Instant Randomness

HOW MANY SWIPES OF YOUR SPOON will it take to stir your milk into your coffee? How many shuffles of a deck of cards does it take to randomize them? In many systems like these, the onset of randomness turns out to be quite sudden. Mathematically speaking, the system goes from unmixed to mixed in the blink of an eye.

The name for this abrupt mixing behavior is the "cutoff phenomenon," and the time T when mixing occurs is called the mixing time. To be more precise, when you stir your coffee or shuffle your cards a little bit less than the prescribed amount of time, your system is still a long way from being mixed. If you go a little past the mixing time, your system is essentially completely mixed, and further stirring or shuffling will not make very much difference.

The cutoff phenomenon was first observed in the 1980s, in some models chosen more for their mathematical tractability and symmetry than their relevance to physics. But recently, a team of mathematicians at Microsoft Research showed that cutoff occurs in several of the most important models of statistical physics, such as the Ising model (which simulates ferromagnetism), the Potts model, and the hard-core gas model. The result, announced by Eyal Lubetzky and Allan Sly, lends plausibility to a broad conjecture about mixing made a few years ago by Yuval Peres, also of Microsoft Research.

Peres has conjectured that cutoff is a general feature of systems in which the time for a small, local deviation from randomness to become smoothed out (called the "relaxation time") is less than the time for the entire system to achieve randomness (called the "mixing time"). Intuitively, the idea behind Lubetzky and Sly's work is this: if relaxation happens faster than mixing, then small deviations from randomness in different locations (such as small patches of milk in your coffee) do not have time to communicate with each other. Thus they evolve almost independently, and they are governed by the statistical laws of independent processes. In a nearly psychic way, once one patch becomes randomized, nearly all of them do. "My feeling, and conjecture, is that cutoff represents what happens to temporal dynamics when you have relative spatial independence," says Peres.

"When I first heard about Peres' conjecture, I said, 'Give me a break.' I didn't believe it," says Persi Diaconis of Stanford University. "Now we're proving it in example after example."

Although Peres' hypothesis remains unconfirmed in general, for the Ising model—perhaps the most thoroughly studied system in statistical physics—Lubetzky and Sly's theorem is "hard and original and definitive," Diaconis says. The cutoff phenomenon occurs in every case where you can reasonably expect it, and their work determines the cutoff time T precisely.

Eyal Lubetzky. *(Photo courtesy of E. Lubetzky and A. Sly.)*

Allan Sly. *(Photo courtesy of E. Lubetzky and A. Sly.)*

Simple Yet Confounding

The Ising model is named after Ernst Ising, who wrote about it in his 1924 doctoral thesis; sometimes it is called the Lenz-Ising model, because Ising's dissertation advisor Wilhelm Lenz had described it earlier in 1920. It is one of the simplest examples of a physical system that exhibits a phase transition. Even though it does not literally describe what happens in a magnet, it has served physicists well as an analogue or paradigm of the ferromagnetic phase transition.

In the Ising model, the magnet is considered to be a lattice of point atoms, each one of which can have a "spin" of $+1$ or -1. Neighboring atoms in the lattice like to have the same spin, and so an energy penalty accrues for each neighboring pair with opposite spins. For simplicity, the energy of a matching pair can be assumed to be -1 and the energy of a pair with opposing spins is $+1$. The energy of an entire configuration of spins σ, denoted by $H(\sigma)$, is the sum of the energy of all the adjoining pairs:

$$H(\sigma) = -\sum_{u\sim v} \sigma(u)\sigma(v).$$

(Here the sum is taken over all pairs of adjacent vertices in the lattice; $u \sim v$ signifies that the vertices u and v are adjacent; and $\sigma(u)$ is the spin at vertex u, either $+1$ or -1. Note that $\sigma(u)\sigma(v) = 1$ when the spins at u and v match.)

Ising and Lenz assumed that the probability of any particular configuration of spins, $\mu(\sigma)$, decreases exponentially as the energy of the configuration increases. High-energy configurations are not impossible; they are merely unlikely. In symbols, their assumption can be written as follows:

$$\mu(\sigma) \propto e^{-\beta H(\sigma)}.$$

Two points should be made about this equation. First, the proportionality constant β plays the role of an *inverse temperature*. When β is large (i.e., the temperature is near zero), high-energy configurations are very improbable. In such a state, the system should have a very strong preference for having all the atoms with the same spin. As β decreases (or the temperature increases), configurations with high energy H become more likely. This means that the system can tolerate a lot of non-matching spins, and there is more disorder—as we would expect at a higher temperature. Finally, when $\beta = 0$ (or the temperature is infinite), the system is completely disordered; every configuration has an equal probability of occurring.

The second point is that the exact probability density function μ, called the Gibbs distribution, is beyond the power of any computer, now or in the future, to compute. The reason is that μ has to be normalized so that the sum of all the probabilities is 1. To do this, you would have to compute the weighting factor $e^{-\beta H(\sigma)}$ for every single configuration σ, and then divide by the sum. (This sum is called the "partition function.") How many configurations are there? Even for a tiny, 10-by-10 lattice, there are 100 vertices and 2^{100} different ways to assign a spin to each of them. To store all of this information you would need a computer the size of the universe. Now imagine trying to do the same thing for a billion-by-billion-by-billion lattice, representing a real-world crystal!