

EMPIRICAL SPECTRAL DISTRIBUTIONS OF SPARSE RANDOM GRAPHS

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ABSTRACT. We study the spectrum of a random multigraph with a degree sequence $\mathbf{D}_n = (D_i)_{i=1}^n$ and average degree $1 \ll \omega_n \ll n$, generated by the configuration model, and also the spectrum of the analogous random simple graph. We show that, when the empirical spectral distribution (ESD) of $\omega_n^{-1} \mathbf{D}_n$ converges weakly to a limit ν , under mild moment assumptions (e.g., D_i/ω_n are i.i.d. with a finite second moment), the ESD of the normalized adjacency matrix converges in probability to $\nu \boxtimes \sigma_{sc}$, the free multiplicative convolution of ν with the semicircle law. Relating this limit with a variant of the Marchenko–Pastur law yields the continuity of its density (away from zero), and an effective procedure for determining its support.

Our proof of convergence is based on a coupling between the random simple graph and multigraph with the same degrees, which might be of independent interest. We further construct and rely on a coupling of the multigraph to an inhomogeneous Erdős–Rényi graph with the target ESD, using three intermediate random graphs, with a negligible fraction of edges modified in each step.

1. INTRODUCTION

We study the spectrum of a random multigraph $\mathbf{G}_n = ([n], \mathbf{E}_n)$ of n vertices of degrees $\{D_i^{(n)}\}_{i=1}^n$, constructed by the configuration model, where the *even*

$$\sum_{i=1}^n D_i^{(n)} = 2|\mathbf{E}_n| = n\omega_n(1 + o(1)), \quad (1.1)$$

is assumed to be such that

$$\omega_n \rightarrow \infty, \quad \omega_n = o(n). \quad (1.2)$$

Specifically, setting $[n] = \{1, 2, \dots, n\}$, equip each vertex $i \in [n]$ with $D_i^{(n)}$ half-edges, whereby the edge set \mathbf{E}_n results from a uniformly chosen perfect matching of the $2|\mathbf{E}_n|$ half-edges. The uniformly chosen *simple* graph $\mathcal{G}_n = ([n], \mathcal{E}_n)$ with the degrees $D_i^{(n)}$ — assuming of course that this degree sequence is graphical (i.e., there exist simple graphs with these degrees) — is similarly described via a uniform perfect matching of half-edges, subject to the constraint of having neither self-loops nor multiple edges.

Our study of the spectrum of the adjacency matrix $\mathbf{A}_{\mathbf{G}_n}$ of the multigraph \mathbf{G}_n , proceeds through a sequence of couplings, relating it to certain “band” matrices, with independent albeit non-identically-distributed entries (adjacency matrices of Erdős–Rényi *inhomogeneous* random graphs). Various spectral features of the latter will then be derived using the powerful tools that have been developed in the last few decades in random matrix theory and free probability.

In Proposition 1.4 we further provide a novel coupling of \mathcal{G}_n and \mathbf{G}_n , which may be of independent interest. Utilizing this coupling we deduce that the uniformly chosen random *simple* graph \mathcal{G}_n , satisfying the same degree assumptions as \mathbf{G}_n , will also have the same limiting spectrum.

For random *regular* graphs—the case of $D_i^{(n)} = d_n$ for all i —it was shown by Tran, Vu and Wang [19] (extending a previous result of [8]) that, whenever $d_n \gg 1$, the empirical spectral distribution (ESD, defined for a symmetric matrix \mathbf{A} with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$ as $\mathcal{L}^{\mathbf{A}} = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$) of the normalized matrix $\hat{\mathbf{A}}_{\mathbf{G}_n} = \frac{1}{\sqrt{d_n}} \mathbf{A}_{\mathbf{G}_n}$ converges weakly, in probability, to σ_{sc} , the standard semicircle law (with support $[-2, 2]$).

The non-regular case with $|\mathbf{E}_n| = O(n)$ has been studied by Bordenave and Lelarge [6] when the graphs \mathbf{G}_n converge in the Benjamini–Schramm sense, translating in the above setup to having $\{D_i^{(n)}\}$ that are i.i.d. in i and uniformly integrable in n . The existence and uniqueness of the limiting ESD was obtained in [6] by relating this ESD to a recursive distributional equation — arising from the Galton–Watson trees that correspond to the local neighborhoods in \mathbf{G}_n — and showing that this equation has a unique fixed point. See also, e.g., [5, 7, 14] and the references therein, for the analysis of the limiting spectrum at $\lambda = 0$ for Erdős–Rényi graphs of constant average degree. Note that (a) this approach relies on the locally-tree-like structure of the graphs, and is thus tailored for low (at most logarithmic) degrees; and (b) very little is known on this limit, even in seemingly simple settings such as when all degrees are either 3 or 4.

At the other extreme, when $|\mathbf{E}_n|$ diverges polynomially with n (whence the tree approximations are invalid), the trace method—the standard tool for establishing the convergence of the ESD of an Erdős–Rényi random graph to σ_{sc} —faces the obstacle of non-negligible dependencies between edges in the configuration model (the trace method can handle dependencies, but here $n^{-1} \text{tr}((\mathbf{E}\hat{\mathbf{A}}_{\mathbf{G}_n})^{2k}) \asymp \omega_n^k$, thus the precise cancellations of many diverging terms are needed for it to work; such cancellations are very difficult to attain in the presence of dependencies).

1.1. Limiting ESD as a free multiplicative convolution. Our assumptions on the triangular sequence $\{D_i^{(n)}\}$ of degrees are that (2) holds, and in addition, for ω_n satisfying (1), the normalized degrees $\hat{D}_i^{(n)} = D_i^{(n)}/\omega_n$ satisfy that

$$\{\hat{D}_{U_n}^{(n)}\} \quad \text{is uniformly integrable with} \quad \mathbb{E}[(\hat{D}_{U_n}^{(n)})^2] = o(\sqrt{n/\omega_n}), \quad (1.3)$$

where U_n is uniformly chosen in $\{1, \dots, n\}$. Let

$$\hat{\mathbf{A}}_{\mathbf{G}_n} := \omega_n^{-1/2} \mathbf{A}_{\mathbf{G}_n} \quad \text{and} \quad \hat{\mathbf{\Lambda}}_n := \text{diag}(\hat{D}_1^{(n)}, \dots, \hat{D}_n^{(n)}).$$

Call a degree sequence $\{D_i^{(n)}\}$ *graphical* if for every n there exists a simple graph \mathcal{G}_n with such degrees (equivalently, the criterion of the Erdős–Gallai theorem [9] is met).

Theorem 1.1. *Let $\{D_i^{(n)}\}_{i=1}^n$ be a degree sequence satisfying (1)–(3), and further suppose that the ESD $\mathcal{L}^{\hat{\mathbf{A}}_n}$ converges weakly to a limit $\nu_{\hat{D}}$.*

- (a) *The ESD $\mathcal{L}^{\hat{\mathbf{A}}_{\mathbf{G}_n}}$ corresponding to the multigraph $\mathbf{G}_n = ([n], \mathbf{E}_n)$ with degrees $\{D_i^{(n)}\}_{i=1}^n$ (generated via the configuration model), converges weakly, in probability, to $\nu_{\hat{D}} \boxtimes \sigma_{\text{sc}}$.*
- (b) *If $\{D_i^{(n)}\}$ is graphical then the same convergence holds for the ESD $\mathcal{L}^{\hat{\mathbf{A}}_{\mathcal{G}_n}}$ corresponding to a uniformly chosen simple graph $\mathcal{G}_n = ([n], \mathcal{E}_n)$ with this degree sequence.*

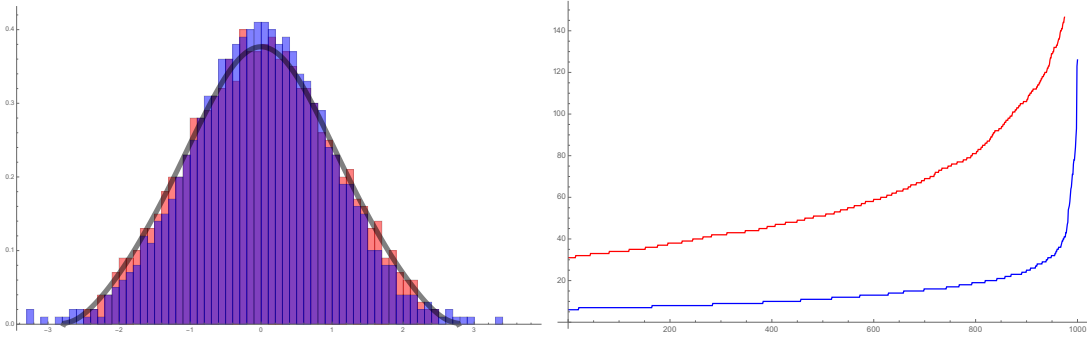


FIGURE 1. Spectra of two random multigraphs on $n = 1000$ vertices with different degree sequences $\{D_i\}$. In red, $D_i = \lceil \tau_i \sqrt{n} \rceil$ for all i , and in blue, $D_i = \lceil \tau_i \log n \rceil$ for $i < n - \sqrt{n}$ and $D_i = \lceil \tau_i \sqrt{n} \rceil$ for $i \geq n - \sqrt{n}$, with $\tau_i \sim 1 + \text{Exp}(1)$ i.i.d. (right plot). The limiting law for the ESD, shown by Theorem 1.1 to be $\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$, is plotted in black (left plot).

In the above theorem, the *free multiplicative convolution* of a symmetric probability measure ψ and a probability measure φ on \mathbb{R}_+ with $\varphi, \psi \neq \delta_0$, denoted $\varphi \boxtimes \psi$, is the unique probability measure such that $S_{\varphi \boxtimes \psi}(z) = S_{\varphi}(z)S_{\psi}(z)$ for z in the common domain of the corresponding S -transforms (see [2, Thm. 7], extending the definition of $\varphi \boxtimes \psi$ from [4] and [20] in case both φ, ψ , are of bounded support and non-zero mean). To define the S -transform, recall that the Cauchy–Stieltjes transform of a probability measure μ on \mathbb{R} , uniquely determining it, is $G_{\mu}(z) := \int [t - z]^{-1} d\mu(t)$. For φ as above, the related

$$m_{\varphi}(z) := z^{-1}G_{\varphi}(z^{-1}) - 1 = \int \frac{zt}{1 - zt} d\varphi(t), \quad (1.4)$$

is invertible as a formal power series in $z \in \mathbb{C}_+$, and the S -transform is defined as

$$S_{\varphi}(w) := (1 + w^{-1})m_{\varphi}^{-1}(w) \quad \text{for } w \in m_{\varphi}(\mathbb{C}_+) \quad (1.5)$$

(cf., e.g., [2, Prop. 1]). Following the extension in [13] of the S -transform to measures of zero mean and finite moments of all order, the S -transform is similarly defined for ψ as above in [2, Thm. 6]. In particular, with σ_{SC} being symmetric and $\nu_{\hat{D}} \neq \delta_0$ supported on \mathbb{R}_+ , the measure $\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$ is well-defined.

Corollary 1.2. *Let $\{\hat{D}_i^{(n)} : 1 \leq i \leq n\}$ be i.i.d. for each n , such that $\mathbb{E}\hat{D}_1^{(n)} = 1$, $\sup_n \mathbb{E}[(\hat{D}_1^{(n)})^2] < \infty$, and the law of $\hat{D}_1^{(n)}$ converges weakly to some $\nu_{\hat{D}}$. For $\omega_n \rightarrow \infty$ such that $\omega_n = o(n)$, let G_n denote the uniform multigraph of degrees $D_i^{(n)} = \lceil \omega_n \hat{D}_i^{(n)} \rceil$ (modifying $D_n^{(n)}$ by one if needed for an even sum). Further, for any integers $\bar{d}_n = o(n)$ with $\omega_n = o(\bar{d}_n)$, the truncated degrees $\lceil \omega_n \hat{D}_i^{(n)} \rceil \wedge \bar{d}_n$ are graphical WHP (after increasing the minimal degree by one, if needed, for an even sum).*

Denoting by \mathcal{G}_n the uniform simple graph, both ESDs $\mathcal{L}^{\hat{A}_{G_n}}$ and $\mathcal{L}^{\hat{A}_{\mathcal{G}_n}}$ converge weakly, in probability, to $\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$.

Remark 1.3. The reason for the appearance of $\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$ in our context is due to the fact that it is the limiting ESD of $\mathbf{B}_n := \hat{\Lambda}_n^{1/2} \mathbf{X}_n \hat{\Lambda}_n^{1/2}$ when $\max_i \hat{D}_i^{(n)} = O(1)$ and \mathbf{X}_n is a standard GOE random matrix. Indeed, as its name suggest, the free multiplicative convolution $\varphi \boxtimes \psi$ is the law of the product ab of free, bounded, random non-commutative operators a of law φ and b of law ψ (cf. [1, Defn. 5.2.1, 5.2.3, 5.3.1, 5.3.28] for the precise meaning of all this). This extends to the limiting ESD for the product of *asymptotically free* matrices: two sequences $\mathbf{X}_n, \mathbf{Y}_n$ of random self-adjoint, matrices are asymptotically free if $\mathbb{E}[\text{tr}_n(f_1(\mathbf{X}_n)g_1(\mathbf{Y}_n) \cdots f_k(\mathbf{X}_n)g_k(\mathbf{Y}_n))] = o(1)$ for the normalized trace $\text{tr}_n(\cdot) = \frac{1}{n} \text{tr}(\cdot)$ and any collections $(f_i)_{i=1}^k$ and $(g_i)_{i=1}^k$ of polynomials (with k fixed) that satisfy $\mathbb{E}[\text{tr}_n(f_i(\mathbf{X}_n))] = o(1)$ and $\mathbb{E}[\text{tr}_n(g_i(\mathbf{Y}_n))] = o(1)$ for all $1 \leq i \leq k$ (see [1, Defn. 5.4.1] or [17, §2.5]). It is known that the GOE \mathbf{X}_n is asymptotically free of any uniformly bounded diagonal $\hat{\Lambda}_n$ (see, *e.g.*, [1, Theorem 5.4.5]), which in turn implies that $\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$ is the weak limit of the ESD for the random matrices \mathbf{B}_n (the spectral radius of the GOE \mathbf{X}_n is $O(1)$ with high probability, so by a standard truncation argument we arrive at the bounded case of [1, Corollary 5.4.11(iii)]).

Theorem 1.1 and Corollary 1.2 are proved in §2. This is achieved by first analyzing the ESD of the random multigraph \mathbf{G}_n ; the move from multigraphs to simple graphs is achieved via the following proposition, which we prove in §3.

Proposition 1.4. *Fixing graphical degrees $D_1 \geq D_2 \geq \cdots \geq D_n$, let \mathbf{G}_n and \mathcal{G}_n be the corresponding random multigraph and uniform simple graph, respectively. There exists a coupling μ between the matchings which yield \mathbf{G}_n and \mathcal{G}_n so the number $\Delta_n \leq 2|\mathbf{E}_n|$ of half-edges whose matching links are different between the two graphs, satisfies*

$$\mathbb{E}_\mu[\Delta_n(\Delta_n - 1)] \leq 4 \sum_{i=1}^{n-1} \sum_{j=i+1}^{i+D_i} (2D_i D_j - D_i - D_j). \quad (1.6)$$

Remark 1.5. A crude, yet already useful, upper bound on the RHS of (6) is

$$8\sqrt{2|\mathbf{E}_n|} \sum_{i=1}^n D_i^2. \quad (1.7)$$

(Indeed, $(\sum_{j=i+1}^{i+D_i} D_j)^2 \leq D_i \sum_{j=1}^n D_j^2$ by Cauchy–Schwarz for any $i \in [n]$; thus, again by Cauchy–Schwarz, the RHS of (6) is at most $8(\sum_i D_i^2)(\sum_i D_i)^{\frac{1}{2}}$, and $\sum_i D_i = 2|\mathbf{E}_n|$.) In general, the RHS of (6) can be replaced by any bound on the expected number of pairs of half-edges $e \neq f$ on a which a “switch” would yield a non-simple graph.

Remark 1.6. The proof of Theorem 1.1(a) extends to the dense regime, where ω_n/n is bounded below (and above). However, the minimal expected edit distance between \mathbf{G}_n and \mathcal{G}_n exceeds the expected number $O(\omega_n^2)$ of parallel edges in \mathbf{G}_n , which in the dense regime is already $O(|\mathbf{E}_n|)$, thereby blocking in the dense regime our route to Theorem 1.1(b) as a consequence of part (a). Further, our assumption (3) allows having a maximal degree that far exceeds n (indeed, prior to truncation this happens

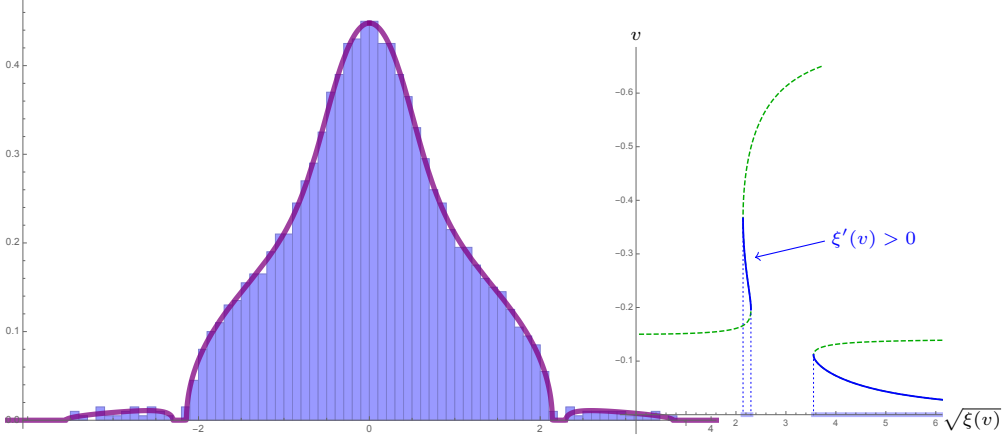


FIGURE 2. Recovering the support of the limiting ESD. Left: ESD of the random multigraph on $n = 1000$ vertices with degrees \sqrt{n} , $3\sqrt{n}$, $15\sqrt{n}$ in frequencies 0.5, 0.49, 0.01, resp. Right: $\xi(v)$ from Remark 1.9.

in the i.i.d. setting of Corollary 1.2). Even for specified graphical degrees, the number of simple graphs \mathcal{G}_n oscillates widely as the degrees change, so (3) might not suffice for the statement of Theorem 1.1(b) to be true in the dense regime. Going back to the sparse regime, assumptions à la (3) have little to do with controlling extreme eigenvalues, or with bringing the corresponding local law to the celebrated GOE-universality class of homogeneous Erdős-Rényi graphs. Indeed, one must further restrict $\hat{\mathbf{A}}_n$, in order to have any hope of transferring the many fine results on extreme eigenvalues and local laws that are available for the GOE, via \mathbf{B}_n of Remark 1.3 to $\hat{\mathbf{A}}_{\mathbf{G}_n}$.

1.2. Properties of the limiting ESD. The next two propositions, proved in §4, relate the limiting measure $\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$ with a Marchenko–Pastur law, and thereby, via [16], yield its support and density regularity.

Proposition 1.7. *For the law $\nu_{\hat{D}}$ of a nonnegative random variable \hat{D} with $\mathbb{E}\hat{D} = 1$, let μ_{MP} be the Marchenko–Pastur limit (on \mathbb{R}_+) of the ESD of $n^{-1}\mathbf{\Lambda}_n\tilde{\mathbf{X}}_n\tilde{\mathbf{X}}_n^*\mathbf{\Lambda}_n$, in which the non-symmetric $\tilde{\mathbf{X}}_n$ has standard i.i.d. complex Gaussian entries and $\mathcal{L}^{\mathbf{\Lambda}_n} \Rightarrow \nu$ for non-negative diagonal matrices $\mathbf{\Lambda}_n$ and the size-biased ν with $\frac{d\nu}{d\nu_{\hat{D}}}(x) = x$ on \mathbb{R}_+ . The free multiplicative convolution $\mu = \nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$ has the Cauchy–Stieltjes transform*

$$G_{\mu}(z) = -z^{-1} \left[1 + G_{\tilde{\mu}}(z)^2 \right], \quad \forall z \in \mathbb{C}_+, \quad (1.8)$$

where $\tilde{\mu}$ is the law of the symmetric X such that X^2 is distributed according to μ_{MP} .

Recall [16, Lemma 3.1, Lemma 3.2] that $h(z) := G_{\mu_{\text{MP}}}(z)$ is uniformly bounded on \mathbb{C}_+ away from the imaginary axis, and [16, Theorem 1.1] that $h(z) \rightarrow h(x)$ whenever $z \in \mathbb{C}_+$ converges to $x \in \mathbb{R} \setminus \{0\}$. Further, the \mathbb{C}_+ -valued function $h(x)$ is continuous on $\mathbb{R} \setminus \{0\}$ and the continuous density

$$\rho_{\text{MP}}(x) := \frac{d\mu_{\text{MP}}}{dx} = \frac{1}{\pi} \Im(h(x)), \quad (1.9)$$

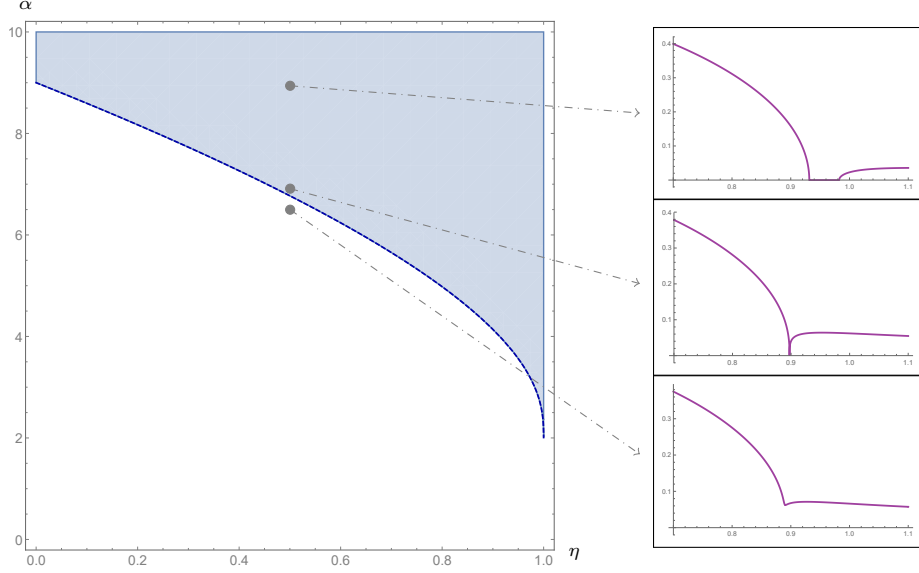


FIGURE 3. Phase diagram for the existence of holes in the limiting ESD when $\nu_{\hat{D}}$ is supported on two atoms $\alpha > \eta > 0$ (see Corollary 1.10). Left: the region (12) (where $\text{supp}(\mu)$ is not connected) highlighted in blue. Right: zooming in on the emergence of a hole as α varies at $\eta = \frac{1}{2}$.

is real analytic at any $x \neq 0$ where it is positive. The density $\tilde{\rho}(x) = |x|\rho_{\text{MP}}(x^2)$ of $\tilde{\mu}$ inherits these regularity properties. Bounding $\tilde{\rho}$ uniformly and analyzing the effect of (8) we next make similar conclusions about the density $\rho(x)$ of μ , now also at $x = 0$.

Proposition 1.8. *In the setting of Proposition 1.7, for $x \neq 0$ there is density*

$$\rho(x) := \frac{d\mu}{dx} = -2\Re(h(x^2))\tilde{\rho}(x), \quad (1.10)$$

which is continuous, symmetric, and moreover real analytic where positive. The support of μ is $\text{supp}(\mu) := \{x \in \mathbb{R} : \rho(x) > 0\} = \text{supp}(\tilde{\mu})$, which up to the mapping $x \mapsto x^2$ further matches $\text{supp}(\mu_{\text{MP}})$. In addition $\pi\tilde{\rho}(x) \leq 1 \wedge (2/|x|)$, $\pi\rho(x) \leq (\mathbb{E}\hat{D}^{-2})^{1/2} \wedge (4/|x|^3)$ and if $\nu_{\hat{D}}(\{0\}) = 0$ then μ is absolutely continuous (i.e., $\mu(\{0\}) = 0$).

Remark 1.9. Recall the unique inverse of h on $h(\mathbb{C}_+)$ given by

$$\xi(h) := -\frac{1}{h} + \mathbb{E}\left[\frac{\hat{D}^2}{1+h\hat{D}}\right], \quad (1.11)$$

namely $\xi(h(z)) = z$ on \mathbb{C}_+ (see [16, Eqn. (1.4)]); this inverse extends analytically to a neighborhood of $\mathbb{C}_+ \cup \Gamma$ for $\Gamma := \{h \in \mathbb{R} : h \neq 0, -h^{-1} \in \text{supp}(\hat{\nu}_D)^c\}$ and [16, Theorems 4.1 and 4.2] show that $x \in \text{supp}(\mu_{\text{MP}})^c$ iff $\xi'(v) > 0$ for $v \in \Gamma$, where $v = h(x)$ and $x = \xi(v)$ (thus validating the characterization of $\text{supp}(\mu_{\text{MP}})$ which has been proposed in [12]). We show in Lemma 4.2 that $\Re(h(x^2)) < 0$ everywhere, hence

the behavior of $\rho(x)$ at the soft-edges of $\text{supp}(\mu)$ can be read from the soft-edges of $\text{supp}(\mu_{\text{MP}})$ (as in [11, Prop. 2.3]), depicted in Figure 3.

Corollary 1.10. *Suppose $\nu_{\hat{D}}$ of mean one is supported on two atoms $\alpha > \eta > 0$. The support $\text{supp}(\mu)$ of $\mu = \nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$ is then disconnected iff*

$$\alpha > \eta \left[\frac{3}{1 - (1 - \eta)^{1/3}} - 1 \right]. \quad (1.12)$$

Moreover, when (12) holds, $\text{supp}(\mu) \cap \mathbb{R}_+$ consists of exactly two disjoint intervals.

2. CONVERGENCE OF THE ESD'S

The proof of Theorem 1.1 will use the following standard lemma.

Lemma 2.1. *Let $\{\mathbf{M}_{n,r}\}_{n,r \in \mathbb{N}}$ be a family of matrices of order n , define $\mu_{n,r} := \mathcal{L}^{\mathbf{M}_{n,r}}$ and $\eta(r) := \limsup_{n \rightarrow \infty} \frac{1}{n} \text{tr}((\mathbf{M}_{n,r} - \mathbf{M}_{n,\infty})^2)$. Let $\{\mu_r : r \in \mathbb{N}\}$ denote a family of measures such that*

$$\mu_{n,r} \Rightarrow \mu_r \text{ as } n \rightarrow \infty \text{ for every } r \in \mathbb{N}, \quad (2.1)$$

$$\mu_{n,\infty} \text{ is tight}, \quad (2.2)$$

$$\eta(r) \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (2.3)$$

Then the weak limit of $\mu_{n,\infty}$ as $n \rightarrow \infty$ exists and equals $\lim_{r \rightarrow \infty} \mu_r$.

Proof. Let μ_∞ be a limit point of $\mu_{n,\infty}$, the existence of which is guaranteed by the tightness assumption (14). A standard consequence of the Hoffman–Wielandt bound (cf. [1, Lemma 2.1.19]) and Cauchy–Schwarz is that for matrices \mathbf{A} and \mathbf{B} of order n ,

$$d_{\text{BL}}(\mathcal{L}^{\mathbf{A}}, \mathcal{L}^{\mathbf{B}})^2 \leq \frac{1}{n} \text{tr}((\mathbf{A} - \mathbf{B})^2),$$

where d_{BL} is the bounded-Lipchitz metric on the space $M_1(\mathbb{R}_+)$ of probability measures on \mathbb{R}_+ (see the proof of [1, Theorem 2.1.21]). Thus, by (13) and the triangle-inequality for d_{BL} , it follows that

$$\eta(r) \geq d_{\text{BL}}(\mu_\infty, \mu_r)^2.$$

Consequently, $\mu_r \rightarrow \mu_\infty$ as $r \rightarrow \infty$, from which the uniqueness of μ_∞ also follows. ■

Proof of Theorem 1.1. **In Step I** we reduce the proof to treating the *single-adjacency* matrix \mathbf{A}_n of \mathbb{G}_n , where multiple copies of an edge/loop are replaced by a single one (that is, $\mathbf{A}_n = \mathbf{A}_{\mathbb{G}_n} \wedge 1$ entry-wise), and further $\{\omega_n^{-1} D_i^{(n)}\} \subseteq \mathcal{S}$ for some fixed finite set \mathcal{S} . Scaling $\hat{\mathbf{A}}_n := \omega_n^{-1/2} \mathbf{A}_n$ we rely in **Step II**, on Proposition 2.3 to replace the limit points of $\mathcal{L}^{\hat{\mathbf{A}}_n}$ by those of $\mathcal{L}^{\omega_n^{-1/2} \tilde{\mathbf{A}}_n}$ for symmetric matrices $\tilde{\mathbf{A}}_n$ of independent Bernoulli entries. **Using** the moment method, **Step III** relates the latter limit points to the limit of $\mathcal{L}^{\mathbf{B}_n}$ for the matrices \mathbf{B}_n of Remark 1.3.

Step I. We claim that if $\mathcal{L}^{\hat{\mathbf{A}}_n} \Rightarrow \mu$ in probability, then the same applies for $\mathcal{L}^{\hat{\mathbf{A}}_{\mathbf{G}_n}}$. This will follow from Lemma 2.1 with $\mathbf{M}_{n,r} = \hat{\mathbf{A}}_n$ and $\mathbf{M}_{n,\infty} = \hat{\mathbf{A}}_{\mathbf{G}_n}$ upon verifying that

$$\xi_n := \mathbb{E} \left[\frac{1}{n} \operatorname{tr} \left((\hat{\mathbf{A}}_{\mathbf{G}_n} - \hat{\mathbf{A}}_n)^2 \right) \right] \rightarrow 0. \quad (2.4)$$

Indeed, Condition (13) has been assumed; Condition (14) follows from the fact that

$$\frac{1}{2n} \operatorname{tr} \left(\hat{\mathbf{A}}_{\mathbf{G}_n}^2 \right) \leq \frac{1}{n} \operatorname{tr} \left((\hat{\mathbf{A}}_{\mathbf{G}_n} - \hat{\mathbf{A}}_n)^2 \right) + \frac{1}{n} \operatorname{tr} (\hat{\mathbf{A}}_n^2) \leq \frac{1}{n} \operatorname{tr} \left((\hat{\mathbf{A}}_{\mathbf{G}_n} - \hat{\mathbf{A}}_n)^2 \right) + \frac{2|\mathbf{E}_n|}{n\omega_n},$$

so in particular $\mathbb{E}[\frac{1}{n} \operatorname{tr} (\hat{\mathbf{A}}_{\mathbf{G}_n}^2)] \leq \xi_n + 1 + o(1)$, yielding tightness; and Condition (15) holds in probability by (16) and Markov's inequality. Recall that the stochastic ordering $X \preceq X'$ denotes that $\mathbb{P}(X > x) \leq \mathbb{P}(X' > x)$ for all $x \in \mathbb{R}$, or equivalently, that there exists a coupling of (X, X') such that $\mathbb{P}(X \leq X') = 1$. To establish (16), observe that, for every i and j we have $(\mathbf{A}_{\mathbf{G}_n})_{i,j} \preceq \operatorname{Bin}(m, q)$ for $m = D_i^{(n)}$ and $q = (D_j^{(n)} - 1_{i=j}) / (2|\mathbf{E}_n| - 1)$, whereas $\operatorname{Bin}(m, q) \preceq Y_\lambda \sim \operatorname{Po}(\lambda)$ for every m and λ such that $1 - q \geq e^{-\lambda/m}$. Thus,

$$\mathbb{E} [(\mathbf{A}_{\mathbf{G}_n} - \mathbf{A}_n)_{i,j}^2] \leq \mathbb{E} [(Y_\lambda - 1)_+^2] \leq \lambda^2.$$

Since $q \leq \frac{1+o(1)}{\omega_n}$ uniformly over i, j , we take WLOG $\lambda = mD_j^{(n)} / |\mathbf{E}_n|$, yielding for n large

$$\xi_n \leq \frac{2}{n\omega_n} \sum_{i,j=1}^n \left[\frac{D_i^{(n)} D_j^{(n)}}{|\mathbf{E}_n|} \right]^2 \leq \frac{4\omega_n}{n} \left[\frac{1}{n} \sum_{i=1}^n (\hat{D}_i^{(n)})^2 \right] \rightarrow 0,$$

by our assumption that $\mathbb{E}[(\hat{D}_{U_n}^{(n)})^2] = o(\sqrt{n/\omega_n})$. Considering hereafter only single-adjacency matrices, we proceed to reduce the problem to the case where the variables $\hat{D}_i^{(n)}$ are all supported on a finite set. To this end, let $\ell = 2r^2$ for $r \in \mathbb{N}$ and

$$\hat{D}_i^{(n,r)} = \Psi_r(\hat{D}_i^{(n)}) \quad \text{for} \quad \Psi_r(x) := \sum_{j=1}^{\ell} d_j^{(r)} \mathbb{1}_{[d_j^{(r)}, d_{j+1}^{(r)})}(x),$$

where $0 = d_1^{(r)} < \dots < d_{\ell+1}^{(r)}$ are continuity points of $\nu_{\hat{D}}$ of interdistances in $[\frac{1}{2r}, \frac{1}{r}]$, which are furthermore in $\varepsilon_r \mathbb{Z}$ for some irrational $\varepsilon_r > 0$. Let

$$D_i^{(n,r)} = \omega_{n,r} \hat{D}_i^{(n,r)} \in \mathbb{Z}_+ \quad \text{for} \quad \omega_{n,r} := \frac{[\varepsilon_r \omega_n]}{\varepsilon_r},$$

possibly deleting one half-edge from $D_n^{(n,r)}$ if needed to make $\sum_{i=1}^n D_i^{(n,r)}$ even.

Observation 2.2. *Let $\{d_i\}_{i=1}^n, \{d'_i\}_{i=1}^n$ be degree sequences with $d'_i \leq d_i$, and let \mathbf{G} be a random multigraph with degrees $\{d_i\}$ generated by the configuration model. Generate \mathbf{H} by (a) marking a subset of d'_i half-edges of vertex i BLUE, chosen independently of the matching that generated \mathbf{G} ; (b) retaining every edge that has two BLUE endpoints; and (c) adding an independent uniform matching on all other BLUE half-edges. Then \mathbf{H} has the law of the random multigraph with degrees $\{d'_i\}$ generated by the configuration model.*

Proof. Since the configuration model matches the half-edges in \mathbf{G} via a uniformly chosen perfect matching, and the coloring step (a) is performed independently of this matching, it follows that the induced matching on the subset of BLUE half-edges that are matched to BLUE counterparts—namely, the edges retained in step (b)—is uniform. \blacksquare

Using this, and noting $D_i^{(n,r)} \leq D_i^{(n)}$ for all i , let $\mathbf{G}_n^{(r)} = ([n], \mathbf{E}_n^{(r)})$ be the following random multigraph with degrees $\{D_i^{(n,r)}\}$, coupled to the already-constructed \mathbf{G}_n :

- (a) For each i , mark a uniformly chosen subset of $D_i^{(n,r)}$ half-edges incident to vertex i as BLUE in \mathbf{G}_n .
- (b) Retain in $\mathbf{G}_n^{(r)}$ every edge of \mathbf{E}_n where both parts are BLUE.
- (c) Complete the construction of $\mathbf{G}_n^{(r)}$ via a uniformly chosen matching of all unmatched half-edges.

Let $\hat{\mathbf{A}}_n^{(r)} = \omega_n^{-1/2} \mathbf{A}_n^{(r)}$ for $\mathbf{A}_n^{(r)}$, the single-adjacency matrix of $\mathbf{G}_n^{(r)}$. We next control the difference between $\mathcal{L}^{\hat{\mathbf{A}}_n}$ and $\mathcal{L}^{\hat{\mathbf{A}}_n^{(r)}}$. Indeed, by the definition of the coupling of \mathbf{G}_n and $\mathbf{G}_n^{(r)}$, the cardinality of the symmetric $\mathbf{E}_n \triangle \mathbf{E}_n^{(r)}$ is at most twice the number of *unmarked* half-edges in \mathbf{G}_n . Thus,

$$\begin{aligned} \frac{1}{4n} \operatorname{tr} ((\hat{\mathbf{A}}_n - \hat{\mathbf{A}}_n^{(r)})^2) &\leq \frac{1}{2n\omega_n} |\mathbf{E}_n \triangle \mathbf{E}_n^{(r)}| \leq \frac{1}{n\omega_n} \sum_{i=1}^n (D_i^{(n)} - D_i^{(n,r)}) \\ &\leq \frac{1+o(1)}{\varepsilon_r \omega_n} + \frac{1}{r} + \frac{1}{n} \sum_{i=1}^n \hat{D}_i^{(n)} \mathbb{1}_{\{\hat{D}_i^{(n)} \geq r\}} =: \eta(n, r), \end{aligned} \quad (2.5)$$

where the first term in $\eta(n, r)$ accounts for the discrepancy between ω_n and $\omega_{n,r}$, the term $1/r$ accounts for the degree quantization, while the last term accounts for degree truncation (since $d_{\ell+1}^{(r)} \geq r$). Thanks to the uniform integrability of $\{\hat{D}_U^{(n)}\}$ from (3), we have that $\eta(r) := \limsup_{n \rightarrow \infty} \eta(n, r)$ satisfies $\eta(r) \rightarrow 0$ as $r \rightarrow \infty$. Furthermore,

$$\int x^2 d\mathcal{L}^{\hat{\mathbf{A}}_n} = \frac{1}{n} \operatorname{tr}(\hat{\mathbf{A}}_n^2) \leq 1 + o(1)$$

by the choice of ω_n in (1), yielding the tightness of $\mu_{n,\infty} := \mathcal{L}^{\hat{\mathbf{A}}_n}$. Altogether, we conclude from Lemma 2.1 that, if $\mathcal{L}^{\hat{\mathbf{A}}_n^{(r)}} \Rightarrow \mu_r$, then $\mathcal{L}^{\hat{\mathbf{A}}_n} \Rightarrow \lim_{r \rightarrow \infty} \mu_r$.

Next, let $\omega_n^{(r)} = 2|\mathbf{E}_n^{(r)}|/n$ (as in (1) but for the multigraph $\mathbf{G}_n^{(r)}$). Since (see (17)),

$$\limsup_{n \rightarrow \infty} \left| 1 - \frac{\omega_n^{(r)}}{\omega_n} \right| \leq \eta(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

WLOG we replace ω_n by $\omega_n^{(r)}$ in the definition of $\hat{\mathbf{A}}_n^{(r)}$, i.e., starting with

$$\hat{D}_i^{(n,r)} \in \{d_1^{(r)}, \dots, d_\ell^{(r)}\} =: \mathcal{S}_r.$$

Further, note that the hypothesis $\mathcal{L}^{\hat{\mathbf{A}}_n} \Rightarrow \nu_{\hat{D}}$ as $n \rightarrow \infty$, together with our choice of \mathcal{S}_r , implies that $\mathcal{L}^{\hat{\mathbf{A}}_n^{(r)}}$ (corresponding to $\hat{\mathbf{A}}_n^{(r)} = \operatorname{diag}(\hat{D}_1^{(n,r)}, \dots, \hat{D}_n^{(n,r)})$) converges weakly for each r to some $\nu_{\hat{D}_r} \neq \delta_0$, supported on \mathbb{R}_+ , and further, $\nu_{\hat{D}_r} \Rightarrow \nu_{\hat{D}} \neq \delta_0$, as $r \rightarrow \infty$.

Let $\mu^{(2)}$ denote hereafter the pushforward of the measure μ by the mapping $x \mapsto x^2$ (that is, the measure given by $B \mapsto \mu(f^{-1}(B))$ for $f(x) = x^2$.) It is known that, for probability measures on \mathbb{R}_+ , the free multiplicative convolution is continuous w.r.t. weak convergence; that is, $\nu_k \boxtimes \nu'_k \Rightarrow \nu \boxtimes \nu'$ provided $\nu_k \Rightarrow \nu \neq \delta_0$, $\nu'_k \Rightarrow \nu' \neq \delta_0$ all of which are supported on \mathbb{R}_+ (see, *e.g.*, [2, Prop. 3]). Applying this twice, we find that

$$\nu_{\hat{D}_r} \boxtimes \sigma_{\text{SC}}^{(2)} \boxtimes \nu_{\hat{D}_r} \Rightarrow \nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}^{(2)} \boxtimes \nu_{\hat{D}}. \quad (2.6)$$

From this we next deduce that $\nu_{\hat{D}_r} \boxtimes \sigma_{\text{SC}} \Rightarrow \nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$. Indeed, recall [2, Lemma 8] that the LHS of (18) equals $(\nu_{\hat{D}_r} \boxtimes \sigma_{\text{SC}})^{(2)}$, while likewise its RHS equals $(\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}})^{(2)}$. For any $f \in C_b(\mathbb{R})$, the function $g(x) = \frac{1}{2}[f(\sqrt{x}) + f(-\sqrt{x})]$ is in $C_b(\mathbb{R}_+)$. Thus, the weak convergence $(\nu_{\hat{D}_r} \boxtimes \sigma_{\text{SC}})^{(2)} \Rightarrow (\nu_{\hat{D}} \boxtimes \sigma_{\text{SC}})^{(2)}$, implies that $\nu_{\hat{D}_r} \boxtimes \sigma_{\text{SC}} \Rightarrow \nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$ for the corresponding symmetric source measures of the map $x \mapsto x^2$. In conclusion, it suffices hereafter to prove the theorem for the case where $\hat{D}_i^{(n)} \in \mathcal{S}$, a fixed finite set, for all n .

Step II. For $1 \leq a \leq \ell$, let $m_a^{(n)} = |\mathbf{V}_n^a|$ where $\mathbf{V}_n^a = \{v \in [n] : \deg(v) = d_a \omega_n\}$ is the set of vertices of degree $d_a \omega_n$ in \mathbf{G}_n . By assumption, $m_a^{(n)}/n \rightarrow \nu_a$ for $\nu_a := \nu_{\hat{D}}(\{d_a\})$. (Observe that our choice of ω_n dictates that $\sum_a d_a \nu_a = 1$.) For all $1 \leq a, b \leq \ell$, set

$$q_{a,b} := d_a d_b \nu_b.$$

Let $\mathbf{H}_n = \cup_{a \leq b} \mathbf{H}_{a,b}^{(n)}$ for the edge-disjoint multigraphs $\mathbf{H}_{a,b}^{(n)}$ that are generated by the configuration model in the following way.

- For $1 \leq a \leq \ell$, let $\mathbf{H}_{(a,a)}^{(n)}$ be the random $D_{a,a}^{(n)}$ -regular multigraph on \mathbf{V}_n^a , where $D_{a,a}^{(n)} m_a^{(n)}$ is even and $\hat{D}_{a,a}^{(n)} := D_{a,a}^{(n)}/\omega_n$ converges to $q_{a,a}$ as $n \rightarrow \infty$.
- For $1 \leq a < b \leq \ell$, let $\mathbf{H}_{a,b}^{(n)}$ be the random bipartite multigraph with sides $(\mathbf{V}_n^a, \mathbf{V}_n^b)$ and degrees $D_{a,b}^{(n)}$ in \mathbf{V}_n^a and $D_{b,a}^{(n)}$ in \mathbf{V}_n^b , such that the detailed balance

$$D_{a,b}^{(n)} m_a^{(n)} = D_{b,a}^{(n)} m_b^{(n)}$$

holds, and $\hat{D}_{a,b}^{(n)} := D_{a,b}^{(n)}/\omega_n$ tends to $q_{a,b}$ as $n \rightarrow \infty$ (hence, $\hat{D}_{b,a}^{(n)} \rightarrow q_{b,a}$).

Finally, setting

$$\lambda_{a,b}^{(n)} := \frac{\omega_n}{n} d_a d_b, \quad (2.7)$$

let $\tilde{\mathbf{A}}_n$ denote the single-adjacency matrix of the multigraph $\tilde{\mathbf{H}}_n = \cup_{a \leq b} \tilde{\mathbf{H}}_{a,b}^{(n)}$, where the edge-disjoint multigraphs $\tilde{\mathbf{H}}_{a,b}^{(n)}$ are defined as follows.

- For $1 \leq a \leq b \leq \ell$, mutually independently set the multiplicity of the edge between distinct $i \in \mathbf{V}_n^a$ and $j \in \mathbf{V}_n^b$ in $\tilde{\mathbf{H}}_{a,b}^{(n)}$ to be a $\text{Po}(\lambda_{a,b}^{(n)})$ random variable.
- For $1 \leq a \leq \ell$, mutually independently set the number of loops incident to $i \in \mathbf{V}_n^a$ to be a $\text{Po}(\frac{1}{2} \lambda_{a,a}^{(n)})$ random variable.

Our next proposition shows that $\mathcal{L}^{\hat{\mathbf{A}}_n} \Rightarrow \nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}$, in probability, whenever

$$\mathcal{L}^{\omega_n^{-1/2} \tilde{\mathbf{A}}_n} \Rightarrow \nu_{\hat{D}} \boxtimes \sigma_{\text{SC}}, \quad \text{in probability.} \quad (2.8)$$

Proposition 2.3. *The empirical spectral measures of $\mathbf{A}_n, \mathbf{A}'_n$ and $\tilde{\mathbf{A}}_n$, the respective single-adjacency matrices of $\mathbf{G}_n, \mathbf{H}_n$ and $\tilde{\mathbf{H}}_n$, satisfy*

$$d_{\text{BL}}\left(\mathcal{L}^{\omega_n^{-1/2}}\mathbf{A}_n, \mathcal{L}^{\omega_n^{-1/2}}\mathbf{A}'_n\right) = o(1) \quad \text{and} \quad d_{\text{BL}}\left(\mathcal{L}^{\omega_n^{-1/2}}\mathbf{A}'_n, \mathcal{L}^{\omega_n^{-1/2}}\tilde{\mathbf{A}}_n\right) = o(1),$$

in probability, as $n \rightarrow \infty$.

Proof. Setting

$$\mathbf{G}_n^{(0)} = \mathbf{G}_n, \quad \mathbf{G}_n^{(2)} = \mathbf{H}_n, \quad \mathbf{G}_n^{(4)} = \tilde{\mathbf{H}}_n,$$

associate with each multigraph its sub-degrees (accounting for edge multiplicities),

$$D_{i,b}^{(n,k)} := \sum_{j \in \mathbf{V}_n^b} (\mathbf{A}_{\mathbf{G}_n^{(k)}})_{i,j}, \quad i \in [n], \quad 1 \leq b \leq \ell,$$

so in particular $D_{i,b}^{(n,2)} = D_{a(i),b}^{(n)}$ where $a(i)$ is such that $i \in \mathbf{V}_n^a$. Of course, for $k = 0, 2, 4$,

$$m_{a,b}^{(n,k)} := \sum_{i \in \mathbf{V}_n^a} D_{i,b}^{(n,k)} = m_{b,a}^{(n,k)}, \quad m_{a,a}^{(n,k)} \text{ is even}, \quad 1 \leq a, b \leq \ell. \quad (2.9)$$

Claim 2.4. *Conditional on a given sequence of sub-degrees $\{D_{i,b}^{(n,k)}\}$, the adjacency matrices $\mathbf{A}_{\mathbf{G}_n^{(k)}}$ for $k \in \{0, 2, 4\}$ all have the same conditional law.*

Proof. Observe that $\mathbf{G}_n = \mathbf{G}_n^{(0)}$ gives the same weight to each perfect matching of its half-edges, thus conditioning on $\{D_{i,b}^{(n,k)}\}$ amounts to specifying a subset of permissible matchings, on which the conditional distribution would be uniform. The same applies to the graphs $\mathbf{H}_{(a,b)}^{(n)}$ for all $1 \leq a \leq b \leq \ell$, each being an independently drawn uniform multigraph, and hence to their union $\mathbf{H}_n = \mathbf{G}_n^{(2)}$, thus establishing the claim for $k = 0, 2$. To treat $k = 4$, notice that the probability that the multigraph $\mathbf{H}_{(a,b)}^{(n)}$, $a \neq b$, given the sub-degrees $\{D_{i,b}^{(n,k)}\}$, features the adjacency matrix $\mathbf{a} := (a_{i,j})$ ($i \in \mathbf{V}_n^a$, $j \in \mathbf{V}_n^b$), is

$$\frac{1}{m_{a,b}^{(n,k)}!} \left(\prod_{i \in \mathbf{V}_n^a} \frac{D_{i,b}^{(n,k)}!}{\prod_{j \in \mathbf{V}_n^b} a_{i,j}!} \right) \left(\prod_{j \in \mathbf{V}_n^b} D_{j,a}^{(n,k)}! \right) \propto \prod_{i \in \mathbf{V}_n^a} \prod_{j \in \mathbf{V}_n^b} \frac{1}{a_{i,j}!}$$

by the definition of the configuration model. As the distribution of a vector of t i.i.d. Poisson variables with mean λ , conditional on their sum being m , is multinomial with parameters $(m, \frac{1}{t}, \dots, \frac{1}{t})$, the analogous conditional probability under $\tilde{\mathbf{H}}_{(a,b)}^{(n)}$ is

$$\prod_{i \in \mathbf{V}_n^a} \frac{D_{i,b}^{(n,k)}!}{\prod_{j \in \mathbf{V}_n^b} a_{i,j}!} |\mathbf{V}_n^b|^{-D_{i,b}^{(n,k)}} \propto \prod_{i \in \mathbf{V}_n^a} \prod_{j \in \mathbf{V}_n^b} \frac{1}{a_{i,j}!}.$$

Lastly, the probability that $\mathbf{H}_{(a,a)}^{(n)}$, conditional on $\{D_{i,b}^{(n,k)}\}$, assigns to $\mathbf{a} = (a_{i,j})$ is

$$\prod_{i \in \mathbf{V}_n^a} \frac{D_i!}{2^{a_{i,i}}} \prod_{\substack{j \in \mathbf{V}_n^a \\ j > i}} \frac{1}{a_{i,j}!} \propto 2^{-\sum_i a_{i,i}} \prod_{\substack{i,j \in \mathbf{V}_n^a \\ j > i}} \frac{1}{a_{i,j}!},$$

whereas the analogous conditional probability under $\tilde{\mathbf{H}}_{(a,b)}^{(n)}$ (now involving a vector that is multinomial with parameters $(D_{i,b}^{(n,k)}, \frac{1}{2t+1}, \frac{2}{2t+1}, \dots, \frac{2}{2t+1})$ for $t = |\{j \in \mathbf{V}_n^a : j \geq i\}|$, recalling the factor of 2 in the definition of the rate of loops under $\tilde{H}_{(a,a)}^{(n)}$), is

$$\prod_{i \in \mathbf{V}_n^a} \frac{D_{i,b}^{(n,k)}!}{\prod_{\substack{j \in \mathbf{V}_n^a \\ j > i}} a_{i,j}!} 2^{-a_{i,i}} \left(\frac{2}{|\{j \in \mathbf{V}_n^a : j \geq i\}|} \right)^{-D_{i,b}^{(n,k)}} \propto 2^{-\sum_i a_{i,i}} \prod_{\substack{i,j \in \mathbf{V}_n^a \\ j > i}} \frac{1}{a_{i,j}!}.$$

This completes the proof of the claim. \blacksquare

We will introduce two auxiliary multigraphs $\mathbf{G}_n^{(1)}$ and $\mathbf{G}_n^{(3)}$ having the latter property, and further, the corresponding single-adjacency matrices (or single-edge sets $\mathbf{E}_n^{(k)}$), can be coupled in such a way that

$$\sum_{k=1}^4 \mathbb{E} \left[|\mathbf{E}_n^{(k)} \triangle \mathbf{E}_n^{(k-1)}| \right] = o(n\omega_n). \quad (2.10)$$

It follows that, under the resulting coupling, both $\mathbb{E}[\text{tr}((\mathbf{A}_n - \mathbf{A}'_n)^2)] = o(n\omega_n)$ and $\mathbb{E}[\text{tr}((\mathbf{A}'_n - \tilde{\mathbf{A}}_n)^2)] = o(n\omega_n)$, yielding Proposition 2.3 via the Hoffman–Wielandt bound.

Proceeding to construct the multigraph $\mathbf{G}_n^{(1)}$, write, for all $i \in [n]$ and $1 \leq b \leq \ell$,

$$D_{i,b}^{(n,1)} = D_{i,b}^{(n,0)} \wedge D_{i,b}^{(n,2)}, \quad (2.11)$$

then further uniformly reduce the number of potential half-edges in $\mathbf{G}_n^{(1)}$ until achieving (21) for $k = 1$. That is, if (23) yields $m_{a,b}^{(n,1)} > m_{b,a}^{(n,1)}$ for some $a \neq b$, we uniformly choose and eliminate $m_{a,b}^{(n,1)} - m_{b,a}^{(n,1)}$ potential half-edges leading from \mathbf{V}_n^a to \mathbf{V}_n^b and accordingly adjust $\{D_{i,b}^{(n,1)}, i \in \mathbf{V}_n^a\}$, an operation which only affects the constraint (21) for that particular $a \neq b$. With Observation 2.2 in mind, construct two *bridge* copies of the random multigraph $\mathbf{G}_n^{(1)}$ with the adjusted sub-degrees $\{D_{i,b}^{(n,1)}\}$, as follows:

- For each i and b , mark as BLUE(b) a uniformly chosen subset of $D_{i,b}^{(n,1)}$ half-edges incident to vertex i , the other part of which is, according to $\mathbf{G}_n^{(0)}$, in \mathbf{V}_n^b .
- Retain for $\mathbf{G}_n^{(1)}$ every edge of $\mathbf{G}_n^{(0)}$ where both parts are marked with BLUE.
- After removing all non-BLUE half-edges of $\mathbf{G}_n^{(0)}$, complete the construction of $\mathbf{G}_n^{(1)}$ by uniformly matching, for each $a \geq b$, all unmatched BLUE(b) half-edges of \mathbf{V}_n^a to all unmatched BLUE(a) half-edges of \mathbf{V}_n^b .
- A second copy of $\mathbf{G}_n^{(1)}$ is obtained by repeating the preceding construction, now with $\mathbf{G}_n^{(2)}$ taking the role of $\mathbf{G}_n^{(0)}$.

Replacing in the above procedure the multigraph $\mathbf{G}_n^{(0)}$ by the multigraph $\mathbf{G}_n^{(4)}$, the same construction produces a multigraph $\mathbf{G}_n^{(3)}$ having sub-degrees

$$D_{i,b}^{(n,3)} \leq D_{i,b}^{(n,2)} \wedge D_{i,b}^{(n,4)}, \quad (2.12)$$

and two *bridge* copies of $\mathbf{G}_n^{(3)}$ which are coupled (using such BLUE marking), to $\mathbf{G}_n^{(2)}$ and $\mathbf{G}_n^{(4)}$, respectively.

Next, as for (22), recall that $|\mathbf{E}_n^{(k)} \Delta \mathbf{E}_n^{(k-1)}| \leq |\mathbf{E}_{\mathbf{G}_n^{(k)}} \Delta \mathbf{E}_{\mathbf{G}_n^{(k-1)}}|$, which under our coupling is at most the number of edges of $\mathbf{G}_n^{(2\lceil k/2 \rceil)}$ that had at least one non-BLUE part. This in turn is at most

$$\Delta^{(n)} := \sum_{a,b=1}^{\ell} |m_{a,b}^{(n,k)} - m_{a,b}^{(n,k-1)}|.$$

Our construction is such that $m_{a,b}^{(n,0)} \wedge m_{a,b}^{(n,2)} \geq m_{a,b}^{(n,1)}$ and $m_{a,b}^{(n,4)} \wedge m_{a,b}^{(n,2)} \geq m_{a,b}^{(n,3)}$. Further, if the sub-degrees of bridge multigraphs were set by (23), then

$$m_{a,b}^{(n,0)} + m_{a,b}^{(n,2)} - 2m_{a,b}^{(n,1)} = \sum_{i \in \mathbf{V}_n^a} |D_{i,b}^{(n,0)} - D_{a,b}^{(n)}| := \Delta_{a,b}^{(n,1)},$$

for any $1 \leq a, b \leq \ell$, with analogous identities relating $m_{a,b}^{(n,3)}$ and $\Delta_{a,b}^{(n,3)}$. Since (21) holds for $k = 0, 2, 4$, while $m_{a,b}^{(n,1)} \wedge m_{b,a}^{(n,1)}$, $b < a$ are not changed by the $\mathbf{G}_n^{(1)}$ sub-degree adjustments (and similarly for the $\mathbf{G}_n^{(3)}$ sub-degree adjustments), we deduce that

$$\Delta^{(n)} \leq 2 \sum_{a,b=1}^{\ell} \Delta_{a,b}^{(n,1)} + 2 \sum_{a,b=1}^{\ell} \Delta_{a,b}^{(n,3)}.$$

Thus, we have (22) as soon as we show that for any $1 \leq a, b \leq \ell$,

$$\mathbb{E} \Delta_{a,b}^{(n,1)} + \mathbb{E} \Delta_{a,b}^{(n,3)} = o(n\omega_n),$$

which by our choice of $\{D_{a,b}^{(n)}\}$ follows from having for any fixed $i \in \mathbf{V}_n^a$,

$$\mathbb{E} |\omega_n^{-1} D_{i,b}^{(n,0)} - q_{a,b}| + \mathbb{E} |\omega_n^{-1} D_{i,b}^{(n,4)} - q_{a,b}| = o(1). \quad (2.13)$$

For $i \in \mathbf{V}_n^a$ the variable $D_{i,b}^{(n,4)}$ is Poisson with mean $(1+o(1))\lambda_{a,b}^{(n)} m_b^{(n)} = \omega_n q_{a,b}(1+o(1))$ (see (19)), hence $\mathbb{E} |\omega_n^{-1} D_{i,b}^{(n,4)} - q_{a,b}| \rightarrow 0$. Similarly, $D_{i,b}^{(n,0)}$ counts how many of the $d_a \omega_n$ half-edges emanating from such i , are paired by the uniform matching of the half-edges of \mathbf{G}_n , with half-edges from the subset \mathbf{E}_n^b of those incident to \mathbf{V}_n^b . With $|\mathbf{E}_n^b| = d_b \omega_n m_b^{(n)}$, the probability of a specific half-edge paired with an element of \mathbf{E}_n^b is $\mu_n = (|\mathbf{E}_n^b| - 1_{\{a=b\}}) / (2|\mathbf{E}_{\mathbf{G}_n}| - 1) \rightarrow d_b \nu_b$, hence $\omega_n^{-1} \mathbb{E} D_{i,b}^{(n,0)} = d_a \mu_n \rightarrow q_{a,b}$. It is not hard to verify that two specific half-edges incident to $i \in \mathbf{V}_n^a$ are both paired with elements of \mathbf{E}_n^b with probability $v_n = \mu_n^2(1+o(1))$. Consequently,

$$\text{Var}(\omega_n^{-1} D_{i,b}^{(n,0)}) \leq d_a \frac{\mu_n}{\omega_n} + d_a^2 (v_n - \mu_n^2) \rightarrow 0,$$

yielding the L^2 -convergence of $\omega_n^{-1} D_{i,b}^{(n,0)}$ to $q_{a,b}$ and thereby establishing (25). \blacksquare

Step III. We proceed to verify (20) for the single-adjacency matrices $\tilde{\mathbf{A}}_n$ of $\tilde{\mathbf{H}}_n$. To this end, as argued before, such weak convergence as in (20) is not affected by changing $o(n\omega_n)$ of the entries of $\tilde{\mathbf{A}}_n$, so WLOG we modify the law of number of loops in $\tilde{\mathbf{H}}_n$ incident to each $i \in \mathbf{V}_n^a$ to be a $\text{Po}(\lambda_{a,a}^{(n)})$ variable, yielding the symmetric matrix $\tilde{\mathbf{A}}_n$ of independent upper triangular Bernoulli($p_{a,b}^{(n)}$) entries, where $p_{a,b}^{(n)} = 1 - \exp(-\lambda_{a,b}^{(n)})$ when $i \in \mathbf{V}_n^a$ and $j \in \mathbf{V}_n^b$. In particular, the rank of $\mathbb{E}\tilde{\mathbf{A}}_n$ is at most ℓ , so by Lidskii's theorem we get (20) upon proving that $\mathcal{L}^{\hat{\mathbf{B}}_n} \Rightarrow \nu_{\hat{D}} \boxtimes \sigma_{\text{sc}}$ in probability, for $\hat{\mathbf{B}}_n := \omega_n^{-1/2}(\tilde{\mathbf{A}}_n - \mathbb{E}\tilde{\mathbf{A}}_n)$, a symmetric matrix of uniformly (in n) bounded, independent upper-triangular entries $\{\hat{Z}_{ij}\}$, having zero mean and variance $v_{a,b}^{(n)} := \omega_n^{-1}p_{a,b}^{(n)}(1 - p_{a,b}^{(n)}) = \frac{1}{n}d_a d_b(1 + o(1))$ when $i \in \mathbf{V}_n^a$, $j \in \mathbf{V}_n^b$. As a special case of Remark 1.3 (corresponding to piecewise-constant diagonal matrices with values $\{d_a\}_{a=1}^\ell$), such convergence holds for the symmetric matrices \mathbf{B}_n , whose independent centered Gaussian entries Z_{ij} have variance $v_{a,b}^{(n)}$ when $i \in \mathbf{V}_n^a$ and $j \in \mathbf{V}_n^b$, subject to on-diagonal rescaling $\mathbb{E}Z_{ii}^2 = 2v_{a(i),a(i)}^{(n)}$. As in the classical proof of Wigner's theorem by the moment's method (cf. [1, Sec. 2.1.3]), it is easy to check that for any fixed $k = 1, 2, \dots$,

$$\mathbb{E}\left[\frac{1}{n} \text{tr}(\hat{\mathbf{B}}_n^k)\right] = \mathbb{E}\left[\frac{1}{n} \text{tr}(\mathbf{B}_n^k)\right](1 + o(1)),$$

since both expressions are dominated by those cycles of length k that pass via each entry of the relevant matrix exactly twice (or not at all). Further, adapting the concentration argument of [1, Sec. 2.1.4] we deduce that as in the Wigner's case, $\langle x^k, \mathcal{L}^{\hat{\mathbf{B}}_n} - \mathbb{E}\mathcal{L}^{\hat{\mathbf{B}}_n} \rangle \rightarrow 0$ in probability, for each fixed k , thereby completing the proof of Theorem 1.1(a).

To prove Theorem 1.1(b), recall that $|\mathbf{E}_n \triangle \mathcal{E}_n| \leq \Delta_n$ for any coupling of the pair of matching which generate the graphs \mathbf{G}_n and \mathcal{G}_n . Appealing to Proposition 1.4 and the bound (7) following it, we get that under the coupling μ provided by that proposition,

$$\mathbb{E}_\mu[|\mathbf{E}_n \triangle \mathcal{E}_n|] \leq \mathbb{E}_\mu[\Delta_n] \leq \sqrt{2\mathbb{E}_\mu[\Delta_n(\Delta_n - 1)]} \leq 4b_n,$$

where (recalling from (1) that $\omega_n = (2 + o(1))|\mathbf{E}_n|/n$)

$$\begin{aligned} b_n^2 &:= \sqrt{2|\mathbf{E}_n|} \sum_{j=1}^n D_j^2 = (1 + o(1))n^{3/2} \sqrt{\omega_n} \mathbb{E}_{U_n}(D_{U_n}^{(n)})^2 \\ &= (1 + o(1))n^{3/2} \omega_n^{5/2} \mathbb{E}_{U_n}(\hat{D}_{U_n}^{(n)})^2 = o(n^2 \omega_n^2) \end{aligned}$$

via our assumption on the RHS of (3); thus, $\mathbb{E}_\mu[|\mathbf{E}_n \triangle \mathcal{E}_n|] = o(n\omega_n)$. We claim that Lemma 2.1 then concludes the proof. To see this, set $\hat{\mathbf{B}}'_n \equiv \omega_n^{-1/2} \mathbf{A}_{\mathcal{G}_n}$ and further let $\hat{\mathbf{A}}_n \equiv \omega_n^{-1/2} \mathbf{A}_n$ for the single-adjacency matrix \mathbf{A}_n associated with $\mathbf{A}_{\mathbf{G}_n}$. Since the entries of \mathbf{A}_n and $\mathbf{A}_{\mathcal{G}_n}$ may differ at most by one from each other, (2) implies that

$$\mathbb{E}_\mu \left[\frac{1}{n} \text{tr}((\hat{\mathbf{A}}_n - \hat{\mathbf{B}}'_n)^2) \right] \leq \frac{2}{n\omega_n} \mathbb{E}_\mu[|\mathbf{E}_n \triangle \mathcal{E}_n|] \rightarrow 0,$$

as required for Lemma 2.1. ■

Proof of Corollary 1.2. The assumed growth of ω_n yields (2) out of (1). In case of \mathbf{G}_n , the latter amounts to

$$\frac{1}{n} \sum_{i=1}^n \hat{D}_i^{(n)} \rightarrow 1, \quad \text{in probability,} \quad (2.14)$$

which we get by applying the L^2 -WLLN for triangular arrays with uniformly bounded second moments. The same reasoning yields the required uniform integrability in (3), namely, that when $n \rightarrow \infty$ followed by $r \rightarrow \infty$

$$\frac{1}{n} \sum_{i=1}^n \hat{D}_i^{(n)} \mathbb{1}_{\{\hat{D}_i^{(n)} \geq r\}} \rightarrow 0, \quad \text{in probability.} \quad (2.15)$$

Further, applying the weak law for non-negative triangular arrays $\{(\hat{D}_i^{(n)})^2\}$ of uniformly bounded mean, at truncation level $b_n \gg n$, it is not hard to deduce that

$$\frac{1}{b_n} \sum_{i=1}^n (\hat{D}_i^{(n)})^2 \rightarrow 0, \quad \text{in probability,} \quad (2.16)$$

whereupon, considering $b_n = n/\sqrt{\omega_n/n}$ results with the RHS of (3). Next, recall that the empirical measures $\mathcal{L}^{\hat{\Lambda}_n}$ of i.i.d. $\hat{D}_i^{(n)}$ converge in probability to the weak limit $\nu_{\hat{D}}$ of the laws of $\hat{D}_1^{(n)}$. Thus, Theorem 1.1(a) applies for \mathbf{G}_n of degrees $[\omega_n \hat{D}_i^{(n)}]$, yielding Corollary 1.2 in this case.

Turning to the case of uniform simple graphs, thanks to (27), truncating the degrees $[\omega_n \hat{D}_i^{(n)}]$ at some $\bar{d}_n \gg \omega_n$ removes at most $o(n\omega_n)$ edges from \mathbf{E}_n . Thus, such truncation neither affects (1), nor the preceding verification of (3). Further, such truncation alters only $o(n)$ degrees, yielding the same limit $\nu_{\hat{D}}$ for $\mathcal{L}^{\hat{\Lambda}_n}$. In view of Theorem 1.1(b), the stated convergence of $\mathcal{L}^{\hat{\Lambda}_{g_n}}$ holds, provided that $\{[\omega_n \hat{D}_i^{(n)}] \wedge \bar{d}_n\}$ are graphical WHP as $n \rightarrow \infty$. To this end, inspired by the proof of [3, Theorem 1(d)], recall from the Erdős-Gallai theorem, that integers $d_1 \geq d_2 \geq \dots \geq d_n \geq 0$ are graphical if

$$2 \sum_{i=1}^j d_i \leq j(j-1) + \sum_{i=1}^n \min(j, d_i), \quad \forall 1 \leq j \leq n. \quad (2.17)$$

Thanks to (2) we can fix $j_n = o(n)$ such that $j_n/\sqrt{n\omega_n} \rightarrow \infty$. The LHS of (29) is in our setting at most $2\omega_n \sum_i \hat{D}_i^{(n)}$, which in view of (26) is for $j > j_n$ negligible in comparison with the term $j(j-1)$ on the RHS of (29). Denoting by $o_p(1)$ the LHS of (28) at $b_n = n^2/j_n \gg n$, we further have here that the LHS of (29) is at most

$$2 \min \left(j \bar{d}_n, n\omega_n o_p(1) \right), \quad \forall 1 \leq j \leq j_n. \quad (2.18)$$

The Paley–Zygmund inequality yields $\inf_n \mathbb{P}(\hat{D}_i^{(n)} \geq 2/3) \geq 2\delta$, for some $\delta > 0$. Hence,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{D_i^{(n)} \geq \omega_n/3\}} > \delta, \quad \text{in probability.}$$

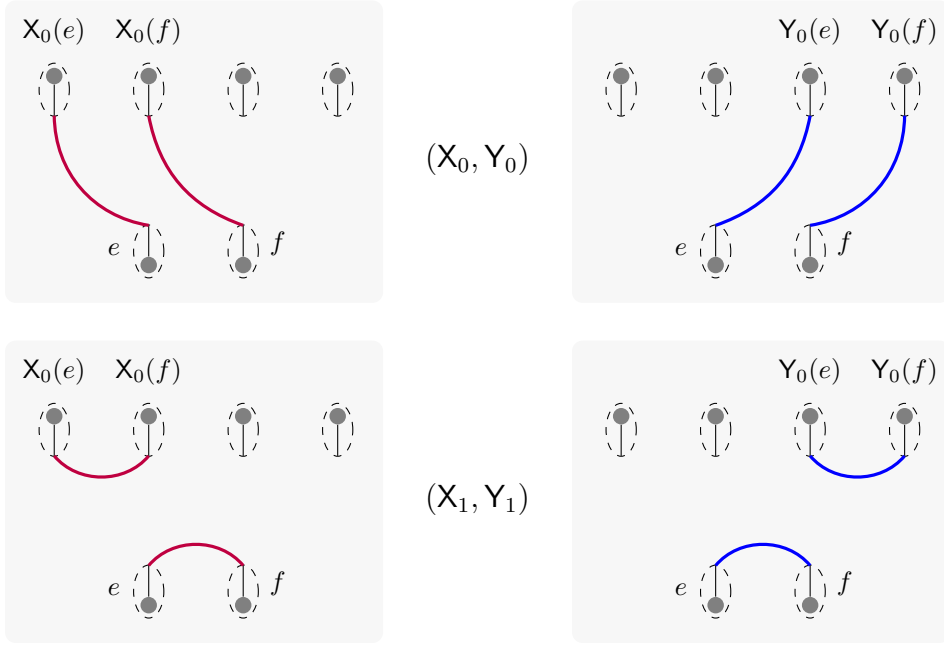


FIGURE 4. Coupling of the chains (X_t, Y_t) corresponding to (G_n, \mathcal{G}_n) .

This yields that the right-most term in (29) is for all large n and $j \in [n]$, at least

$$\delta \min(jn, n\omega_n/3),$$

which in turn exceeds (30) (as $\bar{d}_n = o(n)$), thus completing the proof. \blacksquare

3. COUPLING SIMPLE GRAPHS AND MULTIGRAPHS: PROOF OF PROPOSITION 1.4

Fixing *graphical* degrees $D_1 \geq D_2 \geq \dots \geq D_n$, let $m_n := \sum_i D_i = 2|E_n|$. Enumerate the m_n half-edges as follows: each half-edge e is identified with a vertex $v(e) \in [n]$; the first D_1 half-edges have $v(e) = 1$, the next D_2 have $v(e) = 2$ and so on. A matching of half-edges $m : [m_n] \mapsto [m_n]$ is an involution without fixed points (i.e., $m(e) = m^{-1}(e)$ and $m(e) \neq e$ for all $e \in [m_n]$). A coupled pair of multigraphs (G_n, \mathcal{G}_n) is hereby represented by a pair of matching (X, Y) , restricting $Y(\cdot)$ to the non-empty collection of matching that correspond to a simple graph; namely, $v(e) \neq v(Y(e))$ (no loops) and $\{v(e), v(Y(e))\} \neq \{v(f), v(Y(f))\}$ (no multiple edges) for any $f \neq \{e, Y(e)\}$.

Starting from any such pair of matching (X_0, Y_0) , consider the switching Markov chain (X_k, Y_k) that proceeds as following (see also Figure 4):

- Uniformly choose $e \neq f \in [m_n]$ and disconnect their matching in X_k and Y_k ;
- Reconnect e with f , and $X_k(e)$ with $X_k(f)$, to get the match X_{k+1} ;
- If reconnecting e with f and $Y_k(e)$ with $Y_k(f)$ yields a simple graph, set this to be Y_{k+1} . Otherwise, leave $Y_{k+1} = Y_k$ unchanged.

We say that coupling succeeds in the k -th step if the proposed move to Y_{k+1} results in a simple graph, otherwise saying that the coupling failed (in the k -th step).

The marginal (X_k) evolves as a Markov chain in the space of all matching, with the marginal (Y_k) likewise evolving as a Markov chain in the non-empty subset of all matching that correspond to simple graphs with the specified degrees. These switching chains are further reversible with respect to the corresponding uniform measures. Both marginal chains have been extensively studied as means of sampling uniform graphs subject to given degrees. In particular, it is well-known ([18]; cf. also the recent work [10]) that each of these marginals is an irreducible Markov chain. Having a non-empty finite state space, the Markov chain (X_k, Y_k) admits an invariant probability measure μ , and by the preceding, any such μ is a coupling between the random multigraph G_n and the corresponding uniformly simple graph \mathcal{G}_n of the specified degrees.

Denoting by

$$C_k \equiv \{e \in [m_n] : X_k(e) = Y_k(e)\},$$

the common part of the two matching X_k, Y_k , note that under an invariance measure $\mathbb{E}_\mu[|C_k|]$ must be independent of k . We further have the following lower bound on the change between $|C_{k+1}|$ and $|C_k|$:

$$|C_{k+1}| - |C_k| \geq 2\mathbb{1}_{\{e, f \notin C_k\}} - 4\mathbb{1}_{\{\text{coupling fails in step } k\}}. \quad (3.1)$$

Indeed, (31) is verified by enumerating over the seven possible cases for $e, f \in [m_n]$:

- I. $X_0(e) = Y_0(e) = f$;
- II. $X_0(e) = Y_0(e) \neq f, X_0(f) = Y_0(f)$;
- III. $X_0(e) = Y_0(e) \neq f, X_0(f) \neq Y_0(f)$ or $X_0(f) = Y_0(f) \neq e, X_0(e) \neq Y_0(e)$;
- IV. $X_0(e) = f \neq Y_0(e)$ or $Y_0(e) = f \neq X_0(e)$;
- V. $X_0(e) = Y_0(f), X_0(f) = Y_0(e)$;
- VI. $X_0(e) = Y_0(f), X_0(f) \neq Y_0(e)$ or $X_0(f) = Y_0(e), X_0(e) \neq Y_0(f)$;
- VII. $e, f, X_0(e), X_0(f), Y_0(e), Y_0(f)$ are six distinct half-edges.

The corresponding value of $|C_1| - |C_0|$ in each of these cases are given in Table 1, from which it follows that under an invariant measure μ ,

$$0 = \mathbb{E}[|C_1| - |C_0|] \geq 2\mathbb{P}(e, f \notin C_0) - 4\mathbb{P}(\text{coupling fails}). \quad (3.2)$$

For the first term on the RHS of (32),

$$\mathbb{P}(e, f \notin C_0 \mid C_0) = \left(\frac{m_n - |C_0|}{m_n}\right) \left(\frac{m_n - |C_0| - 1}{m_n - 1}\right). \quad (3.3)$$

Combining these, we get that the LHS of (6) is at most $2m_n(m_n - 1)\mathbb{P}(\text{coupling fails})$.

For the latter, note that the coupling fails only under one of the following scenarios:

- (a) *introducing a loop*: $v(e) = v(f)$ or $v(Y_0(e)) = v(Y_0(f))$;
- (b) *introducing multiple edges*: $v(e)$ is connected to $v(f)$ in $Y_0 \setminus \{(e, Y_0(e)), (f, Y_0(f))\}$, or $v(Y_0(e))$ is connected to $v(Y_0(f))$ in $Y_0 \setminus \{(e, Y_0(e)), (f, Y_0(f))\}$.

As $(Y_0(e), Y_0(f))$ has the same (uniform) distribution as (e, f) , we thus deduce that

$$\frac{1}{2}\mathbb{P}(\text{coupling fails}) \leq \mathbb{P}(v(e) = v(f)) + \mathbb{P}(v(e) \text{ connected to } v(f)).$$

Case	Criterion		$ C_1 - C_0 $	success failure
I	$X_0(e) = f = Y_0(e)$		0	—
II	$X_0(e) = Y_0(e) \neq f$ $X_0(f) = Y_0(f)$		0	-4
III	$X_0(e) = Y_0(e) \neq f$ $X_0(f) \neq Y_0(f)$		0	-2
	$X_0(f) = Y_0(f) \neq e$ $X_0(e) \neq Y_0(e)$			
IV	$X_0(e) = f \neq Y_0(e)$		(*) ≥ 2	0
	$Y_0(e) = f \neq X_0(e)$			
V	$X_0(e) = Y_0(f)$ $X_0(f) = Y_0(e)$		4	0
VI	$X_0(e) = Y_0(f)$ $X_0(f) \neq Y_0(e)$		2	0
	$X_0(f) = Y_0(e)$ $X_0(e) \neq Y_0(f)$			
VII	$e, X_0(e), Y_0(e),$ $f, X_0(f), Y_0(f)$ are all distinct		(*) ≥ 2	(*) ≥ 0

TABLE 1. Analysis of the change in the size of the common part of the two matchings after one step of the coupling. In cases marked by (*), the difference could be larger if $X_0(Y_0(e)) = Y_0(f)$ or $Y_0(X_0(e)) = X_0(f)$.

With q_{ij} denoting the probability that $i \neq j$ are adjacent in Y_0 , clearly

$$\mathbb{P}(v(e) \text{ connected to } v(f)) = \sum_{i \neq j} \frac{(D_i - 1)(D_j - 1)}{m_n(m_n - 1)} q_{ij}.$$

Similarly, recalling that $\sum_j q_{ij} = D_i$ for any $i \in [n]$, we have that

$$\mathbb{P}(v(e) = v(f)) = \sum_{i=1}^n \frac{D_i(D_i - 1)}{\mathfrak{m}_n(\mathfrak{m}_n - 1)} = \sum_{i \neq j} \frac{(D_i + D_j)/2 - 1}{\mathfrak{m}_n(\mathfrak{m}_n - 1)} q_{ij}.$$

Adding these expressions and reducing the sum by symmetry to $j > i$, we arrive at

$$\frac{1}{2} \mathbb{P}(\text{coupling fails}) \leq \frac{1}{\mathfrak{m}_n(\mathfrak{m}_n - 1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n (2D_i D_j - D_i - D_j) q_{ij}. \quad (3.4)$$

With $j \mapsto D_j$ non-decreasing and $\sum_{j>i} q_{ij} \leq D_i$, by replacing q_{ij} with $\mathbb{1}_{\{j \leq i + D_i\}}$ we upper bound the RHS of (3.4). Combining this with (32)–(33) establishes (6), thereby concluding the proof of Proposition 1.4. \blacksquare

4. ANALYSIS OF THE LIMITING DENSITY

Remark 4.1. With $\nu^{(2)}$ denoting the pushforward of ν by the map $x \mapsto x^2$ (that is, the weak limit of $\mathcal{L}^{\Lambda_n^2}$), we have similarly to Remark 1.3 that $\mu_{\text{MP}} = \nu^{(2)} \boxtimes \sigma_{\text{SC}}^{(2)}$, where the pushforward $\sigma_{\text{SC}}^{(2)}$ (of density $(2\pi)^{-1} \sqrt{4/x - 1}$ on $[0, 4]$), is the limiting empirical distribution of singular values of $n^{-1/2} \tilde{\mathbf{X}}_n$.

Proof of Proposition 1.7. The matrix $\mathbf{M}_n := n^{-1} \tilde{\mathbf{X}}_n \Lambda_n^2 \tilde{\mathbf{X}}_n^*$ has the same ESD as $n^{-1} \Lambda_n \tilde{\mathbf{X}}_n \tilde{\mathbf{X}}_n^* \Lambda_n$. Thus, μ_{MP} is also the limiting ESD for \mathbf{M}_n (see [12, 15]). Taking $\mathcal{L}^{\Lambda_n} \Rightarrow \nu$ with $d\nu/d\nu_{\hat{D}}(x) = x$ yields the Cauchy–Stieltjes transform $G_{\mu_{\text{MP}}}(z) = h(z)$ which is the unique decaying to zero as $|z| \rightarrow \infty$, \mathbb{C}_+ -valued analytic on \mathbb{C}_+ , solution of

$$h = \left(\mathbb{E} \left[\frac{\hat{D}^2}{1+h\hat{D}} \right] - z \right)^{-1} = -z^{-1} \mathbb{E} \left[\frac{\hat{D}}{1+h\hat{D}} \right]. \quad (4.1)$$

Indeed, the LHS of (35) merely re-writes the fact that $\xi(\cdot)$ of (11) is such that $\xi(h(z)) = z$ on \mathbb{C}_+ , while having $\int x d\nu_{\hat{D}} = 1$, one thereby gets the RHS of (35) by elementary algebra. Recall [2, Prop. 5(a)] that the Cauchy–Stieltjes transform of the symmetric measure $\tilde{\mu}$ having the pushforward $\tilde{\mu}^{(2)} = \mu_{\text{MP}}$ under the map $x \mapsto x^2$, is given for $\Re(z) > 0$ by $g(z) = zh(z^2) : \mathbb{C}_+ \mapsto \mathbb{C}_+$, which by the RHS of (35) satisfies for $\Re(z) > 0$,

$$g = -\mathbb{E} \left[\frac{\hat{D}}{z + g\hat{D}} \right]. \quad (4.2)$$

By the symmetry of the measure $\tilde{\mu}$ on \mathbb{R} we know that $g(-\bar{z}) = -\bar{g}(z)$ thereby extending the validity of (36) to all $z \in \mathbb{C}_+$. Applying the implicit function theorem in a suitable neighborhood of $(-z^{-1}, g) = (0, 0)$ we further deduce that $g(z) = G_{\tilde{\mu}}(z)$ is the unique \mathbb{C}_+ -valued, analytic on \mathbb{C}_+ solution of (36) tending to zero as $\Im(z) \rightarrow \infty$. Recall the S -transform defined via (4)–(5) for $\varphi \neq \delta_0$ supported on \mathbb{R}_+ and similarly for symmetric measure ψ . In particular (see [2, Eqn. (20)],

$$S_{\sigma_{\text{SC}}}(w) = w^{-1/2}.$$

Further, from (4) we see that (36) results with $m_{\nu_{\hat{D}}}(-z^{-1}g) = g^2$, yielding

$$S_{\nu_{\hat{D}}}(g^2) = -(1 + g^{-2})z^{-1}g.$$

Since $S_{\mu}(w) = S_{\nu_{\hat{D}}}(w)S_{\sigma_{\text{sc}}}(w)$, we get $S_{\mu}(g^2) = -(1 + g^{-2})z^{-1}$ and consequently $m_{\mu}(-z^{-1}) = g^2$. The latter amounts to

$$f(z) := -z^{-1}(1 + g^2) = \int \frac{1}{-t - z} d\mu(t), \quad (4.3)$$

which since μ is symmetric, matches the stated relation $f(z) = G_{\mu}(z)$ of (8). \blacksquare

Proof of Proposition 1.8. Recall from (37) that $f(z) = -zh(z^2)^2 - z^{-1}$ for $z \in \mathbb{C}_+$ and $\Re(z) > 0$. When $z \rightarrow x \in (0, \infty)$ we further have that $h(z^2) \rightarrow h(x^2)$ and hence

$$\frac{1}{\pi} \Im(f(z)) \rightarrow -\frac{1}{\pi} \Im(xh(x^2)^2) = -2\Re(h(x^2))\tilde{\rho}(x), \quad (4.4)$$

where the last identity is due to (9). Thus, for a.e. $x > 0$ the density $\rho(x)$ exists and given by Plemelj formula, namely the RHS of (38). The continuity of $x \mapsto h(x)$ implies the same for the symmetric density $\rho(x)$, thereby we deduce the validity of (10) at every $x \neq 0$. While proving [16, Thm. 1.1] it was shown that $h(z)$ extends analytically around each $x \in \mathbb{R} \setminus \{0\}$ where $\Im(h(x)) > 0$ (see also Remark 1.9). In particular, (10) implies that $\rho(x)$ is real analytic at any $x \neq 0$ where it is positive. Further, in view of (10), the support identity $\text{supp}(\mu) = \text{supp}(\tilde{\mu})$ is an immediate consequence of having $\Re(h(x)) < 0$ for all $x > 0$ (as shown in Lemma 4.2). Similarly, the stated relation with $\text{supp}(\mu_{\text{MP}})$ follows from the explicit relation $\tilde{\rho}(x) = |x|\rho_{\text{MP}}(x^2)$. Finally, Lemma 4.2 provides the stated bounds on $\tilde{\rho}$ and ρ (see (39) and (40), respectively), while showing that if $\nu_{\hat{D}}(\{0\}) = 0$ then μ is absolutely continuous. \blacksquare

Our next lemma provides the estimates we deferred when proving Proposition 1.8.

Lemma 4.2. *The function $g(z) = G_{\tilde{\mu}}(z)$ satisfies*

$$|g(z)| \leq 1 \wedge \frac{2}{|\Re(z)|}, \quad \forall z \in \mathbb{C}_+ \cup \mathbb{R} \quad (4.5)$$

and (36) holds for $z \in \mathbb{C}_+ \cup \mathbb{R} \setminus \{0\}$, resulting with $\Re(h(x)) < 0$ for $x > 0$. In addition

$$\rho(x) \leq \frac{1}{\pi} (\mathbb{E}\hat{D}^{-2})^{1/2} \wedge 4|x|^{-3} \quad \forall x \in \mathbb{R}, \quad (4.6)$$

and if $\nu_{\hat{D}}(\{0\}) = 0$, then $\mu(\{0\}) = 0$.

Proof. As explained when proving Proposition 1.7, by the symmetry of $\tilde{\mu}$, we only need to consider $\Re(z) \geq 0$. Starting with $z \in \mathbb{C}_+$, let

$$\begin{aligned} z &= x + i\eta && \text{for } x \geq 0 \text{ and } \eta > 0, \\ g(z) &= -y + i\gamma && \text{for } y \in \mathbb{R} \text{ and } \gamma > 0. \end{aligned}$$

Then, separating the real and imaginary parts of (36) gives

$$y = \mathbb{E} \left[\hat{D}(x - y\hat{D})\hat{W}^{-2} \right], \quad \gamma = \mathbb{E} \left[\hat{D}(\eta + \gamma\hat{D})\hat{W}^{-2} \right], \quad (4.7)$$

where $\hat{W} := |z + g(z)\hat{D}|$ must be a.s. strictly positive (or else $\gamma = \infty$). Next, defining

$$A = A(z) := \mathbb{E}[\hat{D}\hat{W}^{-2}], \quad B = B(z) := \mathbb{E}[\hat{D}^2\hat{W}^{-2}], \quad (4.8)$$

both of which are positive and finite (or else $\gamma = \infty$), translates (41) into

$$y = Ax - By, \quad \gamma = A\eta + B\gamma.$$

Therefore,

$$y = \frac{Ax}{1+B}, \quad \gamma = \frac{A\eta}{1-B}. \quad (4.9)$$

Since $\gamma > 0$, necessarily $0 < B < 1$ and $y \geq 0$ is strictly positive iff $x > 0$. Next, by (36), Jensen's inequality and (42),

$$|g| \leq \mathbb{E} \left[\hat{D}\hat{W}^{-1} \right] := V(z) \leq \sqrt{B} \leq 1. \quad (4.10)$$

Further, letting $D \sim \nu$ be the size-biasing of \hat{D} and $W := |z + g(z)D|$, we have that

$$g(z) = -\mathbb{E}[(z + g(z)D)^{-1}], \quad V = \mathbb{E}[W^{-1}], \quad A = \mathbb{E}[W^{-2}]. \quad (4.11)$$

With $B < 1$ we thus have by (43), (45) and Jensen's inequality, that

$$\frac{|x|A}{2} \leq \frac{|x|A}{1+B} = |y| \leq |g| \leq V \leq \sqrt{A}.$$

Consequently, $|g(z)| \leq \sqrt{A} \leq 2/|x|$ as claimed. Next, recall [16, Theorem 1.1] that $h(z) \rightarrow h(x)$ whenever $z \rightarrow x \neq 0$, hence same applies to $g(\cdot)$ with (39) and the bound $B(z) \leq 1$, also applicable throughout $\mathbb{R} \setminus \{0\}$. Further, having $z_n \rightarrow x \neq 0$ implies that $|\Re(z_n)|$ is bounded away from zero, hence $\{A(z_n)\}$ are uniformly bounded. In view of (45), this yields the uniform integrability of $(z_n + g(z_n)D)^{-1}$ and thereby its L_1 -convergence to the absolutely-integrable $(x + g(x)D)^{-1}$. Appealing to the representation (45) of $g(z)$ we conclude that (36) extends to $\mathbb{R} \setminus \{0\}$. Utilizing (36) at $z = x > 0$ we see that $0 < |g(x)|^2 \leq A(x)$ due to (45). Hence, from (41) we have as claimed,

$$\Re(h(x^2)) = x^{-1}\Re(g(x)) = \frac{-A(x)}{1+B(x)} < 0.$$

From (43) we have that $g(z) = i\gamma$ when $z = i\eta$, where by (36), for any $\delta > 0$,

$$\gamma = \mathbb{E} \left[\frac{\hat{D}}{\eta + \gamma\hat{D}} \right] \geq \frac{\delta}{\eta + \gamma\delta} \nu_{\hat{D}}([\delta, \infty)).$$

Taking $\eta \downarrow 0$ followed by $\delta \downarrow 0$ we see that $\gamma(i\eta) \rightarrow \gamma(0) = 1$, provided $\nu_{\hat{D}}(\{0\}) = 0$. By definition of the Cauchy–Stieltjes transform and bounded convergence, we have then

$$\mu(\{0\}) = -\lim_{\eta \downarrow 0} \Re(i\eta f(i\eta)) = 1 - [\lim_{\eta \downarrow 0} \gamma(i\eta)]^2 = 0,$$

due to (37) (and having $\Re(g(i\eta)) = 0$). Finally, from (36) and the LHS of (37) we have that $f(z) = -\mathbb{E}[(z + g(z)\hat{D})^{-1}]$ throughout \mathbb{C}_+ , hence by Cauchy–Schwarz

$$|f(z)| \leq \mathbb{E}[\hat{W}^{-1}] \leq \sqrt{B(z)\mathbb{E}[\hat{D}^{-2}]} \leq \mathbb{E}[\hat{D}^{-2}]^{1/2}$$

is uniformly bounded when $\mathbb{E}\hat{D}^{-2}$ is finite. Up to factor π^{-1} this yields the stated uniform bound on $\rho(x)$, namely the RHS of (38). At any $x > 0$ the latter is bounded above also by $\frac{1}{\pi x}|g(x)|^2$, with (40) thus a consequence of (39). \blacksquare

Proof of Corollary 1.10. Fixing $\alpha > \eta > 0$ we have that

$$\nu_{\hat{D}}(\{\alpha\}) = q_o, \quad \nu_{\hat{D}}(\{\eta\}) = 1 - q_o$$

and since $1 = \mathbb{E}\hat{D} = \alpha q_o + \eta(1 - q_o)$, further $\alpha > 1 > \eta$. By Remark 1.9 we identify $\text{supp}(\mu)$ upon examining the regions in which $\xi'(-v) > 0$ for \mathbb{R} -valued $v \notin \{0, \alpha^{-1}, \eta^{-1}\}$. Since $\Re(h(x)) < 0$ for $x > 0$ (see Lemma 4.2), for $\text{supp}(\mu) \cap \mathbb{R}_+$ it suffices to consider the sign of

$$\xi'(-v) = \frac{1}{v^2} - \frac{q\alpha^2}{(1 - v\alpha)^2} - \frac{(1 - q)\eta^2}{(1 - v\eta)^2},$$

when $v \in (0, \infty) \setminus \{\alpha^{-1}, \eta^{-1}\}$ and $q := \alpha q_o$. Observe that $\xi'(-v) > 0$ for such v iff

$$\begin{aligned} P(v) &:= av^3 + bv^2 + cv + d \\ &= -2\alpha\eta(q\eta + (1 - q)\alpha)v^3 + (q\eta^2 + 4\alpha\eta + (1 - q)\alpha^2)v^2 - 2(\alpha + \eta)v + 1 > 0. \end{aligned}$$

Noting that $\lim_{v \rightarrow \infty} P(v) = -\infty$ and $\lim_{v \downarrow 0} P(v) = 1$, we infer from Remark 1.9 that $\text{supp}(\mu)$ has holes iff $P(v)$ has three distinct positive roots. As Descartes' rule of signs is satisfied ($a, c < 0$ and $b, d > 0$), the latter occurs iff the discriminant $D(P)$ is positive. Evaluating $D(P)$ shows that

$$D(P) = b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2 = 4q(1 - q)(\alpha - \eta)^2(\alpha\phi - q\theta),$$

where

$$\theta := (\alpha - \eta)(\alpha + \eta)^3, \quad \phi := (\alpha - 2\eta)^3.$$

Having $q = \alpha q_o$ and $\theta > 0$ we conclude that $D(P) > 0$ iff $\phi/\theta > q_o$. That is

$$\frac{\phi}{\theta} = \frac{(\alpha - 2\eta)^3}{(\alpha - \eta)(\alpha + \eta)^3} > \frac{1 - \eta}{\alpha - \eta} = q_o.$$

For $\varphi := 3\eta/(\alpha + \eta)$ and $\eta \in (0, 1)$ this translates into $1 - \varphi > (1 - \eta)^{1/3}$, or equivalently

$$\frac{\alpha}{\eta} + 1 = \frac{3}{\varphi} > \frac{3}{1 - (1 - \eta)^{1/3}},$$

as stated in (12). \blacksquare

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