $3 \to 3$ map we can recover the rates relative to α :

modulo: P_α gives info on conf inside !!

- BLUE path dominated by [for Max of I_{BLUE}]

$$P\left(\vdots\ ?\,? \right) = \frac{\rho}{\rho+(1-\rho)q} + \frac{(1-\rho)\cancel{q}}{\rho+(1-\rho)q} \cdot \frac{1}{\cancel{q}} \approx 1 - \boxed{(q-1)\, e^{-\beta}}$$

- Non RED path dominated by [for Min of I_{BLUE}]

$$P\left(\vdots\ ?\,? \right) = \frac{\rho}{\rho+(1-\rho)q} + \frac{(1-\rho)\cancel{q}}{\rho+(1-\rho)q} \cdot \frac{q-1}{\cancel{q}} \approx 1 - \boxed{e^{-\beta}}$$

non red

- ω-Conn path dominated by [for Min of I_{Top}]

$$P\left(\vdots \right) = \frac{\rho}{\rho+(1-\rho)q} \approx 1 - \boxed{q\, e^{-\beta}}$$

□