Cover time of a random graph with a degree sequence II: Allowing vertices of degree two

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Abstract

We study the cover time of a random graph chosen uniformly at random from the set of graphs with vertex set $[n]$ and degree sequence $\mathbf{d} = (d_i)_{i=1}^n$. In a previous work [1], the asymptotic cover time was obtained under a number of assumptions on $\mathbf{d}$, the most significant being that $d_i \geq 3$ for all i. Here we replace this assumption by $d_i \geq 2$. As a corollary, we establish the asymptotic cover time for the 2-core of the emerging giant component of $\mathcal{G}(n,p)$.

1 Introduction

Let $G = (V,E)$ be a connected graph with n vertices and m edges. For $v \in V$, let C_v be the expected time for a simple random walk $\mathcal{W}_v$ on G starting at v, to visit every vertex of G. The *(vertex) cover time* $T_{\mathrm{COV}}(G)$ of G is defined as $T_{\mathrm{COV}}(G) = \max_{v \in V} C_v$. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [3] that $T_{\mathrm{COV}}(G) \leq 2m(n-1)$. Feige [16, 17] showed that the cover time of any connected graph G satisfies $(1-o(1))n \ln n \leq T_{\mathrm{COV}}(G) \leq (1+o(1))\frac{4}{27}n^3$. Between these two extremes, the cover time, both exact and asymptotic, has been extensively studied for different classes of graphs (see, e.g., [2] for an introduction to the topic).

In the context of random graphs, a basic question is to understand the cover time for the giant component $\mathcal{C}_1$ of the celebrated Erdős-Rényi [15] random graph model $\mathcal{G}(n,p)$. Decomposing the giant $\mathcal{C}_1$ into the *2-core* $\mathcal{C}_1^{(2)}$ (its maximal subgraph of minimum degree 2) and collection of trees decorating $\mathcal{C}_1^{(2)}$, much is known about their structure (see, e.g., the characterization theorems in the recent works [12, 13]). However, our understanding of the cover time for these remains incomplete.

It is well-known that for $G \sim \mathcal{G}(n, p = c/n)$ with $c > 1$ fixed, the giant component $\mathcal{C}_1$ is roughly of size xn where $x = x(c)$ is the solution in $(0,1)$ of $x = 1 - e^{-cx}$. Cooper and Frieze [9] showed that in this regime

$$T_{\mathrm{COV}}(\mathcal{C}_1) \sim \frac{cx(2-x)}{4(cx - \ln c)} n \ln^2 n \quad \text{and} \quad T_{\mathrm{COV}}(\mathcal{C}_1^{(2)}) \sim \frac{cx^2}{16(cx - \ln c)} n \ln^2 n \tag{1.1}$$

with high probability (w.h.p.), i.e., with probability tending to 1 as $n \to \infty$. However, analogous results for $p = (1+\varepsilon)/n$ with $\varepsilon = o(1)$, $\varepsilon^3 n \to \infty$ (the *emerging* giant component) were unavailable.

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Barlow *et al.* [4] showed that when $p = (1+\varepsilon)/n$ with $n^{-1/3} \ll \varepsilon \ll 1$ (here and in what follows we let $A_N \ll B_N$ denote $\lim_{N\to\infty} A_N/B_N = 0$) the cover time $T_{\text{COV}}(\mathcal{C}_1)$ is of order $n\log^2(\varepsilon^3 n)$. With this in mind, substituting $c = 1+\varepsilon$ with $\varepsilon > 0$ in the estimates of (1.1), and noting that the aforementioned $x(c)$ becomes $2\varepsilon + O(\varepsilon^2)$, shows that for small *fixed* $\varepsilon > 0$, w.h.p.

$$T_{\text{COV}}(\mathcal{C}_1) = (1+O(\varepsilon))n\ln^2(\varepsilon^3 n) \quad \text{and} \quad T_{\text{COV}}(\mathcal{C}_1^{(2)}) = \frac{\varepsilon + O(\varepsilon^2)}{4} n \ln^2(\varepsilon^3 n)\,, \tag{1.2}$$

and one may expect these results to hold throughout the emerging giant regime of $n^{-1/3} \ll \varepsilon \ll 1$.

A natural step towards this goal is to exploit the well-known characterizations of $\mathcal{C}_1$, its 2-core and its *kernel*: as mentioned above, by stripping the giant component of its attached trees one arrives at the 2-core $\mathcal{C}_1^{(2)}$. By further shrinking every induced path in $\mathcal{C}_1^{(2)}$ into a single edge one arrives at the kernel K (see §2.1 for more details). It was shown by Łuczak [22] that the kernel of the emerging giant component is a random multi-graph on a certain degree sequence, and so, potentially, the cover times of K, $\mathcal{C}_1^{(2)}$ and $\mathcal{C}_1$ could all be determined as a consequence of general results on the cover-time of random graphs with a given degree sequence.

Promising in that regard is a framework developed by Cooper and Frieze, which was already successful in tackling this problem for a variety of random graph models, notably including random regular graphs [6] and random graphs with certain degree sequences [1] (also see [6–10]). However, among the various conditions on the degree sequence in [1], a main caveat was the requirement that the minimal degree should be at least 3, rendering this machinery useless for analyzing the 2-core.

In this paper we eliminate this restriction and allow vertices of degree 2 in the degree sequence. Of course, if our degree sequence $\mathbf{d}$ features linearly many degrees that are 2 — as in the case of the 2-core of the emerging giant — a uniformly chosen graph with these degrees will typically contain linearly many isolated cycles, which would have to be removed. To avoid this issue, we let the degree 2 vertices arise as they do in the giant component, as subdivision of kernel edges:

- Given $\mathbf{d} = (d_1 \le d_2 \le \cdots \le d_n)$ with $d_i \ge 2$ for all i, let ν_2 be the number of degree 2 vertices, and let $\mathbf{d}_3$ be the degree sequence restricted to all i such that $d_i \ge 3$.
- Choose the kernel $K_{\mathbf{d}} \sim \mathcal{G}_{\mathbf{d}_3}$, i.e., uniformly from all multi-graphs with degree sequence $\mathbf{d}_3$.
- Replace each edge e of $K_{\mathbf{d}}$ by a path P_e of length ℓ_e (edges), where the values of $\{\ell_e : e \in E(K_{\mathbf{d}})\}$ are uniform over all $\binom{\nu_2+|E(K_{\mathbf{d}})|-1}{\nu_2}$ possible choices, to obtain the final graph $G_{\mathbf{d}}$.

Under several natural conditions on $\mathbf{d}$ (e.g., satisfied when it has a power law/exponential tail, as in the 2-core of $\mathcal{C}_1$), detailed next, we can determine the asymptotic cover time of $T_{\text{COV}}(G_{\mathbf{d}})$.

Definition 1.1. *Let $\boldsymbol{d} = (d_i)_{i=1}^n$ and let $\nu_j = \#\{i : d_i = j\}$ count the degree-j vertices in $\boldsymbol{d}$. Let N, M, d be the number of vertices, number of edges and minimum degree in the associated kernel:*

$$N = \textstyle\sum_{j\ge 3} \nu_j\,, \qquad M = \frac{1}{2}\sum_{j\ge3} j\nu_j \qquad d = \min\{j \ge 3 : \nu_j \ne 0\}\,.$$

We say that $\boldsymbol{d}$ is nice *(and similarly, $\mathcal{G}_{\boldsymbol{d}}$ is nice) if it satisfies the following conditions:*

$$N \to \infty \text{ as } n \to \infty \qquad \textit{(diverging kernel)}\,, \tag{1.3}$$

$$2 \le d_1 \le d_2 \le \cdots \le d_n \le N^{\zeta_0} \text{ where } \zeta_0 = o(1) \qquad \textit{(sub-poly degrees)}\,, \tag{1.4}$$

$$\textstyle\sum_{j\ge3} j^3\nu_j \le a_0 M \text{ for an absolute constant } a_0 \ge 1 \qquad \textit{(3rd moment bound)}\,, \tag{1.5}$$

$$\nu_d \ge \alpha N \text{ for an absolute constant } \alpha > 0 \qquad \textit{(minimum kernel degree)}\,. \tag{1.6}$$

Observe that without condition (1.3), the graph $G_{\mathbf{d}}$ would be disconnected w.h.p. The upper bound in (1.4) is for convenience, and we can assume without loss of generality that

$$\zeta_0 \gg \frac{\ln \ln N}{\ln N}. \tag{1.7}$$

Condition (1.5) allows us to work directly with the configuration model of Bollobás [5]. It does, however, restrict our attention to cases where the average degree in the kernel (thus overall) is bounded, as Jensen's inequality implies that $\sum_{j\geq 3} j^3 \nu_j \geq N(2M/N)^3$ and so

$$\frac{2M}{N} \leq \left(\frac{a_0}{2\sqrt{2}}\right)^{1/2} \leq a_0. \tag{1.8}$$

Finally, the minimum kernel degree d (the focus of (1.6)) will be featured in the statement of our main theorem. We note that some of the assumptions above can be relaxed at the cost of some extra technicalities that would detract from the main new ideas of the paper.

The following two important classes of degree sequence are nice:

(i) Exponential tail: there exist real non-negative constants α, β with $\beta < 1$ and a positive integer $j_0 \geq 3$ such that $\nu_j/N \leq \alpha\beta^j$ for $j \geq j_0$.

(ii) Power law (moderate): there exist real positive constants c, γ with $\gamma \geq 3$ and a positive integer $j_0 \geq 3$ such that $\nu_j/N \leq cj^{-\gamma}$ for $j \geq j_0$, and the maximum degree is $N^{o(1)}$.

This of course includes degree sequences with bounded maximum degree Δ_0.

The main result of this paper is the following.

Theorem 1. *Let $\boldsymbol{d}$ be a nice degree sequence as per Definition 1.1. The following hold w.h.p.*

(a) If $\nu_2 = M^{o(1)}$ then

$$T_{\mathrm{COV}}(G_{\boldsymbol{d}}) \sim \frac{2(d-1)}{d(d-2)} M \ln M\,.$$

(b) If $\nu_2 = M^{\alpha}$ for some fixed $0 < \alpha < 1$ then

$$T_{\mathrm{COV}}(G_{\boldsymbol{d}}) \sim \max\left\{\frac{2(d-1)}{d(d-2)}\,, \phi_{\alpha,d}\right\} M \ln M\,,$$

where

$$\phi_{\alpha,d} = \min\left\{\tau : \min_{k=1,2,\ldots}\left\{(1-\alpha)k + \frac{\tau}{2}\left(\frac{1}{\lfloor (k+1)/2\rfloor + \frac{1}{d-2}} + \frac{1}{\lceil (k+1)/2\rceil + \frac{1}{d-2}}\right)\right\} \geq 1\right\}.$$

(c) If $\nu_2 = \Omega(M^{1-o(1)})$ then

$$T_{\mathrm{COV}}(G_{\boldsymbol{d}}) \sim \frac{m \ln^2 M}{-8\ln(1-\xi)}\,,$$

where $m = |E(G_{\boldsymbol{d}})| = \nu_2 + M$ and

$$\xi = M/m\,. \tag{1.9}$$

Note that as $\alpha \to 1$ we will have $\phi_{\alpha,d} \sim \frac{1}{8(1-\alpha)}$ and $-\ln(1-\xi) \sim (1-\alpha)\ln M$. So, as $\alpha \to 1$ we see that Cases (b) and (c) are consistent. Finally, observe that the condition in Case (c) can also be written as $-\ln(1-\xi) = o(\ln M)$.

Going back to the cover time of $\mathcal{C}_1^{(2)}$, the 2-core of $\mathcal{C}_1$, we see immediately that the estimate of [9] on its cover time (see (1.1)) readily follows from Case (c) of Theorem 1, whence

$$\nu_2 \sim c^2x^2e^{-cx}n/2 \quad \text{and} \quad M \sim cx^2(1-ce^{-cx})n/2.$$

Furthermore, Theorem 1 implies that the estimate for $T_{\mathrm{COV}}(\mathcal{C}_1^{(2)})$ in case $p = (1+\varepsilon)/n$ with $\varepsilon > 0$ fixed (see (1.2)) extends to the entire emerging supercritical regime. Indeed, by known characterizations of the 2-core (see, e.g., [12]) this case corresponds to $M \sim 2\varepsilon^3 n$ and $\nu_2 \sim 2\varepsilon^2 n$.

Corollary 2. *Let $p = (1+\varepsilon)/n$ where $\varepsilon = o(1)$ and $\varepsilon^3 n \to \infty$. Then w.h.p.,*

$$T_{\mathrm{COV}}(\mathcal{C}_1^{(2)}) \sim \frac{\varepsilon}{4} n \ln^2(\varepsilon^3 n).$$

We conclude with an open problem. While this work eliminated the restrictive assumption of minimum degree 3 for the degree sequence under consideration, vertices of degree 1 still pose a significant barrier in the analysis. It would be interesting to extend Theorem 1 to degree sequences that do include a linear number of such vertices, towards establishing the following conjecture for the cover time of the emerging giant component.

Conjecture. *Let $p = (1+\varepsilon)/n$ where $\varepsilon = o(1)$ and $\varepsilon^3 n \to \infty$. Then w.h.p.,*

$$T_{\mathrm{COV}}(\mathcal{C}_1) \sim n \ln^2(\varepsilon^3 n).$$

Outline of the paper

We begin with those arguments that are common to all parts of Theorem 1. Section 2.1 describes the configuration model of graphs with a fixed degree sequence that we will use throughout. Section 2.2 describes the distribution of the number of vertices $(\ell_e - 1)$ that are placed on each edge e of the kernel. Section 2.3 shows that most vertices have tree like neighbourhoods. Rapid mixing is an important property of our graphs and Section 2.4 gives an initial analysis of conductance.

Lemma 3.1 is our main tool in proving an upper bound on cover time. Let T be a "mixing time". Fix a vertex v and let π_v denote the steady state probability that a random walk on a graph G is at v. Let R_v be the expected number of returns to v of a random walk, started at v, within time T. Broadly speaking, Lemma 3.1 says that if we define the event

$$\mathcal{A}_t(v) = \{\text{vertex } v \text{ is not visited by the walk during the interval } [T,t]\} \tag{1.10}$$

then, if $T\pi_v = o(1)$ and another more technical condition holds, then to all intents and purposes,

$$\mathbb{P}(\mathcal{A}_t(v)) \approx e^{-t\pi_v/R_v}.$$

The above inequality has been used to prove an upper bound in [1, 7–11] and several other papers. In this paper we use it in inequality (4.4) below.

- **The case where ν_2 is not too large:** We begin the proof of Case (c) of Theorem 1 in Section 4.1, where we consider the case of ν_2 "close" to M; this will be Case (c1). In this range, ξ is not too small and Lemma 3.1 is sufficient to the task. We have $T = O(\ln^{O(1)} M/\xi^2)$ and $\pi_v = O(\ln M/(\xi M))$ and $T\pi_v = o(1)$. Section 4.1.1 proves this and verifies the more technical condition. So, Lemma 3.1 can be applied directly in this case. Given this, the main task that arises is in estimating the values, R_v. The number of returns to v is related in a strong way to the electrical resistance of its "local neighbourhood". This reduces to estimating the resistance $R(T)$ of a bounded depth binary tree T where the resistance of an edge is equal to a geometric random variable with success probability ξ. This is the content of Section 4.1.3. We only prove bounds on the probability that $R(T)$ is large.

- **The case where ν_2 is large:** Section 4.2 deals with the case where ν_2 is large with respect to M; we split this into Case (c2) where ν_2 is large but not "too large" and Case (c3) where ν_2 is very large. We will see that Case (c2) takes up most of our time and that Case (c3) can easily be reduced to the former case. We immediately run into a problem in using Lemma 3.1. As ν_2 grows, the mixing time of a walk grows like $(\nu_2/M)^2$ and the steady state values decrease like $1/(\nu_2 M)$. This means that for ν_2 large, $T\pi_v \gg 1$. This is where we need some new ideas. We choose some $\omega = N^{o(1)}$ and define $\ell^* = 1/\xi\omega$. A typical edge e of the kernel will give rise to a path P_e of length $\ell_e = \Theta(1/\xi)$. We divide P_e into $\Theta(\omega)$ sub-paths of length $\ell \in [\ell^*, 2\ell^*]$. (Because ℓ^* does not necessarily divide ℓ_e, the value of ℓ may vary from sub-path to sub-path). We then replace these sub-paths by edges of weight ℓ^*/ℓ to create an edge-weighted graph G_0. We consider a random walk $\mathcal{W}_0$ where at a vertex, we choose the next edge to cross with probability proportional to weight. We argue that the *edge* cover time of $\mathcal{W}_0$ is approximately $(\ell^*)^2$ times the cover time we are interested in.

  At first glance, this should eliminate the $T\pi_v \to \infty$ problem, as T should be $O(\ln^{O(1)} M/\omega^2)$ and so $\pi_v = O(\ln M/(\omega M)$. Unfortunately, this bound on T is false: the problem comes from edges of the kernel for which $\ell_e < \ell^*$. These edges give rise to single edges of weight ℓ^*/ℓ_e in G_0. In the worst-case we have $\ell_e = 1$ and we have an edge $f = (w_1, w_2)$ of weight ℓ^*. The walk $\mathcal{W}_0$ could spend a lot of time travelling back and forth from w_1 to w_2 and vice-versa. In any case, such an edge can reduce the conductance of the walk $\mathcal{W}_0$ to $O(1/(\ell^*)^2)$ undoing all of our work. Our solution to this is to modify the walk so that it "races along" edges of high weight. This will give us a walk that satisfies the conditions of the lemma. We then have to bound the time we ignored, to which end we apply a concentration inequality of Gillman [19].

  Section 4.2.1 deals with structural properties associated with this case. In particular showing that there are relatively few vertices of high weight. It also deals in some detail with properties that are needed for estimates of the conductance of our modified walk. Section 4.2.2 deals in detail as to how we make edges out of sub-paths. The goal from now on is to estimate $\mathbb{P}(\mathcal{A}_t(f))$ where f is some edge of G_0. We deal with each f separately in the sense that we create a graph G for each f. Splitting f by adding a vertex v_f to its middle. Then visiting v_f will be equivalent to crossing f. Section 4.3.4 uses Gillman's theorem to show that we have not ignored too many steps.

The remainder of the paper is organized as follows. Sections 4.5 and 4.6 deal with Cases (b) and (a) of Theorem 1. They are easier to prove than Case (c), being closer in spirit to earlier papers.

Section 5 deals with matching lower bounds on the cover time. Section 5.3 uses the Matthews bound, see for example [21]. Section 5.2 and Section 5.1 follow a pattern established in the earlier mentioned papers. We choose a time t that is a little bit less than our estimated cover time. We identify a set of vertices S that have not been visited up to time t. The size of S is large in expectation and Chebyshev inequality combined with Lemma 3.1 to show that $S \neq \emptyset$ w.h.p.

2 Structural properties

Recall that for a degree sequence $\mathbf{d} = (d_1 \leq \ldots \leq d_n)$ we let ν_j count the number of vertices of degree j. It will be useful to further define $V_j = \{i \in V : d_i = j\}$ (so that $\nu_j = |V_j|$) as well as

$$D_k = \sum_{j \geq 3} j^k \nu_j$$

(so that $N = D_0$ and $M = D_1/2$ are the number of vertices and edges in the kernel, respectively).

2.1 Configuration model

We make our calculations in the configuration model, see Bollobás [5]. Let $W = [2m]$ be our set of *configuration points* and let $W_i = [d_1 + \cdots + d_{i-1} + 1, d_1 + \cdots + d_i]$, $i \in [n]$, partition W. The function $\phi : W \to [n]$ is defined by $w \in W_{\phi(w)}$. Given a pairing F (i.e., a partition of W into m pairs) we obtain a (multi-)graph G_F with vertex set $[n]$ and an edge $(\phi(u), \phi(v))$ for each $\{u, v\} \in F$. Choosing a pairing F uniformly at random from among all possible pairings Ω_W of the points of W produces a random (multi-)graph G_F. Let

$$\mathcal{F}(2m) = \frac{(2m)!}{m!2^m}. \tag{2.1}$$

This is the number of pairings F of the points in W.

The *kernel* K_F is obtained from G_F by repeatedly replacing *induced* paths of length two by edges. The number of vertices in the kernel is N, the number of vertices of degree at least three and the number of edges in the kernel is $M \leq D_3/2 \leq a_0 N/2$ by (1.5).

Let

$$\sigma = \frac{1}{2m} \sum_{j=1}^{n} d_j(d_j - 1) \leq \frac{2\nu_2 + D_2}{2\nu_2 + 2M} = O(1)$$

by Assumption (c).

Assuming that $d_n = o(m^{1/3})$ (as it will be for nice sequences), the probability that G_F is simple (no loops or multiple edges) is given by

$$P_S = \mathbb{P}(G_F \text{ is simple}) \sim e^{-\sigma/2 - \sigma^2/4} = \Omega(1). \tag{2.2}$$

See e.g. [24]. Furthermore each simple graph $G \in \mathcal{G}_{\mathbf{d}}$ is equiprobable. We can therefore use G_F as a replacement model for $G_{\mathbf{d}}$ in the sense that any event that occurs w.h.p. in G_F will occur w.h.p. in $G_{\mathbf{d}}$.

We argue next that:

Lemma 2.1. *The distribution of K_F is that of a configuration model where W is replaced by $\widehat{W} = W_{\nu_2+1} \cup W_{\nu_2+2} \cup \cdots \cup W_n$.*

Proof. Indeed, we can define a map $\psi : \Omega_W \to \Omega_{\widehat{W}}$ such that for all $F_1, F_2 \in \Omega_{\widehat{W}}$ we have $|\psi^{-1}(F_1)| = |\psi^{-1}(F_2)|$. Each induced path P of G_F comes from a set of pairs $e_i = \{x_i, y_i\}$, $i = 1, 2, \ldots, r$ where (i) $\phi(x_1), \phi(y_r) \notin V_2$ (= the set of vertices of degree two) and (ii) $\phi(z) \in V_2$ for $z \in \{x_2, \ldots, x_r, y_1, \ldots, y_{r-1}\}$. Replacing e_i, $i = 1, 2, \ldots, r$ by $\{x_1, y_r\}$ defines $\psi(F) \in \Omega_{\widehat{W}}$. The number of $F \in \Omega_W$ that map onto a fixed $F' \in \Omega_{\widehat{W}}$ depends only on ν_2, m and N. This implies the lemma. □

2.2 Distribution of vertices of degree two

We can therefore obtain $F \in \Omega_W$ by first randomly choosing $F' \in \Omega_{\widehat{W}}$ and then replacing each edge e of $G_{F'}$ by a path P_e. The next thing to tackle is the distribution of the lengths of these paths. Let ℓ_e be the length of the path P_e. Suppose now that the edges of F' are $e_1, e_2, \ldots, e_M$ and write ℓ_j for ℓ_{e_j}.

Lemma 2.2. *The vector $(\ell_1, \ell_2, \ldots, \ell_M)$ is chosen uniformly from*

$$\{\ell_i \geq 1, i = 1, 2, \ldots, M \text{ and } \ell_1 + \ell_2 + \cdots + \ell_M = \nu_2 + M\}.$$

Proof. Each such vector arises in $\nu_2!$ ways. Indeed, we order V_2 and then assign the associated vertices in order, $\ell_1 - 1$ to e_1 to create P_{e_1}, $\ell_2 - 1$ to e_2 to create P_{e_2} and so on. □

Some calculations can be made simpler if we observe the alternative description of the distribution of $(\ell_1, \ell_2, \ldots, \ell_M)$.

Lemma 2.3. *Let Z be a geometric random variable with success probability ξ. (ξ can be any value between 0 and 1) here). Then $(\ell_1, \ell_2, \ldots, \ell_M)$ is distributed as $Z_1, Z_2, \ldots, Z_M$ subject to $Z_1 + Z_2 + \cdots + Z_M = \nu_2 + M$, where $Z_1, Z_2, \ldots, Z_M$ are independent copies of Z.*

Proof.

$$\begin{aligned}
&\mathbb{P}((Z_1, Z_2, \ldots, Z_M) = (x_1, x_2, \ldots, x_M) \mid Z_1 + Z_2 + \cdots + Z_M = \nu_2 + M) \\
&= \frac{\prod_{i=1}^M (1-\xi)^{x_i - 1}\xi}{\sum_{y_1+y_2+\cdots+y_M = \nu_2+M} \prod_{i=1}^M (1-\xi)^{y_i-1}\xi} \\
&= \frac{(1-\xi)^{\nu_2}\xi^M}{\binom{M+\nu_2-1}{M-1}(1-\xi)^{\nu_2}\xi^M} \\
&= \frac{1}{\binom{M+\nu_2-1}{M-1}}.
\end{aligned}$$

□

The best choice for ξ will be that for which $\mathbb{E}(Z_1 + Z_2 + \cdots + Z_M) = \nu_2 + M$, i.e. $M\xi^{-1} = \nu_2 + M$. We therefore take ξ as in (1.9).

Pursuing this line, let $\widehat{\mathbb{P}}$ refer to probabilities of events involving $Z_1, Z_2, \ldots, Z_M$ *without* the conditioning $Z_1 + Z_2 + \cdots + Z_M = \nu_2 + M$. (Although $\mathbb{P}$ and $\widehat{\mathbb{P}}$ refer to the same probability space, this will have some notational conveneience later).

Lemma 2.4. *Let* $\xi = \frac{M}{M+\nu_2}$ *and* $M, \nu_2 \to \infty$.

(a) Let $\zeta = z_1 + z_2 + \cdots + z_k$ *and* $k = o(M)$ *where* $k\zeta = o(M + \nu_2)$,

$$\mathbb{P}(Z_1 = z_1, Z_2 = z_2, \cdots, Z_k = z_k \mid Z_1 + Z_2 + \cdots + Z_M = \nu_2 + M) \leq \\ \widehat{\mathbb{P}}(Z_1 = z_1, Z_2 = z_2, \cdots, Z_k = z_k)(1 + \varepsilon) = \xi^k (1 - \xi)^{\zeta - k}(1 + \varepsilon),$$

where

$$\varepsilon = \frac{3k\zeta}{\nu_2 + M}. \tag{2.3}$$

(b) If $k \in \{1, 2\}$ *and* $\zeta = z_1 + \cdots + z_k = o(\nu_2)$ *then*

$$\mathbb{P}(Z_i = z_i,\, i = 1, \ldots, k \mid Z_1 + Z_2 + \cdots + Z_M = \nu_2 + M) = \xi^k (1 - \xi)^{\zeta - k}(1 + \eta)$$

where

$$1 + \eta = \left(1 + O\left(\frac{\zeta^2 M}{\nu_2(\nu_2 + M)}\right) + O\left(\frac{\zeta}{\nu_2 + M}\right)\right).$$

(c) Let $\ell_{\max} = \frac{4(M+\nu_2)\ln M}{M} = 4\xi^{-1}\ln M$. *Then*

$$\mathbb{P}(\exists e:\ \ell_e \geq \ell_{\max}) = o(1).$$

(d) Let $\ell_{\min} = \left\lceil \frac{M+\nu_2}{M^2 \ln M} \right\rceil = \left\lceil \frac{1}{\xi M \ln M} \right\rceil$ *and suppose that* $\nu_2 / M \ln M \to \infty$ *then*

$$\mathbb{P}(\exists e:\ \ell_e < \ell_{\min}) = o(1).$$

Proof. (a) Observe that

$$\begin{aligned}
&\mathbb{P}(Z_1 = z_1, Z_2 = z_2, \cdots, Z_k = z_k \mid Z_1 + Z_{k+2} + \cdots + Z_M = \nu_2 + M) \\
&= \frac{\mathbb{P}((Z_1 = z_1, Z_2 = z_2, \cdots, Z_k = z_k) \wedge (Z_{k+1} + Z_2 + \cdots + Z_M = \nu_2 + M - \zeta)}{\mathbb{P}(Z_1 + Z_{k+2} + \cdots + Z_M = \nu_2 + M)} \\
&= \frac{\mathbb{P}(Z_1 = z_1, Z_2 = z_2, \cdots, Z_k = z_k)\mathbb{P}(Z_{k+1} + Z_{k+2} + \cdots + Z_M = \nu_2 + M - \zeta)}{\mathbb{P}(Z_1 + Z_2 + \cdots + Z_M = \nu_2 + M)} \quad = \frac{\binom{\nu_2 + M - \zeta - 1}{M - k - 1}}{\binom{\nu_2 + M - 1}{M - 1}},
\end{aligned} \tag{2.4}$$

which, since $\zeta \geq k$, equals

$$\prod_{i=1}^{k} \frac{M - i}{\nu_2 + M - i} \times \prod_{i=1}^{\zeta - k} \frac{\nu_2 - i + 1}{\nu_2 + M - k - i} \tag{2.5}$$

$$\leq \xi^k \prod_{i=1}^{\zeta - k} \frac{\nu_2 - i + 1}{\nu_2 + M - k - i} = \xi^k (1 - \xi)^{\zeta - k} \prod_{i=1}^{\zeta - k} \left(1 + \frac{(k+1)\nu_2 - (i-1)M}{(\nu_2 + M - k - i)\nu_2}\right)$$

$$\leq \xi^k (1 - \xi)^{\zeta - k} \left(1 + \frac{(1 + o(1))(k+1)}{\nu_2 + M}\right)^{\zeta - k} \tag{2.6}$$

$$\leq \xi^k (1 - \xi)^{\zeta - k}(1 + \varepsilon).$$

(b) Going back to (2.5) with $k=2$ we use

$$\prod_{i=1}^{k} \frac{M-i}{\nu_2+M-i} = \xi^k \left(1+O\left(\frac{1}{\nu_2+M}\right)\right)$$

and

$$\begin{aligned}
&\prod_{i=1}^{\zeta-k} \frac{\nu_2-i+1}{\nu_2+M-k-i} \\
&= \frac{\nu_2(\nu_2-1)\cdots(\nu_2-k)}{(\nu_2+M-\zeta+k)\cdots(\nu_2+M-\zeta+1)(\nu_2+M-\zeta)} \times \prod_{j=k+1}^{\zeta-k-1} \frac{\nu_2-j}{\nu_2+M-j} \\
&= \left(1+O\left(\frac{\zeta}{\nu_2+M}\right)\right) \times (1-\xi)^{\zeta-k} \times \prod_{j=k+1}^{\zeta-k-1} \left(1-\frac{jM}{\nu_2(\nu_2+M)} + O\left(\frac{j^2M}{\nu_2(\nu_2+M)^2}\right)\right) \\
&= (1-\xi)^{\zeta-k} \times \left(1+O\left(\frac{\zeta^2 M}{\nu_2(\nu_2+M)}\right) + O\left(\frac{\zeta}{\nu_2+M}\right)\right).
\end{aligned}$$

(c) It follows from (2.4) with $k=1$ that

$$\begin{aligned}
\mathbb{P}(\exists e:\ \ell_e \geq \ell_{\max}) &\leq M \sum_{\zeta=\ell_{\max}}^{\nu_2} \frac{\binom{M+\nu_2-\zeta-1}{M-2}}{\binom{M+\nu_2-1}{M-1}} \\
&\leq \frac{2M^2}{\nu_2} \sum_{\zeta=\ell_{\max}}^{\nu_2} \left(1-\frac{\zeta}{M+\nu_2-1}\right)^{M-2} \\
&\leq \frac{2M^2}{\nu_2} \sum_{\zeta=\ell_{\max}}^{\nu_2} \exp\left\{-\frac{(M-2)\zeta}{M+\nu_2-1}\right\} \\
&\leq \frac{2M^2}{\nu_2} \cdot \exp\left\{-\frac{(M-2)\ell_{\max}}{M+\nu_2-1}\right\} \frac{1}{1-e^{-(M-2)/(M+\nu_2-1)}} \qquad (2.7) \\
&\leq \frac{2M^2}{\nu_2} \cdot \frac{2}{M^4} \cdot \frac{2(M+\nu_2)}{M} \\
&= o(1).
\end{aligned}$$

(d) It follows from (a) with $k=1$ and $\zeta < \ell_{\min}$ that

$$\mathbb{P}(\exists e:\ \ell_e < \ell_{\min}) \leq 2M\ell_{\min}\xi = o(1). \qquad \square$$

2.3 Tree like vertices

Let a vertex x of K_F be *locally tree like* if its K_F-neighborhood up to depth

$$L_0 = \delta_0 \ln N \tag{2.8}$$

contains no cycles.

Here

$$\delta_0 \gg \zeta_0 \gg \frac{\ln\ln N}{\ln N} \tag{2.9}$$

where ζ_0 is as in (1.4).

A vertex of G_F is locally tree like if it lies on a path P_e where $e = (v, w)$ and v, w are both locally tree like. An edge of G_F is locally tree like if both of its endpoints are locally tree like.

Lemma 2.5. *With L_0 as defined in (2.8) we have that for the graph K_F:*

(a) *W.h.p. there are at most $N^{10\delta_0 \ln a_0}$ non locally tree like vertices, where a_0 is as in (1.5).*

(b) *W.h.p. there is at most one cycle contained in the $(2L_0)$-neighborhood of any vertex.*

Proof. (a) The expected number of vertices that are within distance $2L_0$ of a cycle of length at most $2L_0$ in the graph K_F can be bounded from above by

$$\sum_{l=0}^{2L_0}\sum_{k=3}^{2L_0}\sum_{\substack{v_1,\ldots,v_k\\ w_1,\ldots,w_l}} \mathrm{d}(v_1)\prod_{i=1}^{k}\frac{\mathrm{d}(v_i)^2}{M}\prod_{j=1}^{l}\frac{\mathrm{d}(w_j)^2}{M} \leq \sum_{l=0}^{2L_0}\sum_{k=3}^{2L_0}\frac{D_3}{M}\left(\frac{D_2}{M}\right)^{k+l-1}$$
$$\leq \sum_{l=0}^{2L_0}\sum_{k=3}^{2L_0} a_0^{k+l} \leq N^{5\delta_0 \ln a_0}. \quad (2.10)$$

where

$\mathrm{d}(v)$ denotes the degree of vertex $v \in V$ in the graph G_F.

Markov's inequality implies that there are fewer than $N^{10\delta_0 \ln a_0}$ such vertices w.h.p.

Explanation of (2.10): We choose $v_1, v_2, \ldots, v_k$ as the vertices of the cycle and $w_1, w_2, \ldots, w_l$ as the vertices of a path joining the cycle at v_1. The probability that the implied edges exist in K_F can be bounded by

$$\frac{\mathrm{d}(v_1)\mathrm{d}(v_2)}{2M-1}\cdot\frac{(\mathrm{d}(v_2)-1)\mathrm{d}(v_3)}{2M-3}\cdots\frac{(\mathrm{d}(v_k)-1)(\mathrm{d}(v_1)-1)}{2M-2k+1}\cdot$$
$$\frac{(\mathrm{d}(v_1)-2)\mathrm{d}(w_1)}{2M-2k-1}\cdot\frac{(\mathrm{d}(w_1)-1)\mathrm{d}(w_2)}{2M-2k-3}\cdots\frac{(\mathrm{d}(w_{k-1})-1)\mathrm{d}(w_k)}{2M-2l-2k+1}$$

(b) If the condition in (b) fails then there exist two small cycles that are close together. More precisely, there exists a path $P = (v_1, v_2, \ldots, v_k)$ where $k \leq 5L_0$ plus two additional edges (v_1, v_i) and (v_k, v_j) where $1 < i, j < k$. The probability that such a path exists can be bounded by

$$\sum_{k=4}^{5L_0}\sum_{1<i,j<k}\sum_{v_1,\ldots,v_k}\frac{\mathrm{d}(v_1)\mathrm{d}(v_i)}{M}\cdot\frac{\mathrm{d}(v_k)\mathrm{d}(v_j)}{M}\cdot\prod_{l=1}^{k}\frac{\mathrm{d}(v_l)^2}{M} \leq \sum_{k=4}^{5L_0}\frac{k^2 D_3^2 D_2^{k-1}}{M^{k+2}}$$
$$= O(N^{o(1)-1}) = o(1). \quad (2.11)$$

Part (b) follows. □

2.4 Conductance

Given a connected graph $G = (V, E)$ let $\pi(v) = \frac{\mathrm{d}(v)}{2|E|}$ denote the steady state probability of being at v. The *conductance* $\Phi(G)$ of a random walk $\mathcal{W}_u$ on G is defined by

$$\Phi(G) = \min_{S:\pi(S)\leq 1/2}\Phi(S) \text{ where } \Phi(S) = \frac{|\partial S|}{\mathrm{d}(S)} \quad (2.12)$$

and where $\mathrm{d}(S) = \sum_{v \in S} \mathrm{d}(v)$ and $\pi(S) = \sum_{v \in S} \pi(v)$ and ∂S denotes the set of edges with one endpoint in S and the other not in S. (We consider the conductance of random walks on edge-weighted graphs in Section 4.2.2).

The following lemma follows directly from Lemma 10 of [1].

Lemma 2.6. *Let $\boldsymbol{d}$ be a nice degree sequence. Let F be chosen uniformly as in Section 2.1. Let K_F be the kernel of the associated configuration multi-graph. Then with probability $1 - o(n^{-1/9})$,*

$$\Phi(K_F) \geq \frac{1}{100}.$$

Note that $\Phi(K_F) \geq 0.01$ implies that K_F and hence G_F is connected. Using (2.2) we see that the probability that $G_{\mathbf{d}}$ is not connected is $o(n^{-1/9}) = o(1)$.

We will now estimate the conductance of G_F using Lemmas 2.4 (Part (c)) and 2.6.

Lemma 2.7. *Let $\boldsymbol{d}$ be a nice degree sequence. Let F be chosen uniformly as in Section 2.1. Let G_F be the associated configuration multi-graph. Then with probability $1 - o(n^{-1/9})$,*

$$\Phi(G_F) = \Omega\left(\frac{\xi}{\ln M}\right).$$

Proof. Consider a set $S \subseteq [n]$ that induces a connected subgraph of G_F. We can restrict our attention to such sets. Suppose S only contains part of some path P_e. To be specific, suppose $P_e = (v, u_1, \ldots, u_k, w)$ where v, w are of degree three or more and $u_1, u_2, \ldots, u_k$ are of degree two. $k = 1$ is allowed here. Assume that $v \in S$. Then we wish to eliminate the case where $u_1, u_2, \ldots, u_l \in S$ and $u_{l+1} \notin S$ where $l < k$. If we add an edge of P_e that is not contained in S to create S' then $\mathrm{d}(S') > \mathrm{d}(S)$ and $|\partial S'| \leq |\partial S|$. Let S *conform* with the kernel if for all $e \in K_F$ we have either (i) S contains all internal vertices of P_e or (ii) S contains no internal vertices of P_e. Then w.h.p.

$$\Phi(G_F) \geq \min\left\{ \min_{\substack{\pi(S) \leq 1/2 \\ S \text{ conforms with } K_F}} \frac{|\partial S|}{\mathrm{d}(S)}, \min_{\substack{1/2 - \ell_{\max}/m \leq \pi(S) \leq 1/2 \\ S \text{ conforms with } K_F}} \frac{|\partial S|}{\mathrm{d}(S) + 2\ell_{\max}} \right\}. \tag{2.13}$$

The lemma now follows from $\ell_{\max} = o(m)$ and $\mathrm{d}(S) \leq \ell_{\max} \mathrm{d}(S \cap V(K_F))$. $\square$

We note a result from Jerrum and Sinclair [20], that

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2}(1 - \Phi^2/2)^t. \tag{2.14}$$

There is a technical point here. The result (2.14) assumes that the walk is lazy. A lazy walk moves to a neighbour with probability 1/2 at any step. This assumption halves the conductance. Asymptotically, the cover time is also doubled. Otherwise, the lazy assumption has a negligible effect on the analysis, see Remark 3.2. We will ignore this assumption for the rest of the paper; and continue as though there are no lazy steps.

3 Estimating first visit probabilities

In this section G denotes a fixed connected graph with ν vertices and μ edges. A random walk $\mathcal{W}_u$ is started from a vertex u. Let $\mathcal{W}_u(t)$ be the vertex reached at step t, let P be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbb{P}(\mathcal{W}_u(t) = v)$. We assume that the random walk $\mathcal{W}_u$ on G is ergodic with stationary distribution π, where $\pi_v = \mathrm{d}(v)/(2\mu)$, and $\mathrm{d}(v)$ is the degree of vertex v.

Let

$$d(t) = \max_{u,x \in V} |P_u^{(t)}(x) - \pi_x|, \tag{3.1}$$

and let T_{MIX} be a positive integer such that for $t \geq T_{\text{MIX}}$

$$\max_{u,x \in V} |P_u^{(t)}(x) - \pi_x| \leq \nu^{-10}. \tag{3.2}$$

Consider the walk $\mathcal{W}_v$, starting at vertex v. Let $r_t = r_t(v) = \mathbb{P}(\mathcal{W}_v(t) = v)$ be the probability that this walk returns to v at step $t = 0, 1, \ldots$. Let

$$R_{T_{\text{MIX}}}(z) = \sum_{j=0}^{T_{\text{MIX}}-1} r_j z^j \tag{3.3}$$

and let

$$R_v = R_{T_{\text{MIX}}}(1).$$

A proof of the following lemma can be found in [9].

Lemma 3.1. *Let $G = (V, E)$ and let $u, v \in V$ be fixed and let $T = T_{\text{MIX}}(G)$. Suppose that*

$$T\pi_v = o(1)\,, \tag{3.4}$$

$$\min_{|z|=1+\lambda} |R_{T_{\text{MIX}}}(z)| \geq \theta \qquad \textit{for some constant } \theta > 0. \tag{3.5}$$

Then there exists a constant K and values $\psi_1, \psi_2 = O(T\pi_v)$ such that if

$$\lambda = \frac{1}{KT_{\text{MIX}}}. \tag{3.6}$$

and

$$p_v = \frac{\pi_v}{R_v(1+\psi_1)}\,. \tag{3.7}$$

then for all $t \geq T$,

$$\mathbb{P}_u(\mathcal{A}_t(v)) = \frac{1+\psi_2}{(1+p_v)^t} + O(T\pi_v e^{-\lambda t/2})\,. \tag{3.8}$$

where $\mathcal{A}_t(v)$ is defined in (1.10).

Remark 3.2. *One effect of making the walk lazy is to (asymptotically) double R_v. Later in the analysis, this would double our upper bound on the cover time, as it should. Thus it is legitimate to ignore this technicality required for (2.14).*

Using Lemma 2.7 and (2.14) we see that we can take

$$T_{\text{MIX}}(G_F) = \frac{\ln^4 M}{\xi^2}. \tag{3.9}$$

This is a little larger than one might expect at this stage. We will explain why later.

Lemma 3.1 is our main tool for proving upper bounds on the cover time.

4 Upper bounds

To begin our analysis we let $G = (V, E)$ be a graph with $\nu = |V|$ and $|E| = O(\nu)$. Assume that $T_{\text{MIX}} = T_{\text{MIX}}(G) \leq \nu$. Let

$$\tau_u(G, \tau) = \min \{t \geq \tau : \mathcal{W}_u \text{ visits every vertex of } G \text{ at least once in the interval } [\tau, t]\}.$$

Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ during steps $[T_{\text{MIX}}, t]$. The following holds:

$$\begin{aligned}
&T_{\text{COV}}(G, u) \leq \mathbb{E}_u(\tau_c(G, T_{\text{MIX}})) \\
&\leq T_{\text{MIX}} + \sum_{t \geq T_{\text{MIX}}} \mathbb{P}_u(\tau_c(G, T_{\text{MIX}}) \geq t), \\
&= T_{\text{MIX}} + \sum_{t \geq T_{\text{MIX}}} \sum_{w \in V} \mathbb{P}_w(\tau_u(G, 0) \geq t - T_{\text{MIX}}) \mathbb{P}_u(\mathcal{W}_u(T_{\text{MIX}}) = w) \\
&\leq T_{\text{MIX}} + \sum_{t \geq T_{\text{MIX}}} \sum_{w \in V} \pi_w \mathbb{P}_w(\tau_u(G, 0) \geq t - T_{\text{MIX}}) + E_1 \\
&\leq 2T_{\text{MIX}} + \sum_{t \geq 2T_{\text{MIX}}} \sum_{w \in V} \pi_w \mathbb{P}_w(\tau_u(G, T_{\text{MIX}}) \geq t - T_{\text{MIX}}) + E_1 \\
&= 2T_{\text{MIX}} + \sum_{t \geq T_{\text{MIX}}} \sum_{w \in V} \pi_w \mathbb{P}_w(\tau_u(G, T_{\text{MIX}}) \geq t) + E_1
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
E_1 = \nu^{-10} \sum_{t \geq T_{\text{MIX}}} \sum_{w \in V} \mathbb{P}_w(\tau_u(G, 0) \geq t - T_{\text{MIX}}) \leq \nu^{-3} + \sum_{t \geq \nu^6} \sum_{w \in V} \mathbb{P}_w(\tau_u(G, 0) \geq \nu^4) \leq \\
\nu^{-3} + \sum_{t \geq \nu^6} \sum_{w \in V} \left(1 - (\pi_w - \nu^{-10})\right)^{t/T_{\text{MIX}}} \leq \nu^{-3} + \sum_{t \geq \nu^6} \sum_{w \in V} e^{-\Omega(t/\nu^5 \log^2 \nu)} = o(1).
\end{aligned} \tag{4.2}$$

Here we use $O(\nu^4 \log \nu)$ as a crude upper bound on the mixing time T_{MIX}. It is obtained from the fact that the conductance of the walk is at least $4/\nu^2$ and $\pi_w = \Omega(1/\nu)$ by assumption.

Now

$$\mathbb{P}_v(\tau_c(G, T_{\text{MIX}}) > t) = \mathbb{P}_v(U_t > 0) \leq \min\{1, \mathbb{E}_v(U_t)\}. \tag{4.3}$$

It follows from (4.1),(4.2),(4.3) that for all $t \gg T_{\text{MIX}}$

$$T_{\text{COV}}(G, u) \leq t + o(t) + \sum_{s \geq t} \sum_{w} \pi_w \mathbb{E}_w(U_s) = t + o(t) + \sum_{w \in V} \pi_w \sum_{v \in V} \sum_{s \geq t} \mathbb{P}_w(\mathcal{A}_s(v)). \tag{4.4}$$

We will choose a value t and then use Lemma 3.1 to estimate $\mathbb{P}_w(\mathcal{A}_s(v))$ and show that the double sum is $o(t)$. It then follows that $T_{\text{COV}}(G,u) \leq t + o(t)$.

The final expression in (4.4) leads us to define the random variable

$$\Psi(S,t) = \sum_{v\in V, w\in S} \sum_{s\geq t} \pi_v \mathbb{P}_v(\mathcal{A}_s(w))$$

for any $S \subseteq V$, $t \geq 0$. (Here Ψ is a random variable on the space of graphs G).

We can use (4.4) if we have a good estimate for $\mathbb{P}_v(\mathcal{A}_s(w))$. For this we will use Lemma 3.1. Let

$$\delta_1 = \delta_0/100 \tag{4.5}$$

4.1 Case (c1): $M^{1-o(1)} \leq \nu_2 \leq M^{1+\delta_1}$

We first check that Lemma 3.1 is applicable.

4.1.1 Conditions of Lemma 3.1 for G

Checking (3.4) for G_F:
By assumption, the maximum degree in G_F is at most $N^{o(1)}$. So for $v \in [n]$ we have from (3.9),

$$T_{\text{MIX}}\pi_v \leq_b \frac{(M+\nu_2)^2 \ln^4 M}{M^2} \cdot \frac{N^{o(1)}}{M+\nu_2} = o(1)$$

where we use $A \leq_b B$ to denote $A = O(B)$. So, (3.4) holds.

Checking (3.5) for G_F:
Suppose that v is one of the vertices that are placed on an edge $f = (w_1, w_2)$ of K_F. We will say that f *contains* v. We allow $v = w_1$ here and then for convenience we say that v is contained in one of the edges incident with v of K_F. We remind the reader that w.h.p. all K_F-neighborhoods up to depth $2L_0$ contain at most one cycle, see Lemma 2.5(b). Let X_f be the set of kernel vertices that are within kernel distance L_0 of f in K_F. Let Λ_f be the sub-graph of G obtained as follows: Let H_f be the subgraph of the kernel induced by X_f. Thus f is an edge of H_f. To create Λ_f add the vertices of degree two to the edges of H_f as in the construction of G_F. The vertices of X_f that are at kernel distance L_0 from f in K_F are said to be at the frontier of Λ_f. Denote these vertices by Φ_f.

In this paper we consider walks on several distinct graphs. We have for example, $\mathcal{W}_v$, the random walk on G_F, starting at v. We will now write this as $\mathcal{W}_v^{G_F}$. The idea of this notation is to identify explicitly the graph on which the walk is defined.

Let us make Φ_f into absorbing states for a walk $\mathcal{W}_v^{\Lambda_f}$ in Λ_f, starting at v. Let $\beta(z) = \sum_{t=1}^{T_{\text{MIX}}} \beta_t z^t$ where β_t is the probability of a *first* return to v at time $t \leq T_{\text{MIX}} = T_{\text{MIX}}(G_F)$ before reaching Φ_f. Let $\alpha(z) = 1/(1-\beta(z))$, and write $\alpha(z) = \sum_{t=0}^{\infty} \alpha_t z^t$, so that α_t is the probability that the walk $\mathcal{W}_v^{\Lambda_f}$ is at v at time t. We will prove below that the radius of convergence of $\alpha(z)$ is at least $1+\lambda$, where λ is as in (3.6).

We can write

$$\begin{aligned} R_{T_{\text{MIX}}}(z) &= \alpha(z) + Q(z) && (4.6)\\ &= \frac{1}{1-\beta(z)} + Q(z), && (4.7)\end{aligned}$$

where $Q(z) = Q_1(z) + Q_2(z)$, and

$$\begin{aligned} Q_1(z) &= \sum_{t=1}^{T_{\text{MIX}}} (r_t - \alpha_t) z^t \\ Q_2(z) &= -\sum_{t=T_{\text{MIX}}+1}^{\infty} \alpha_t z^t. \end{aligned}$$

We claim that the expression (4.7) is well defined for $|z| \leq 1+\lambda$. We will show below that

$$|Q_2(z)| = o(1) \tag{4.8}$$

for $|z| \leq 1+2\lambda$ and thus the radius of convergence of $Q_2(z)$ (and hence $\alpha(z)$) is greater than $1+\lambda$. This will imply that $|\beta(z)| < 1$ for $|z| \leq 1+\lambda$. For suppose there exists z_0 such that $|\beta(z_0)| \geq 1$. Then $\beta(|z_0|) \geq |\beta(z_0)| \geq 1$ and we can assume (by scaling) that $\beta(|z_0|) = 1$. We have $\beta(0) = 0 < 1$ and so we can assume that $\beta(|z|) < 1$ for $0 \leq |z| < |z_0|$. But as ρ approaches 1 from below, (4.6) is valid for $z = \rho|z_0|$ and then $|R_{T_{\text{MIX}}}(\rho|z_0|)| \to \infty$, contradiction.

Recall that $\lambda = 1/KT_{\text{MIX}}$. Clearly $\beta(1) \leq 1$ (from its definition) and so for $|z| \leq 1+\lambda$

$$\beta(|z|) \leq \beta(1+\lambda) \leq \beta(1)(1+\lambda)^{T_{\text{MIX}}} \leq e^{1/K}.$$

Using $|1/(1-\beta(z))| \geq 1/(1+\beta(|z|))$ we obtain

$$|R_{T_{\text{MIX}}}(z)| \geq \frac{1}{1+\beta(|z|)} - |Q(z)| \geq \frac{1}{1+e^{1/K}} - |Q(z)|. \tag{4.9}$$

We now prove that $|Q(z)| = o(1)$ for $|z| \leq 1+\lambda$ and we will have verified both conditions of Lemma 3.1.

Turning our attention first to $Q_1(z)$, we note that $r_t - \alpha_t$ is at most the probability of a return to v within time T_{MIX}, after a visit to Φ_f for the walk $\mathcal{W}_v^{G_F}$.

Lemma 4.1. *Fix $w \in \Phi_f$. Then*

$$\mathbb{P}(\mathcal{W}_w^{G_F} \textit{ visits } f \textit{ within time } T_{\text{MIX}}) = O(N^{-\delta_0/5}).$$

Proof. Now consider the walk $\mathcal{W}_w$. We will find an upper bound for the probability that it reaches w_1 or w_2, the endpoints of the K_f edge that v was added to. We consider a simple random walk $\mathcal{X}$ on H that starts at w and is reflected when it reaches Φ_f. We show that

$$\mathbb{P}(\mathcal{X} \text{ reaches } w_1 \text{ within time } T_{\text{MIX}}) \leq N^{-\delta_0/6}. \tag{4.10}$$

Let P be one of the at most two paths P, P' from w to w_1 in K_F. $P = P'$ whenever w_1 is locally tree like. Now to get to w_1 the walk $\mathcal{X}$ will have to traverse the complete length of one of two paths, P say. We can ignore the times taken up in excursions outside P. So, we will think of $\mathcal{X}$ as a walk along a path in which there are L_0 points at which the probability of moving away from w_1 is (at least) $2/3$ as opposed to $1/2$. (There could be a couple of places γ_1, γ_2 where P meets P' and then we will have the particle moving further or closer to w_1 with different probabilities). We can also assume that $\ell_e = 1$ for all $e \in P$. This follows from an application of Rayleigh's principle (see, e.g., [14]). We are reducing the resistance of P by increasing the conductance of individual edges. This will increase the (escape) probability of the walk reaching w_1 before returning to w. (Alternatively we can couple the original walk with a walk where we have contracted some edges).

So we next consider a biassed random walk $\mathcal{Y}$ on $[0, L_0]$ where $\mathcal{Y}$ starts at 0 and moves right with probability $1/3$. It follows from Feller [18, p314] that

$$\mathbb{P}(\mathcal{Y} \text{ reaches } L_0 \text{ before returning to } 0) \leq \frac{1}{2^{L_0-2}-1} \leq N^{-\delta_0/2}. \tag{4.11}$$

(We write $L_0 - 2$ instead of L_0 to account for the two possible places γ_1, γ_2, where we can just insist on a move towards w_1).

Let $N_0 = N^{\delta_0/4}$. If we restart $\mathcal{X}$ from w then the probability that we reach w_1 after N_0 restarts is at most $N_0 N^{-\delta_0/2} = N^{-\delta_0/4}$. We observe that $T_{\text{MIX}} = O(N^{2\delta_1} \ln^4 N) \leq N^{\delta_0/40}$, see (2.9), (3.9) and (4.5). To summarise,

$$\mathbb{P}(\mathcal{W}_w \text{ reaches } w_1 \text{ within time } T_{\text{MIX}}) \leq T_{\text{MIX}} N^{-\delta_0/4} \leq N^{-\delta_0/5}. \tag{4.12}$$

By doubling the above estimate in (4.12) to handle w_2, we obtain the lemma. □

Thus,

$$|Q_1(z)| \leq (1+\lambda)^{T_{\text{MIX}}} Q_1(1) \leq 2(1+\lambda)^{T_{\text{MIX}}} N^{-\delta_0/5} T_{\text{MIX}} = o(1). \tag{4.13}$$

We next turn our attention to $Q_2(z)$. Let σ_t be the probability that the walk on Λ_f has not been absorbed by step t. Then $\sigma_t \geq \alpha_t$, and so

$$|Q_2(z)| \leq \sum_{t=T_{\text{MIX}}+1}^{\infty} \sigma_t |z|^t,$$

For each $w \in \Phi_f$ there are one or two paths from v to w. We first consider the number of edges in such a path. It follows from Part (c) of Lemma 2.4 that we can assume that the number of edges in such a path is $L \leq L_0 \ell_{\max}$.

Assume first that v is locally tree like. The distance from v of our walk on Λ_f dominates the distance from the origin of a simple random walk on $\{0, \pm 1, \pm 2, \ldots, \}$ starting at 0. We estimate an upper bound for σ_t as follows: Consider a simple random walk $X_0^{(b)}, X_1^{(b)}, \ldots$ starting at $|b| < L$ on the finite line $(-L, -L+1, ..., 0, 1, ..., L)$, with absorbing states $-L, L$.

$X_m^{(0)}$ is the sum of m independent ± 1 random variables. So the Central Limit Theorem implies that there exists a constant $c > 0$ such that

$$\mathbb{P}(X_{cL^2}^{(0)} \geq L \text{ or } X_{cL^2}^{(0)} \leq -L) \geq 1 - e^{-1/2}.$$

Consequently, for any b with $|b| < L$,

$$\mathbb{P}(|X_{2cL^2}^{(b)}| \geq L) \geq 1 - e^{-1}. \tag{4.14}$$

Hence, for $t > 0$,

$$\sigma_t \leq \mathbb{P}(|X_\tau^{(0)}| < L,\, \tau = 0, 1, \dots, t) \leq e^{-\lfloor t/(2cL^2) \rfloor}. \tag{4.15}$$

Thus the radius of convergence of $Q_2(z)$ is at least $e^{1/(3cL^2)}$. As $L \leq 4L_0\xi^{-1}\ln M$ we have $L^2 \ll T_{\text{MIX}}$, see (3.9). (The need for $L^2 \ll T_{\text{MIX}}$ explains the larger value of T_{MIX} than one might expect in (3.9)). So $e^{1/(3cL^2)} \geq 1 + 2\lambda$ and for $|z| \leq 1 + 2\lambda$,

$$|Q_2(z)| \leq \sum_{t=T_{\text{MIX}}+1}^{\infty} e^{2\lambda t - \lfloor t/(2cL^2) \rfloor} = o(1).$$

This lower bounds the radius of convergence of $\alpha(z)$ by $1 + 2\lambda$, proves (4.8) and then (4.8), (4.9) and (4.13) complete the proof of the case when v is locally tree like.

We now turn to the case where Λ_f contains a unique cycle C. The place where we have used the fact that Λ_f is a tree is in (4.15) which relies on (4.14). Let x be the furthest vertex of C from v in Λ_f. This is the only possible place where the random walk is more likely to get closer to v_1 at the next step. We can see this by considering the breadth first construction of Λ_f. Thus we can compare our walk with random walk on $[-L, L]$ where there is a unique value $d < L$ such that only at $\pm d$ is the walk more likely to move towards the origin and even then this probability is at most $2/3$. The distance of the walk $\mathcal{W}_v^{\Lambda_f}$ from v is dominated by the distance to the origin of a simple random walk, modified at one of two symmetric places P_1, P_2 to move towards the origin with probability $2/3$ instead of $1/2$. A simple coupling shows that making $P_1, P_2 = \pm 1$ keeps the particle closest to the origin. We can then contract $0, \pm 1$ into one node $0'$ with a loop. When at $0'$ the loop is chosen with probability $2/3$. The net effect is to multiply the time spent at the origin by 3, in expectation. We can couple this with a simple random walk by replacing excursions from the origin and back by a loop traversal, with probability $2/3$. In this way, we reduce to the locally tree like case with T_{MIX} inflated by 4 to account for the loop replacements.

We have now established that in the current case, G_F satisfies the conditions of Lemma 3.1.

4.1.2 Analysis of a random walk on G_F

We have a fixed vertex $u \in V$ and a vertex v and we estimate an upper bound for $\mathbb{P}(\mathcal{A}_t(v))$ using Lemma 3.1. For this we need a good upper bound on R_v. Let $f = (w_1, w_2)$ be the edge of K_F containing v.

We write $R_v = R'_v + R''_v$ where R'_v is the expected number of returns to v within time T_{MIX} before the first visit to Φ_f and R''_v is the expected number of visits after the first such visit.

$$R'_v = \mathrm{d}(v) R_P \tag{4.16}$$

where R_P is the *effective resistance* (see, e.g., Levin, Peres and Wilmer [21]) of a network N_v obtained from Λ_f by giving each edge of this graph resistance one and then joining the vertices in Φ_f via edges of resistance zero to a common dummy vertex.

For future reference, we note that (4.16) can be replaced by

$$R'_v = \lambda(v) R_P \tag{4.17}$$

when edges have weight $\lambda(e)$ and vertices have weight equal to the weight of incidence edges and edges are chosen with probability proportional to weight.

If f is locally tree like, let $\widehat{T}_1, \widehat{T}_2$ be the trees in K_F rooted at w_1, w_2 obtained by deleting the edge f from H_f. We then prune away edges of the trees $\widehat{T}_1, \widehat{T}_2$ to make the branching factors of the two trees exactly two, except at the root. We have to be careful here not to delete any edges incident with the roots. Thus one of the trees might have a branching factor at the root that is more than two. Then let T_1, T_2 be obtained from $\widehat{T}_1, \widehat{T}_2$ by placing vertices of degree two on their edges. If f is not locally tree like then we can remove an edge of the unique cycle C in H_f not incident with v from Λ_v and obtain trees $\widehat{T}_1, \widehat{T}_2$ in this way. Having done this, we prune edges and add vertices of degree two to create T_1, T_2 as in the locally tree like case. Removing an edge of C can only increase effective resistance and R_v.

Let R_1, R_2 be the resistances of the pruned trees.

We have
$$\frac{1}{R_P} = \frac{1}{\ell_1 + R_1} + \frac{1}{\ell_2 + R_2}.$$
Here ℓ_i is the number of edges in the path from v to w_i in G_f. If v is a vertex of K_F then we can dispense with ℓ_2, R_2.

Now when $v \notin V(K_F)$ we have, with $\ell = \ell_1 + \ell_2$ and $R = R_1 + R_2$,
$$\frac{1}{\ell_1 + R_1} + \frac{1}{\ell_2 + R_2} \geq \frac{4}{\ell + R} \tag{4.18}$$
which follows from the arithmetic-harmonic mean inequality.

When $v \in V(K_F)$ we have
$$\frac{1}{R_P} = \frac{1}{\ell_1 + R_1} + \frac{1}{\ell_2 + R_2} + \cdots + \frac{1}{\ell_d + R_d} \geq \frac{d^2}{\ell + R},$$
where $d = d(v) \geq 3$ and ℓ_i is the length of the ith induced path incident with v and R_i is the resistance of the tree at the other end of the path.

Let $\mathcal{E}_{\max}$ be the event that $\ell_e \leq \ell_{\max}$ for all $e \in E(K_F)$. With ε as defined in (2.3),
$$\mathbb{P}(R_1 \geq \rho_1, R_2 \geq \rho_2, \ell_1 + \ell_2 = l) \leq (1+\varepsilon)\widehat{\mathbb{P}}(R_1 \geq \rho_1)\widehat{\mathbb{P}}(R_2 \geq \rho_2)\widehat{\mathbb{P}}(\ell_1 + \ell_2 = l). \tag{4.19}$$

This follows from Part (a) of Lemma 2.4. If $\omega \in \{R_1 \geq \rho_1, R_2 \geq \rho_2, \ell_1 + \ell_2 = l\}$ then $k(\omega) \leq 3^{L_0} = M^{o(1)} = o(M)$. Also, if $\mathcal{E}_{\max}$ holds then $\zeta(\omega) \leq k\ell_{\max}$ and so $k\zeta = M^{o(1)}/\xi = o(\nu_2 + M)$. Since $\{R_1 \geq \rho_1\}, \{R_2 \geq \rho_2\}$, $\{\ell_1 + \ell_2 = l\}$ depend on disjoint sets of edges, we can write the product on the RHS of (4.19).

We will implicitly condition on $\mathcal{E}_{\max}$ when using $\mathbb{P}$ and this can only inflate probability estimates by $1 + o(1)$.

We will show in Section 4.1.3 that
$$\widehat{\mathbb{P}}(R_1 \geq \rho) \leq_b \begin{cases} 1 & \rho \leq L_0 \\ 3^{L_0}(1-\xi)^{\rho-2} & \rho > L_0 \end{cases} \tag{4.20}$$

Note that $1-\xi$ can be as small as $N^{-o(1)}$ and so we cannot replace $(1-\xi)^{\rho-2}$ by $(1-\xi)^{\rho}$ without further justification.

We will show in Section 4.1.4 that

$$R_v'' = o(R_v'). \tag{4.21}$$

Let Z_{ℓ,ρ_1,ρ_2} be the random variable that is equal to the number of vertices of G_F with parameters $\ell = \ell_1 + \ell_2, R_1 \geq \rho_1, R_2 \geq \rho_2$. Then we have

$$\mathbb{E}(Z_{\ell,\rho_1,\rho_2}) \leq_b \sum_{v \in V(G_F)} \xi(1-\xi)^{\ell-4} \times 3^{2L_0}(1-\xi)^{\lambda_1\rho_1+\lambda_2\rho_2}. \tag{4.22}$$

where $\lambda_i = 1_{\rho_i \geq L_0}$ for $i = 1, 2$.

For these vertices, we estimate that, with $\rho = \rho_1 + \rho_2$,

$$\mathbb{P}_w(\mathcal{A}_s(v)) \leq \exp\left\{-(1+o(1))\frac{\mathrm{d}(v)}{2m} \cdot s \cdot \frac{1}{\mathrm{d}(v)} \cdot \frac{4}{\ell+\rho}\right\} + O(T_{\text{MIX}}\pi_{\max}e^{-\lambda t/2}) \tag{4.23}$$

using Lemma 3.1 combined with (4.16), (4.17) and (4.18) to bound

$$\frac{1}{R_v} \geq (1-o(1))\frac{1}{\mathrm{d}(v)} \cdot \frac{4}{\ell+\rho}.$$

Using Lemma 3.1 we see that, where $m = M + \nu_2 = |E(G_F)|$,

$$\begin{aligned}\mathbb{E}(\Psi(V,t)) \leq_b \frac{3^{2L_0}\xi}{(1-\xi)^4} \sum_{v \in V(G)} \sum_{s \geq t} \sum_{\ell} \int_{\rho_1,\rho_2} d_{\rho_1} d_{\rho_2} (1-\xi)^{\ell+\rho_1\lambda_1+\rho_2\lambda_2} \times \\ \left(\exp\left\{-(1+o(1))\frac{\mathrm{d}(v)}{2m} \cdot s \cdot \frac{1}{\mathrm{d}(v)} \cdot \frac{4}{\ell+\rho}\right\} + O(T_{\text{MIX}}\pi_{\max}e^{-\lambda t/2})\right).\end{aligned} \tag{4.24}$$

where $\pi_{\max} = \max\{\pi_v : v \in V\}$.

This is to be compared with the expression in (4.4). Here we are summing our estimate for $\mathbb{P}(\mathcal{A}_s(v))$ over vertices v. Notice that the sum over $w \in V$ can be taken care of by the fact that we weight the contributions involving w by π_w. Remember that here w represents the vertex reached by $\mathcal{W}_u$ at time T_{MIX}.

We next remark that with $t = \Omega\left(\frac{m \ln^2 M}{-\ln(1-\xi)}\right)$ the term

$$O(T_{\text{MIX}}\pi_{\max}e^{-\lambda t/2}) = o(e^{-\Omega(M^{1-o(1)}})$$

can be neglected from now on.

We then have

$$\begin{aligned}&\mathbb{E}(\Psi(V,t)) \\ &\leq_b \sum_{v \in V(G)} \frac{3^{2L_0}\xi}{(1-\xi)^{4+2L_0}} \sum_{s \geq t} \sum_{\ell} \int_{\rho_1,\rho_2} d_{\rho_1} d_{\rho_2} \exp\left\{(1+o(1))\left((\ell+\rho_1\lambda_2+\rho_2\lambda_2)\ln(1-\xi) - \frac{2s}{m(\ell+\rho)}\right)\right\} \\ &\leq_b \sum_{v \in V(G)} \frac{3^{2L_0}\xi}{(1-\xi)^4} \sum_{\ell} \int_{\rho_1,\rho_2} d_{\rho_1} d_{\rho_2} \frac{\exp\left\{(1+o(1))\left((\ell+\rho_1\lambda_2+\rho_2\lambda_2)\ln(1-\xi) - \frac{2t}{m(\ell+\rho)}\right)\right\}}{1-\exp\left\{-\frac{2+o(1)}{m(\ell+\rho)}\right\}}.\end{aligned} \tag{4.25}$$

Our estimate for T_{COV} is $\Omega\left(\frac{m\ln^2 M}{-\ln(1-\xi)}\right)$. So, the contribution from $\ell_1,\ell_2,\rho_1,\rho_2$ with $\ell+\rho\le\frac{\gamma\ln M}{-\ln(1-\xi)}$ is negligible for small enough γ. If $\ell+\rho\ge\frac{\gamma\ln M}{-\ln(1-\xi)}$ then $\ell+\rho_1\lambda_2+\rho_2\lambda_2\sim\ell+\rho$, where $A\sim B$ denotes $A=(1+o(1))B$ as $N\to\infty$. Finally observe that the contributions from $\ell+\rho\ge\frac{\gamma^{-1}\ln M}{-\ln(1-\xi)}$ will also be negligible.

Ignoring negligible values we obtain a bound by further replacing the denominator in (4.25) by $\Omega\left(\frac{-\ln(1-\xi)}{m\ln M}\right)$. Thus,

$$
\begin{aligned}
&\mathbb{E}(\Psi(V,t))\\
&\le_b \sum_{v\in V(G)}\frac{m\ln M}{-\ln(1-\xi)}\times\frac{3^{2L_0}\xi}{(1-\xi)^4}\sum_{\ell}\int_{\rho_1,\rho_2}d_{\rho_1}d_{\rho_2}\exp\left\{(1+o(1))(\ell+\rho)\ln(1-\xi)-\frac{2t}{m(\ell+\rho)}\right\}\\
&\le_b \sum_{v\in V(G)}\frac{m\ln M}{-\ln(1-\xi)}\times\frac{3^{2L_0}\xi}{(1-\xi)^4}\sum_{\ell}\int_{\rho_1,\rho_2}d_{\rho_1}d_{\rho_2}\exp\left\{-\sqrt{\frac{(8+o(1))(-\ln(1-\xi))t}{m}}\right\}\\
&\le_b M^{2+o(1)}\exp\left\{-\sqrt{\frac{(8+o(1))(-\ln(1-\xi))t}{m}}\right\}.
\end{aligned}
\tag{4.26}
$$

Putting $t\sim\frac{m\ln^2 M}{8(-\ln(1-\xi))}$, where the implied $o(1)$ term goes to zero sufficiently slowly, we see that the RHS of (4.26) is $o(t)$. (Note that $L_0=o(\ln M)$ and $\ell_{\max},(1-\xi)^{-1},(-\ln(1-\xi))^{-1}=M^{o(1)}$ here). Summarising, if

$$t\ge\frac{(1+o(1))m\ln^2 M}{8(-\ln(1-\xi))} \tag{4.27}$$

then

$$\mathbb{E}(\Psi(V,t))=o(t)$$

and then Markov's inequality implies that w.h.p.

$$\Psi(V,t)=o(t).$$

This completes the proof of the upper bound for Case (c1) of Theorem 1, modulo some claims about R_v.

4.1.3 Estimating R_P

Assume first of all that we are in the locally tree like case. We consider the trees T_1,T_2. Their main variability is in the number of vertices of degree two that are planted on the edges of $\widehat{T}_1,\widehat{T}_2$. Fortunately, we only need to compute an upper bound on $\mathbb{P}(R(T)\ge\rho)$ where $R(T)$ is the resistance of one of these trees. We focus on T_1. Now let the subtrees of T_1 be $T_{1,1},\dots,T_{1,d}$, where $d\ge 2$.

We have

$$\frac{1}{R(T_1)}=\frac{1}{\ell(T_{1,1})+R(T_{1,1})}+\cdots+\frac{1}{\ell(T_{1,d})+R(T_{1,d})}\ge\frac{1}{\ell(T_{1,1})+R(T_{1,1})}+\frac{1}{\ell(T_{1,2})+R(T_{1,2})} \tag{4.28}$$

where $\ell_i = \ell(T_{1,i})$, $i = 1, \ldots, d$ is the resistance of the path in G_f from the root of T_1 to the root of $T_{1,i}$.

It follows from this that

$$\widehat{\mathbb{P}}(R(T_1) \geq \rho) \leq 2\widehat{\mathbb{P}}(\ell_1 + R(T_{1,1}) \geq 2\rho)\widehat{\mathbb{P}}(\ell_2 + R(T_{1,2}) \geq \rho). \tag{4.29}$$

This is because if $R(T_1) \geq \rho$ then (i) both of the $R(T_{1,i}) + \ell_i$, $i = 1, 2$ must be at least ρ and (ii) at least one of them must be at least 2ρ.

Now,

$$\widehat{\mathbb{P}}(\ell_1 = \ell) = \xi(1 - \xi)^{\ell - 1} \tag{4.30}$$

and

$$\widehat{\mathbb{P}}(\ell_1 \geq \ell) \leq (1 - \xi)^{\ell - 1}. \tag{4.31}$$

Let the level of a tree like T_1 be the depth of the tree in K_F from which it is derived. Let R_k be the (random) resistance of a tree of level k, obtained from a binary tree of depth k by the addition of a random number of vertices of degree two to each edge. Putting $R_0 = 0$ we get from (4.29) and (4.31) that

$$\widehat{\mathbb{P}}(R_1 \geq \rho) \leq 2(1 - \xi)^{3\rho - 2}. \tag{4.32}$$

Assume inductively that for $k \geq 1$ and $\rho \geq 1$,

$$\widehat{\mathbb{P}}(R_k \geq \rho) \leq a_k(1 - \xi)^{2\rho - k} \tag{4.33}$$

where $a_k = (2.5)^k$.

This is true for $k = 1$ by (4.32). Using (4.29) we get that

$$\begin{aligned}
\widehat{\mathbb{P}}(R_{k+1} \geq \rho) &\leq 2\left(\sum_{s=1}^{2\rho-1} \widehat{\mathbb{P}}(\ell_1 = s)\widehat{\mathbb{P}}(R_k \geq 2\rho - s) + \widehat{\mathbb{P}}(\ell_1 \geq 2\rho)\right) \\
&\leq 2\left(\sum_{s=1}^{2\rho-1} \xi(1 - \xi)^{s-1} \times a_k(1 - \xi)^{2(2\rho - s) - k} + (1 - \xi)^{2\rho}\right) \\
&= 2\left(a_k\xi(1 - \xi)^{4\rho - k - 1}\sum_{s=1}^{2\rho-1}(1 - \xi)^{-s} + (1 - \xi)^{2\rho}\right) \\
&\leq 2(a_k + 1)(1 - \xi)^{2\rho - k - 1}. \\
&\leq a_{k+1}(1 - \xi)^{2\rho - k - 1}.
\end{aligned} \tag{4.34}$$

This verifies the inductive step for (4.33) and (4.20) follows after taking $k = L_0$, with room to spare.

For the non locally tree like case, the deletion of a cycle edge of H_f to make a tree $\widehat{T}_1$, say, may create one or two vertices of degree two out of kernel vertices. After adding a random number of degree two vertices to each edge of $\widehat{T}_1$ to create T_1 we will in essence have created at most two paths whose path length is (asymptotically) distributed as the sum of two independent copies of Z, see Lemma 2.3. (Such a path arises by concatenating the two paths $P_e, P_{e'}$ for a pair of edges e, e' that are incident with a vertex of degree two of $\widehat{T}_1$). We claim that the resistance of such a tree is maximised in distribution if such paths are incident with the root and the rest of the paths have a distribution as in the tree-like-case. For this we consider moving some resistance ε from one edge closer to the root:

$$\left(a+\varepsilon+\frac{(b-\varepsilon)c}{b-\varepsilon+c}\right)-\left(a+\frac{bc}{b+c}\right)=\varepsilon\left(1-\frac{c^2}{(b-\varepsilon+c)(b+c)}\right)\geq 0$$

for $\varepsilon \leq b$. Here we have an edge (x,y) of resistance a and two edges of resistance b,c incident to y before moving ε of resistance.

The resistance R of $k+1$ levels of such a tree now satisfies

$$\frac{1}{R}=\frac{1}{\rho_1'+\rho_1''+S_1}+\frac{1}{\rho_2'+\rho_2''+S_2} \tag{4.35}$$

where S_1,S_2 are copies of R_k and $\rho_1',\rho_1'',\rho_2',\rho_2''$ are copies of Z.

Now we will use

$$\widehat{\mathbb{P}}(\rho_1'+\rho_1''=\rho)\leq 2\widehat{\mathbb{P}}(\rho_1'\geq\rho/2)\leq 2(1-\xi)^{\rho/2-1} \text{ and } \widehat{\mathbb{P}}(\rho_1'+\rho_1''\geq 2\rho)\leq 2(1-\xi)^{\rho-1}. \tag{4.36}$$

and so arguing as for (4.29) and (4.34), with $\rho\geq L$, and using (4.33),

$$\begin{aligned}\widehat{\mathbb{P}}(R_L\geq\rho) &\leq_b \sum_{s=1}^{2\rho-1}(1-\xi)^{s/2-1}(2.5)^L(1-\xi)^{2(2\rho-s)-1}+(1-\xi)^{\rho-1}\\ &\leq_b (2.5)^L(1-\xi)^{\rho-2}+(1-\xi)^{\rho-1}\\ &\leq_b (2.5)^L(1-\xi)^{\rho-2}.\end{aligned}$$

This completes the verification of (4.20).

4.1.4 Estimating R_v''

It follows from (4.12) that

$$R_v''\leq N^{-\delta_0/5}(R_v'+R_v'')$$

and hence

$$R_v''\leq N^{-\delta_0/6}R_v'. \tag{4.37}$$

The proof of the upper bound for Case (c1) of Theorem 1 is now complete.

For the next case we let

$$\omega=N^{\zeta_1}$$

where (2.9) holds and

$$\zeta_0\ll\zeta_1=o(\delta_0) \text{ and now } \delta_0\zeta_1\log N\gg 1. \tag{4.38}$$

4.2 Case (c2): $M^{1+\delta_1} \le \nu_2 \le e^\omega$

We recommend that the reader re-visits Section 1, where we give an outline of our approach to this case.

It is worth pointing out that

$$\xi = o(1)$$

in this case.

We will be considering several graphs in addition to G_F and K_F and so it will be important to keep track of their edge and vertex sets. For now let

$$V_F = V(G_F), E_F = E(G_F) \text{ and } V_K = V(K_F), E_K = E(K_F).$$

We see an immediate problem in the case where $\nu_2/M \to \infty$ too fast. In this case we have

$$T_{\text{MIX}}\pi_v = \Omega\left(\frac{\ln^4 M}{\xi^2} \cdot \frac{1}{\nu_2}\right) = \Omega\left(\frac{\nu_2 \ln^4 M}{M^2}\right). \tag{4.39}$$

So if $\nu_2 \ge M^2$ then we cannot apply Lemma 3.1 directly. Our main problem has been to find a way around this.

We let

$$\ell^* = \left\lfloor \frac{1}{\xi\omega} \right\rfloor. \tag{4.40}$$

We begin with some structural properties tailored to this case.

4.2.1 Structural Properties

Lemma 4.2. *W.h.p. there is no set $S \subseteq V_K, |S| \le n_0 = N^{1-5000\zeta_0}$ such that $e(S) \ge (1.001)|S|$.*

Proof. The expected number of such sets can be bounded by

$$\begin{aligned}
\sum_{s=4}^{n_0} \sum_{|S|=s} \binom{\mathrm{d}(S)}{(1.001)s} \left(\frac{\mathrm{d}(S)}{M}\right)^{(1.001)s} &\le \sum_{s=4}^{n_0} \sum_{|S|=s} \left(\frac{e\mathrm{d}(S)}{(1.001)s} \cdot \frac{\mathrm{d}(S)}{M}\right)^{(1.001)s} \\
&\le \sum_{s=4}^{n_0} \binom{N}{s} \left(\frac{esN^{2\zeta_0}}{M}\right)^{(1.001)s} \\
&\le \sum_{s=4}^{n_0} \left(\frac{e^{2.001} N^{3\zeta_0} s^{0.001}}{M^{0.001}}\right)^s \\
&= o(1).
\end{aligned} \tag{4.41}$$

Explanation for (4.41): Having chosen a set X of $(1.001)s$ configuration points for $(1.001)s$ distinct edges, we randomly pair them with other configuration points. After pairing i of them, the probability the next point makes an edge in S using only one point of X is $\frac{\mathrm{d}(S)-(1.001)s-i}{2M-2i-1} \le \frac{\mathrm{d}(S)}{M}$. $\square$

An edge e of K_F is *light* if $\ell_{\min} \leq \ell_e \leq \ell^*$. Let

$$\widehat{E}_\sigma = \{e \in E_K : e \text{ is light}\}$$
$$\widehat{V}_\sigma = \left\{v \in V_K : \exists e \in \widehat{E}_\sigma \text{ s.t. } v \in e\right\}$$

Note that

$$\mathbb{P}(e \in \widehat{E}_\sigma) \leq \xi\ell^* \leq \frac{1}{\omega}.$$

Lemma 4.3.

$$d(\widehat{V}_\sigma) \leq \frac{2N}{\omega^{1/3}}, \qquad \textit{with probability at least } 1 - \omega^{-1/3}.$$

Proof. For any value D we have

$$\mathbb{E}\left(\left|\left\{v \in \widehat{V}_\sigma : \mathrm{d}(v) \leq D\right\}\right|\right) \leq \frac{D\,|\{v \in V : \mathrm{d}(v) \leq D\}|}{\omega} \leq \frac{ND}{\omega}.$$

Putting $D = \omega^{1/3}$ and applying Markov's inequality we see that with probability at least $1 - \omega^{-1/3}$.

$$\sum_{v \in \widehat{V}_\sigma : \mathrm{d}(v) \leq \omega^{1/3}} \mathrm{d}(v) \leq \frac{N}{\omega^{1/3}}.$$

In addition we have

$$D^2 \sum_{j \geq D} \nu_j j \leq D_3 \text{ and so } \sum_{v \in \widehat{V}_\sigma : \mathrm{d}(v) \geq \omega^{1/3}} \mathrm{d}(v) \leq \frac{D_3}{\omega^{2/3}} \leq \frac{a_0 D_1}{\omega^{2/3}} \leq \frac{2a_0^{3/2} N}{\omega^{2/3}},$$

where we have used (1.8). □

Now define a sequence $X_0 = \widehat{V}_\sigma, X_1, X_2, \ldots,$ where $X_{i+1} = X_i \cup \{x_{i+1}\}$ and x_{i+1} is any vertex in $V_K \setminus X_i$ that has at least two neighbours in X_i. This continues until we find k for which every vertex in $V_0 \setminus X_k$ has at most one neighbour in X_k. Let $\nu_0 = |X_0| \leq \frac{2N}{\omega^{1/3}}$ w.h.p. Then X_i has $\nu_0 + i$ vertices and at least $2i$ edges. Now (4.38) implies that $\nu_0 = o(n_0)$ (of Lemma 4.2) and so if $i \geq \nu_0$ then we contradict the claim in Lemma 4.2. We let

$$V_\sigma = X_k \text{ and } V_\lambda = V_K \setminus V_\sigma \tag{4.42}$$

and observe that

$$|V_\sigma| \leq \frac{4N}{\omega^{1/3}} \text{ and so } \mathrm{d}(V_\sigma) \leq D_\sigma \text{ where } D_\sigma = \frac{6N^{1+\zeta_0}}{\omega^{1/3}}. \tag{4.43}$$

Note also that V_σ is well defined in the sense that all sequences $x_1, x_2, \ldots,$ lead to the same final set.

We will see in Remark 4.10 why we need V_σ instead of the simpler $\widehat{V}_\sigma$.

Lemma 4.4. *W.h.p. there is no path of length L_0 in K_F with more than $L_0/10$ members of V_σ.*

Proof. First note that if $v_1, v_2, \ldots, v_s \in V_\sigma$ then there is an ordering such that $v_1, v_2, \ldots, v_s$ appears as a sub-sequence of $x_1, x_2, \ldots, x_k$ above. We will assume this ordering and inflate our final estimate by $s!$ to account for the choice.

We continue by asserting (justification below) that for vertices $v_1, v_2, \ldots, v_s, s \leq L_0$,

$$\mathbb{P}(v_1, v_2, \ldots, v_s \in V_\sigma \mid \mathrm{d}(V_\sigma) \leq D_\sigma) \leq \left(\frac{20sN^{6\zeta_0}}{\omega^{2/3}}\right)^s. \tag{4.44}$$

Thus, given $\mathcal{D} = \{\mathrm{d}(V_\sigma) \leq D_\sigma\}$, the expected number of paths in question is bounded by

$$\begin{aligned}\sum_{v_1,\ldots,v_{L_0+1}\in V_K} \prod_{i=1}^{L_0} \frac{\mathrm{d}(v_i)\mathrm{d}(v_{i+1})}{2M} \binom{L_0}{L_0/10} \left(\frac{20L_0N^{6\zeta_0}}{\omega^{2/3}}\right)^{L_0/10} &\leq \\ \sum_{v_1,\ldots,v_{L_0+1}} \frac{\mathrm{d}(v_1)\mathrm{d}(v_{L_0+1})}{M} \prod_{i=2}^{L_0} \frac{\mathrm{d}(v_i)^2}{M} \left(\frac{200L_0eN^{6\zeta_0}}{\omega^{2/3}}\right)^{L_0/10}& \\ \leq \frac{D_1^2D_2^{L_0-1}}{M^{L_0}} \left(\frac{200L_0eN^{6\zeta_0}}{\omega^{2/3}}\right)^{L_0/10} \leq_b N \left(\frac{200L_0ea_0^{10}N^{6\zeta_0}}{\omega^{2/3}}\right)^{L_0/10} &= o(1),\end{aligned}$$

after using (4.38).

Proof of (4.44): Observe first of all that

$$\begin{aligned}&\mathbb{P}(v_{i+1} \in \widehat{V}_\sigma \mid v_1, v_2, \ldots, v_i \in V_\sigma, \mathcal{D}) \\ &= \mathbb{P}(v_{i+1} \in \widehat{V}_\sigma \mid v_1, v_2, \ldots, v_i \in \widehat{V}_\sigma, \mathcal{D})\mathbb{P}(v_1, v_2, \ldots, v_i \in \widehat{V}_\sigma, \mathcal{D} \mid v_1, v_2, \ldots, v_i \in V_\sigma, \mathcal{D}) \\ &\leq \mathbb{P}(v_{i+1} \in \widehat{V}_\sigma \mid v_1, v_2, \ldots, v_i \in \widehat{V}_\sigma, \mathcal{D}) \\ &\leq \frac{iN^{\zeta_0}}{M} + \mathbb{P}(v_{i+1} \in \widehat{V}_\sigma \mid v_1, v_2, \ldots, v_i \in \widehat{V}_\sigma, \mathcal{D}, (v_{i+1}, v_j) \notin \widehat{E}_\sigma, \forall j) &\quad (4.45) \\ &\leq \frac{iN^{\zeta_0}}{M} + \frac{N^{\zeta_0}}{\omega} \\ &\leq \frac{2N^{\zeta_0}}{\omega}. &\quad (4.46)\end{aligned}$$

Explanation of (4.45)**:** The first term iN^{ζ_0}/M is a bound on the probability that v_{i+1} is a neighbour of some $v_j, j < i$. The second term is a bound on the probability that an edge incident with v_{i+1} is light. We deal with the conditioning by first exposing K_F and then exposing the placement of the vertices of degree two.

We will now prove that

$$\mathbb{P}(v_{i+1} \in V_\sigma \setminus \widehat{V}_\sigma \mid v_1, v_2, \ldots, v_i \in V_\sigma) \leq \frac{18N^{6\zeta_0}}{\omega^{2/3}}. \tag{4.47}$$

Recall that we assume the order $v_1, v_2, \ldots, v_i$ is such that v_j can be placed in V_σ once $v_1, v_2, \ldots, v_{j-1}$ have been so placed. Then, using the notation of Section 2.1, we let $\widehat{W} = W \setminus W_{v_{i+1}}$. If $|W_{v_{i+1}}|$ is odd, we first choose a random point $x \in \widehat{W}$ and pair up the remainder of points to create $\widehat{F}$. Suppose now that $W_{v_{i+1}} = \{x_1, x_2, \ldots, x_k\}$. We define a sequence of configuration multi-graphs $\Gamma_0 = \widehat{K}_{\widehat{F}}, \Gamma_1, \ldots, \Gamma_k = K_F$. We obtain Γ_{j+1} from Γ_j as follows: If $k - j$ is odd then we pair up x_j

with the unpaired point in Γ_j. If $k-j$ is even we choose a random pair $\{y,z\}$ in Γ_j and pair x_{j+1} with y or z equally likely, leaving the other point unpaired.

We first claim that $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ are all random pairings of their respective point sets. We do this by induction. It is trivially true for Γ_0. When $k-j$ is odd, the construction is equivalent to choosing a random point to pair with x_{j+1} and then choosing a random configuration (Γ_j) on the remaining points. If $k-j$ is even, then we again pair x_{j+1} with a random point y, say. Then z will be a uniform random point and the remaining configuration will be a random pairing of what is left.

Assume that $\mathrm{d}(V_\sigma(\Gamma_0)) \leq D_\sigma$. Now v_{i+1} will be placed into $V_\sigma(\Gamma_k)$ only if there are two values of j for which x_{j+1} is paired with a point associated with a vertex in $V_\sigma(\Gamma_j)$. Up to this point we will have $V_\sigma(\Gamma_j) \subseteq V_\sigma(\Gamma_0)$. It follows that x_{j+1} is so paired with probability at most

$$\binom{k}{2}\left(\frac{D_\sigma}{M}\right)^2. \tag{4.48}$$

Equation (4.47) (and the lemma) follows from (4.48), after inflating the final estimate by $s!$. □

Consider the following property of $S \subseteq V_\lambda$ (defined in (4.42)): Let $s = |S|$.

$$\text{(i) } S \text{ induces a tree in } K_F; \text{ (ii) } \mathrm{d}(S) \leq s \ln N; \text{ (iii) } e(S:V_\sigma) \geq \eta_s = \max\{3, \lceil s/500 \rceil\}. \tag{4.49}$$

Lemma 4.5. *W.h.p., if S satisfies (4.49) then $|S| \leq s_1$ where*

$$s_1 = \frac{10000 \ln N}{\ln \omega}.$$

Proof. Let Z_s be the number of sets satisfying (4.49) under these circumstances. Assume that $s > s_1$. Then, from (4.43),

$$\mathbb{E}(X_s) \leq (1+o(1)) \sum_{\substack{|S|=s \geq s_1 \\ \mathrm{d}(S) \leq s \ln N}} \binom{\mathrm{d}(S)}{s/500}\left(\frac{D_\sigma}{M}\right)^{s/500}\binom{\mathrm{d}(S)}{s-1}\left(\frac{\mathrm{d}(S)}{M}\right)^{s-1}. \tag{4.50}$$

Explanation: We choose configuration points that will be paired with V_σ in $\binom{\mathrm{d}(S)}{s/500}$ ways. The probability that all these points are paired in V_σ is at most

$$\left(\frac{\mathrm{d}(V_\sigma)}{2M-\mathrm{d}(S)}\right)^{s/500} \leq \left(\frac{D_\sigma}{2M-\mathrm{d}(S)}\right)^{s/500},$$

see Lemma 4.3. We choose $s-1$ configuration points for the edges inside S. The probability they are paired with other points associated with S can be bounded by $\left(\frac{\mathrm{d}(S)}{2M-o(M)}\right)^{s-1}$. The factor $1+o(1)$ arises from the conditioning imposed by assuming (4.43). Also, after conditioning on V_σ we only allow a vertex in V_λ to choose a single neighbour in V_σ. Thus $\binom{d(S)}{s/500}$ is an over-estimate of the number of choices.

Continuing,

$$\mathbb{E}(X_s) \leq \sum_{\substack{|S|=s \geq s_1 \\ \mathrm{d}(S) \leq s \ln N}} (500e \ln N)^{s/500}\left(\frac{6N^{\zeta_0}}{\omega^{1/3}}\right)^{s/500}(e \ln N)^s\left(\frac{s \ln N}{N}\right)^{s-1}$$

$$\leq \left(\frac{Ne}{s}\right)^s (500e\ln N)^{s/500}\left(\frac{6N^{\zeta_0}}{\omega^{1/3}}\right)^{s/500}(e\ln N)^s\left(\frac{s\ln N}{N}\right)^{s-1}$$
$$\leq_b N\left(\frac{CN^{\zeta_0/500}\ln^{2.002+o(1)}N}{\omega^{1/1500}}\right)^{s-1}, \qquad C=e^{2+o(1)}(3000e)^{(1+o(1))/500},$$

see (4.38).

So,

$$\mathbb{E}\left(\sum_{s\geq s_1} X_s\right) \leq_b N\sum_{s\geq s_1}\left(\frac{CN^{\zeta_0/500}\ln^2 N}{\omega^{1/1500}}\right)^{s-1} = o(1).$$

This implies that w.h.p. we have $X_s=0$ for $s\geq s_1$. □

We now wish to show that small sets of K_F-edges do not contain too many vertices of degree two.

Lemma 4.6. *W.h.p. no subset $S\subseteq E_K$ satisfies $|S|\leq \varepsilon M$ and $\ell(S)=\sum_{e\in S}\ell_e\geq L=\varepsilon^{1/2}M/\xi$. provided ε is a sufficiently small positive constant. In particular, this holds for any $\varepsilon\leq\varepsilon_1$ where ε_1 is the solution to $\varepsilon^{3/2}e^{1/(6\varepsilon^{1/2})}=20$.*

Proof. Let S be "bad" if it violates the conditions of the lemma. We can assume w.l.o.g. that $|S|=\varepsilon M$ here. Now using (2.6) to go from the first line to the second,

$$\begin{aligned}
\mathbb{P}(\exists \text{ a bad } S) &\leq \binom{M}{\varepsilon M}\sum_{\ell=L}^{m}\binom{\ell-1}{\varepsilon M-1}\frac{\binom{\nu_2+M-1-\ell}{M-\varepsilon M-1}}{\binom{\nu_2+M-1}{M-1}}\\
&\leq \sum_{\ell=L}^{m}\left(\frac{Me}{\varepsilon M}\right)^{\varepsilon M}\left(\frac{\ell e}{\varepsilon M}\right)^{\varepsilon M}\xi^{\varepsilon M}(1-\xi)^{\ell-\varepsilon M}\left(1+\frac{(1+o(1))\varepsilon M}{\nu_2}\right)^{\ell-\varepsilon M}\\
&= \sum_{\ell=L}^{m}\left(\frac{e^2\ell\xi}{\varepsilon^2M(1-\xi)}\left(1+\frac{(1+o(1))\varepsilon M}{\nu_2}\right)^{-1}\right)^{\varepsilon M}\left((1-\xi)\left(1+\frac{(1+o(1))\varepsilon M}{\nu_2}\right)\right)^{\ell}\\
&\leq \sum_{\ell=L}^{m}\left(\frac{10\ell\xi}{\varepsilon^2 M}\right)^{\varepsilon M}(1-(1-2\varepsilon)\xi)^{\ell}. \qquad (4.51)
\end{aligned}$$

Putting $\ell=AM/\xi$ into the summand u_ℓ of (4.51) we obtain for sufficiently small ε that

$$u_\ell \leq \left(\frac{10Ae^{-A/(2\varepsilon)}}{\varepsilon^2}\right)^{\varepsilon M}\leq e^{-\varepsilon^{1/2}M/3}. \qquad (4.52)$$

Now $A\geq\varepsilon^{1/2}$ and a quick check shows that (4.52) is valid if $\varepsilon^{3/2}e^{1/(6\varepsilon^{1/2})}\geq 10$.

So,

$$\mathbb{P}(\exists \text{ a bad } S)\leq me^{-\varepsilon^{1/2}M/3}=o(1),$$

given our upper bound of $e^{M^{o(1)}}$ for m. □

The next lemma shows that our assumption on degrees implies that a small set of vertices has small total degree.

Lemma 4.7. *If $S \subseteq V_K$ and $|V| \leq \varepsilon N$ then $d(S) \leq 2a_0\varepsilon^{1/3}N$ for $\varepsilon < 1$.*

Proof. Let $S_0 = [N_\varepsilon, N]$ where $N_\varepsilon = N - \varepsilon N + 1$. It is enough to prove the lemma for $S = S_0$. Let $D_\varepsilon = \sum_{j \in S_0} d_j$ and $L = d_{N_\varepsilon}$. Then

$$D_\varepsilon \leq \sum_{k \geq L} k\nu_k \leq \sum_{k \geq L} \frac{k^2}{L}\nu_k \leq \frac{a_0 N}{L}. \tag{4.53}$$

If $L > 1/\varepsilon^{1/3}$ then we are done and so assume that $L \leq 1/\varepsilon^{1/3}$.

Let $S_1 = \left\{j : d_j \geq L/\varepsilon^{1/3}\right\}$. Then, following the argument in (4.53) for S_1 we get

$$D_\varepsilon \leq \frac{\varepsilon N L}{\varepsilon^{1/3}} + \sum_{j \in D_1} d_j \leq \frac{\varepsilon N L}{\varepsilon^{1/3}} + \frac{a_0\varepsilon^{1/3}N}{L}$$

and the result follows. □

4.2.2 Surrogates for G_F

We have seen that we can use (4.4) if we have a good estimate for $\mathbb{P}_v(\mathcal{A}_s(w))$. We have seen in (4.39) that we cannot necessarily apply the lemma directly in this case. So what we will do is find a graph G that satisfies the conditions of Lemma 3.1 and whose cover time is related in some easily computable way to the cover time in G_F. (This statement is only approximately true, but it can be used as motivation for some of what follows).

In the following, we define graphs that will be surrogates for G_F with respect to computing the cover time.

Let e be an edge of K_F. We will break the corresponding path P_e of length $\ell_e = p_e\ell^* + q_e$, $p_e \geq 0, 0 \leq q_e < \ell^*$ in the graph G_F into consecutive sub-paths Q_f, $f \in F_e$. For a typical path, where $p_e \geq 1$ there will be $p_e - 1$ paths of length ℓ^* and one path of length $\ell^* + q_e$. There will however be some cases where e is light and so we have to be a little more careful. When e is light we do nothing to P_e. In this case, P_e is considered as a sub-path of itself in the following and is replaced by a single edge in the graph G_0 defined below. Otherwise we construct $p_e - 1$ paths of length ℓ^* and one path of length $\ell^* + q_e$. Let $\mathcal{Q}_e$ denote the set of sub-paths created from P_e.

We define the graph $G_0 = (V_0, E_0)$ as follows: For each $e \in E_K$, we replace each sub-path $Q \in \mathcal{Q}_e$ of length ℓ_Q by an edge $f = f_Q$ of *weight or conductivity* $\kappa(f) = \ell^*/\ell_Q$. The *resistance* $\rho(f)$ of edge f is given by $1/\kappa(f)$. Note that the total resistance of a heavy edge e is ℓ_e/ℓ^*.

We will use the notation $f \in e$ to indicate that edge f of G_0 is obtained from a sub-path of edge $e \in E_K$.

We now check that the total weight of the edges in G_0 is what we would expect. We remark first that since $M = o(\nu_2)$ and $M = \Theta(N)$ we have

$$m \sim |V(G_F)| \sim \nu_2.$$

Lemma 4.8. *W.h.p.,*

$$\kappa(E_0) \sim |E_0| \sim \frac{|E(G_F)|}{\ell^*} = \frac{\nu_2 + M}{\ell^*} \sim \omega M.$$

Proof. Each edge $e \in E_K$ gives rise to a path of length ℓ_e in G_F. We let

$$K_0 = \{e \in E_K : \ell_e < \ell^*\}, \; K_1 = \left\{e \in E_K : \ell^* \le \ell_e < \ell^* \omega^{1/3}\right\} \text{ and } K_2 = E_0 \setminus (K_0 \cup K_1).$$

Then,

$$|E_0| = \frac{1}{\ell^*} \sum_{e \in K_1 \cup K_2} (\ell_e - q_e) + |K_0| \tag{4.54}$$

$$= \frac{m}{\ell^*} - \frac{|K_0|}{\ell^*} - \frac{1}{\ell^*} \sum_{e \in K_1 \cup K_2} q_e + |K_0|. \tag{4.55}$$

Now for $e \in E_K$ and $0 \le q < \ell^*$, and using Part (b) of Lemma 2.4 with $k = 1, \zeta = q$,

$$\mathbb{P}(q_e = q) \sim \sum_{r \ge 0} \xi(1-\xi)^{r\ell^* + q - 1} = \xi(1-\xi)^{q-1} \cdot \frac{1}{1-(1-\xi)^{\ell^*}} \sim \omega \xi (1-\xi)^{q-1}.$$

So,

$$\mathbb{E}(q_e) \sim \sum_{k=1}^{\ell^*-1} k\omega\xi(1-\xi)^{k-1} \le \frac{\omega\xi}{(1-\xi)^2} \le \ell^*,$$

and

$$\mathbb{E}\left(\sum_{e \in E_K} q_e\right) \le_b \ell^* M. \tag{4.56}$$

So w.h.p.

$$\frac{1}{\ell^*} \sum_{e \in K_1 \cup K_2} q_e = o(M\omega^{1/2}). \tag{4.57}$$

Now

$$\mathbb{E}(|K_0|) \sim M \sum_{q=1}^{\ell^*-1} \xi(1-\xi)^{q-1} = O(\ell^* \xi M) = O(M/\omega).$$

So,

$$|K_0| = o(M) \; w.h.p. \tag{4.58}$$

Going back to (4.55) with (4.57) and (4.58) and

$$\frac{m}{\ell^*} \sim \omega M$$

we see that our expression for $|E_0|$ is correct, w.h.p.

Now w.h.p.

$$\kappa(E_0) = \sum_{e \in K_1 \cup K_2} \left(p_e - 1 + \frac{\ell^*}{\ell^* + q_e}\right) + \sum_{e \in K_0} \frac{\ell^*}{\ell_e}$$

$$= \sum_{e \in K_1 \cup K_2} \left(p_e - \frac{q_e}{\ell^* + q_e}\right) + \sum_{e \in K_0} \frac{\ell^*}{\ell_e}$$

$$= \frac{m}{\ell^*} - \sum_{e \in K_1 \cup K_2} \frac{q_e}{\ell^* + q_e} + \sum_{e \in K_0} \left(\frac{\ell^*}{\ell_e} + \frac{\ell_e}{\ell^*}\right).$$

To finish the proof we show that the terms other than m/ℓ^* contribute $o(\omega M)$ in expectation and then we can apply Markov's inequality. We can use (4.57) to deal with the first sum. We are left with

$$\mathbb{E}\left(\sum_{e\in K_0}\frac{\ell^*}{\ell_e}\right)=\ell^*\sum_{e\in E_K}\sum_{k=1}^{\ell^*-1}\frac{\mathbb{P}(\ell_e=k)}{k}\leq$$

$$(1+o(1))\ell^* M\left(\sum_{k=1}^{\nu_2^{1/3}}\frac{\xi(1-\xi)^{k-1}}{k}+\sum_{k=\nu_2^{1/3}}^{\nu_2}\frac{\binom{M+\nu_2-k-1}{M-2}}{k\binom{M+\nu_2-1}{M-1}}\right)$$

$$\leq_b \frac{M\ln\nu_2}{\omega}+\frac{M^2}{\xi\nu_2}\exp\left\{-\frac{(M-2)\nu_2^{1/3}}{M+\nu_2-1}\right\}=o(\omega M),$$

where to get the final expression we have used the calculations in Part (c) of Lemma 2.4, i.e., (2.7). Of course we can use (4.58) to deal with $\sum_{e\in K_0}\ell_e/\ell^*\leq|K_0|$. □

Since, from (4.54) and the above analysis,

$$\sum_{e\in K_1\cup K_2}(p_e-1)\leq|V_0|\leq|E_0|\leq\sum_{e\in K_1\cup K_2}p_e+o(\omega M)$$

we have that w.h.p.

$$|V_0|\sim|E_0|\sim\omega M.$$

We will analyse the expected time for a random walk $\mathcal{W}^{G_0}$ on G_0 to cross each edge of G_0 at least once. We will be able to couple this with $\mathcal{W}^{G_F\to V_0}$, the projection of $\mathcal{W}^{G_F}$ onto V_0. We will see below that if either walk is at $v\in V_0$ and w is a neighbour of v in G_0 then w has the same probability of being the next V_0-vertex visited in both walks.

It is easy to see that after $\mathcal{W}^{G_0}$ has crossed each edge of G_0, in the coupling, $\mathcal{W}^{G_F}$ will have visited each vertex of G_F.

We must modify G_0 slightly, because we have to cover the *edges* of G_0. Let $f^*=(v_1,v_2)$ be an edge of G_0.

The graph $G_0^*=G_0^*(f^*)$ will be obtained from G_0 by splitting f^*. We give edges (v_1,v_{f^*}) and (v_{f^*},v_2) a weight of $\alpha=\min\{\alpha_f,1\}$ where α_f is the weight of edge f.

$\mathcal{W}^{G_0^*}$ is the random walk on G_0^*, where we choose edges according to weight; $\mathcal{W}^{G_0^*\to V_0}$ is the projection of $\mathcal{W}^{G_0^*}$ onto V_0. This walk is $\mathcal{W}^{G_0^*}$ with visits to v_{f^*} omitted from the sequence of states. This means that time passes more slowly in $\mathcal{W}^{G_0^*}$ than it does in $\mathcal{W}^{G_0^*\to V_0}$. We use G_0^* in order to deal with the edge cover time of G_0, which is what we need, see (4.63) below.

Our goal is to compute a good upper estimate for $\mathbb{P}(\mathcal{A}_s(f^*))$ where $\mathcal{A}_s(f^*)$ is the event that we have not crossed edge f^* in the time interval $[T_{\text{MIX}},s]$. We do this by going to G_0^* and estimating $\mathbb{P}(\mathcal{A}_s(v_{f^*}))$ for the random walk on G. Note that $\mathbb{P}(\mathcal{A}_s(f^*))=\mathbb{P}(\mathcal{A}_s(v_{f^*}))$ if f is a heavy edge and $\mathbb{P}(\mathcal{A}_s(f^*))\leq\mathbb{P}(\mathcal{A}_s(v_{f^*}))$ if f is a light edge. Indeed, in both cases there is a natural coupling of $\mathcal{W}^{G_0^*}$ and $\mathcal{W}^{G_0}$, up until v_1 or v_2 are reached. This is because walks in G_0 and walks in G_0^* that do not contain v_1 or v_2 as a middle vertex have the same probability in both. Having reached v_1

or v_2 there is no lesser chance of crossing f^* in G_0 than there is of visiting v_{f^*} in G_0^*. In the case of a heavy edge, we can extend this coupling up until v_{f^*} is visited. This follows from our choice of weight for the edges $(v_i, v_{f^*}), i = 1, 2$.

There is a problem with respect to using G_0 as a surrogate in that its mixing time can be too large. If the edges of a graph are weighted then the conductance of a set of vertices S is given by

$$\Phi(S) = \frac{\sum_{x \in S, y \in \bar{S}} \kappa(x, y)}{\kappa(S)} = \frac{\kappa(\partial S)}{\kappa(S)}.$$

Consider an edge $e = (u, v) \in E_K$ for which $\ell_e = 1$ and such that (i) u, v both have degree three in K_F and (ii) all edges of E_K other than e incident with u, v are heavy. Let $S = \{u, v\}$. Then in G_0, $\Phi(S) = O(1/\ell^*)$, making $\Phi(G_0)$ too small. The situation cannot be dismissed as only happening with probability $o(1)$.

We remark that if the following conjecture is true, then we will be able to fix the problem of small edges by adding more vertices of degree two. We will be able to do this so that ℓ^* divides ℓ_e for all $e \in E_K$. This would simplify the proof somewhat.

Conjecture 4.9. *Adding extra vertices of degree two to the edges of K_F to make $\ell_e \geq \ell^*$ for all e, does not decrease the cover time.*

In the absence of a proof of this conjecture, we must find a work around. We observe for later that if every edge e has a weight $\kappa(e) \in [\kappa_L, \kappa_U]$ then we have

$$\Phi(S) \geq \frac{\kappa_L \partial S}{\kappa_U \mathrm{d}(S)} \tag{4.59}$$

where ∂S is defined following (2.12).

We now define the graph G. It will have vertex set $V_\lambda^* = V_\lambda \cup \{v_1, v_{f^*}, v_2\}$, see (4.42). A G_0^*-edge f contained in V_λ will give rise to an edge of weight κ_f in G.

Next let N_1' be the set of vertices in V_λ that have K_F-neighbors in V_σ and let $N_1 = N_1' \cup \{v_1, v_{f^*}, v_2\}$. The edges from N_1 to V_σ will also give rise to G edges. For each $x \in V_\sigma \cup N_1$ and $y \in N_1$ we define $\theta(x, y)$ as follows: Consider the random walk $\mathcal{W}_x^{G_0^*}$. This starts at x and it chooses to cross an incident edge of the current vertex with probability proportional to its G_0^*-edge weight. Suppose that this walk follows the sequence $x_0 = x, x_1 \in V_\sigma, x_2, \ldots,$ and that $k, k \geq 1$ is the smallest positive index such that $x_k \notin V_\sigma$. Then, $\theta(x, y) = \mathbb{P}(x_k = y)$. Then for $x \in N_1$ and $z \in V_\sigma$ for which $f = (x, z)$ is an edge of G_0^* and $y \in N_1$ ($y = x$ is allowed) we add a *special* edge, *oriented* from x to y of weight $\kappa_f \theta(z, y)$. We remind the reader that $\kappa_f = \ell^*/\ell_f$.

We have introduced some orientation to the edges. We need to check that the Markov chain we have created is reversible. Then we can use conductance to estimate the mixing time. In verifying this claim we will see that the steady state of the walk is proportional to $\kappa(x)$ for $x \in V_\lambda$. We do this by checking detailed balance. For $x, y \in V_\lambda^*$ we let $P(x, y)$ be the probability of moving in one step from x to y. We let $P(x, y) = P_0(x, y) + P_1(x, y)$ where $P_0(x, y)$ is the probability of following a special edge from x to y. We have $\kappa(x)P_1(x, y) = \kappa(y)P_1(y, x)$ because these quantities are derived from the random walk on G_0^*. As for $P_0(x, y)$, we have

$$\kappa(x)P_0(x, y) = \sum_{z_0 \in V_\sigma} \sum_{z_1, z_2 \ldots z_l} \kappa(x)P_1(x, z_0) \prod_{i=0}^{l-1} P_1(z_i, z_{i+1}) \times P_1(z_l, y)$$

$$
\begin{aligned}
&= \sum_{z_0 \in V_\sigma} \sum_{z_1, z_2 \ldots z_l} \kappa(z_0) P_1(z_0, x) \prod_{i=0}^{l-1} P_1(z_i, z_{i+1}) \times P_1(z_l, y) \\
&\vdots \\
&= \kappa(y) P_0(y, x).
\end{aligned}
$$

As a further step in the construction of G, we remove some loops. In particular, if $x \in N_1$ and $p = P(x,x) > 0$ then

$$
P(x,x) \leftarrow 0 \text{ and } P(x,y) \leftarrow P(x,y)/(1-p) \text{ for } y \in N_1, y \neq x.
$$

Because the chain is reversible we can define an associated electrical network $\mathcal{N}$, which is an undirected graph with an edge (x,y) of weight (conductance) $C_{x,y} = \kappa(x)P(x,y) = \kappa(y)P(y,x)$.

We claim that we can couple $\mathcal{X}_1 = \mathcal{W}^{G_0^* \to V_\lambda}$ and $\mathcal{X}_2 = \mathcal{W}^{G}$ where $\mathcal{W}^{G_0^* \to V_\lambda}$ is the projection of $\mathcal{W}^{G_0^*}$ onto V_λ. This walk is $\mathcal{W}^{G_0^*}$ with visits to V_σ omitted from the sequence of states. Indeed, we have designed G so that for each $v, w \in V_\lambda$

$$
\mathbb{P}(\mathcal{X}_1(t+1) = w \mid \mathcal{X}_1(t) = v) = \mathbb{P}(\mathcal{X}_2(t+1) = w \mid \mathcal{X}_2(t) = v).
$$

Remark 4.10. *The reader can now see why we defined V_σ in the way we did. If we had stopped with $\widehat{V}_\sigma$ then G_0 might contain isolated vertices.*

Coupling $\mathcal{W}^{G_0}, \mathcal{W}^{G}$ and $\mathcal{W}^{G_F}$:

We consider the vertices V_0 of G_0 to be a subset of the vertices of G_F. We couple $\mathcal{W}^{G_F}$ with a random walk $\mathcal{W}^{G_0}$ on G_0. In the walk $\mathcal{W}^{G_0}$ edges are selected with probability proportional to their weight/conductivity. We will now check that there is a natural coupling.

Suppose that $\mathcal{W}^{G_F}$ is at a vertex $v \in V_0$. Suppose that v has neighbours $w_1, w_2, \ldots, w_d$ in G_0 and that $f_i = (v, w_i)$ for $i = 1, 2, \ldots, d$. In G_F there will be corresponding paths P_i from v to w_i. Let $i^* \in [d]$ be the index of the path whose other endpoint is next reached by $\mathcal{W}^{G_F}$. Then if $\ell(P)$ is the length of a path P, we prove below that

$$
\mathbb{P}(i^* = i) = \frac{\ell(P_i)^{-1}}{\ell(P_1)^{-1} + \cdots + \ell(P_d)^{-1}} = \frac{\kappa_i}{\kappa_1 + \cdots + \kappa_d} \tag{4.60}
$$

where $\kappa_i = \kappa(f_i)$.

This can be proved by induction. Let $\ell_i = \ell(P_i)$, $i = 1, 2, \ldots, d$. Our induction is on $L = \ell_1 + \cdots + \ell_d$. The base case where $\ell_i = 1$ for $i = 1, 2, \ldots, d$ is trivial. Now suppose that $\ell_1 \geq 2$. Then if $\Pi = \mathbb{P}(i^* = 1)$,

$$
\Pi = \frac{(\ell_1 - 1)^{-1}}{(\ell_1 - 1)^{-1} + \ell_2^{-1} + \cdots + \ell_d^{-1}} \left(\frac{\ell_1 - 1}{\ell_1} + \frac{\Pi}{\ell_1} \right). \tag{4.61}
$$

Explanation: The factor $\frac{(\ell_1-1)^{-1}}{(\ell_1-1)^{-1}+\ell_2^{-1}+\cdots+\ell_d^{-1}}$ is, by induction, the probability that the walk reaches the penultimate vertex of P_1 and then $\frac{\ell_1-1}{\ell_1}$ is the probability that the walk reaches the end of P_1 before going back to v. The term $\frac{\Pi}{\ell_1}$ is then the probability that $i^* = 1$ in the case that the walk returns to v.

Equation (4.60) follows from (4.61) after a little algebra.

Note that (4.60) is the probability that $\mathcal{W}^{G_0}$ chooses to move to w_i from v. Thus we see that $\mathcal{W}^{G_F}$ and $\mathcal{W}^{G_0}$ can be coupled so that they go through the exact same sequence of vertices in V_0, although $\mathcal{W}^{G_0}$ moves faster.

The expected relative speed of these walks can be handled with the following lemma.

Lemma 4.11. *Suppose that T is a tree consisting of a root v and k paths $P_1, P_2, \ldots, P_k$ with common vertex v and no other common vertices. Path P_i has length ℓ_i for $i = 1, 2, \ldots, k$. A walk $\mathcal{W}$ starts at v.*

(a) The expected time Λ for $\mathcal{W}$ to reach a leaf is given by

$$\Lambda = \frac{\ell_1 + \cdots + \ell_k}{\sum_{i=1}^k \ell_i^{-1}}.$$

(b) If $\ell_i \leq \ell$ for $i = 1, 2, \ldots, k$ then $\Lambda \leq \ell^2$.

Proof. (a) Observe that

$$\mathbb{E}(\text{time to reach a leaf}) + \mathbb{E}(\text{time back to } v) = \frac{2(\ell_1 + \cdots + \ell_k)}{\sum_{i=1}^k \ell_i^{-1}}. \tag{4.62}$$

The RHS is twice the number of edges in T times the effective resistance between v and the set of leaves. (see e.g. [21], Proposition 10.6)

It follows from (4.60) and the fact that a simple random walk takes ℓ^2 steps in expectation to move ℓ steps in distance that

$$\mathbb{E}(\text{time back to } v) = \sum_{i=1}^k \frac{\ell_i^{-1}}{\sum_{i=1}^k \ell_i^{-1}} \times \ell_i^2.$$

Part (a) of the lemma follows.

(b) We simply observe that increasing ℓ_i increases the numerator and decreases the denominator.

This completes the proof. $\square$

We next observe that in this coupling, if $\mathcal{W}^{G_0}$ has covered all of the *edges* of G_0 then $\mathcal{W}^{G_F}$ has covered all of the edges of G_F, and so the edge cover time of G_0, suitably scaled, is an upper bound on the edge and hence vertex cover time of G_F.

It follows from Lemma 4.11(b) and the fact that all sub-paths have length at most $(1 + o(1))\ell^*$ that that if D_u is the expected time for the walk $\mathcal{W}_u$ on G_F to cover all the edges of G_F and D_v^* is the expected time for the walk $\mathcal{W}_v^{G_0}$ on G_0 to cover all the edges of G_0, then

$$T_{\text{COV}} = \max_u C_u \leq \max_u D_u \leq (1 + o(1))(\ell^*)^2 (\max_v D_v^* + 1). \tag{4.63}$$

(The $+1$ accounts for the case when u is in the middle of a sub-path).

In the same way, we can couple $\mathcal{W}^{G_0}$ and $\mathcal{W}^G$, up until the first visit to v_{f^*}, in the following sense. We can consider the latter walk to be the former, where we ignore visits to V_σ. By construction, if $v \in V_\lambda, w \in V_\lambda^*$ then for both walks we have that w has the same probability of being the next vertex in $V_\lambda^* = V_\lambda \cup \{v_{f^*}\}$ that is visited by the walk. We will show in Section 4.3.4 that the time spent in V_σ is negligible.

4.3 Conditions of Lemma 3.1 for G

Checking (3.4) for G:
We first claim that we have

$$T_{\text{MIX}}(G) = O(\omega^2 \ln^5 M). \tag{4.64}$$

Let $\tilde{G} = (V_\lambda, E_\lambda)$ be the subgraph of K_F induced by V_λ. We begin by estimating the conductance of $\tilde{G}$, as in (2.12). Let $\Pi_{\beta,s}, 0 \leq \beta \leq 1 \leq s \leq s_0 = \omega^{-1/3} N^{1+2\zeta_0}$ be the probability that there is a *connected* set $S \subseteq V_\lambda$ with $|S| = s$ and $e_K(S) = \beta \mathrm{d}(S)/2 \geq |S|$ and $e_K(S : V_\sigma) \geq (1-\beta)\mathrm{d}(S)/2$. (Here $e_K(S)$ is the number of G_λ (or K_F) edges contained in S and $e_K(S : V_\sigma)$ is the number of edges joining S and V_σ in K_F).

Lemma 4.12. *The following holds simultaneously and w.h.p. for every set $S \subseteq V_\lambda$ that induces a connected subgraph of $\tilde{G}$: In the following, $e_\lambda(S : \bar{S})$ is the number of G_λ edges joining S to $\bar{S} = V_\lambda \setminus S$. Note that*

(a) If (i) $|S| \leq s_0$ and (ii) $e(S) = \beta d(S)/2 \geq |S|$ then

$$e_\lambda(S : \bar{S}) \geq \frac{(1-\beta)d(S)}{2}.$$

(b) If $e(S) = |S| - 1$ then

$$e_\lambda(S : \bar{S}) \geq \frac{2d(S)}{3s_1},$$

where $s_1 = \frac{10000 \ln N}{\ln \omega}$.

Proof. (a) We estimate $\Pi_{\beta,s}$ from above by

$$\Pi_{\beta,s} \leq \sum_{|S|=s} \binom{\mathrm{d}(S)}{(1-\beta)\mathrm{d}(S)/2} \left(\frac{N^{1-C\zeta_0}}{M}\right)^{(1-\beta)\mathrm{d}(S)/2} \binom{\mathrm{d}(S)}{\beta\mathrm{d}(S)/2} \left(\frac{\mathrm{d}(S)}{M}\right)^{\beta\mathrm{d}(S)/2}. \tag{4.65}$$

where C can be any positive constant.

Explanation: We choose configuration points that will be paired with V_σ in $\binom{\mathrm{d}(S)}{(1-\beta)\mathrm{d}(S)/2}$ ways. The probability that all these points are paired in V_σ is at most

$$\left(\frac{\mathrm{d}(V_\sigma)}{2M - \mathrm{d}(S)}\right)^{(1-\beta)\mathrm{d}(S)/2} \leq \left(\frac{N^{1-C\zeta_0}}{2M - \mathrm{d}(S)}\right)^{(1-\beta)\mathrm{d}(S)/2},$$

see (4.43). We choose $\beta\mathrm{d}(S)/2$ configuration points for the edges inside S. The probability they are paired with other points associated with S can be bounded by $\left(\frac{\mathrm{d}(S)}{2M-o(M)}\right)^{\beta\mathrm{d}(S)/2}$.
Using (4.65) we see that

$$\Pi_{\beta,s} \leq_b \sum_{\delta} \sum_{\substack{|S|=s \\ \mathrm{d}(S)=\delta s}} \left(\frac{2e}{1-\beta}\right)^{(1-\beta)\delta s/2} \left(\frac{N^{1-C\zeta_0}}{M}\right)^{(1-\beta)\delta s/2} \left(\frac{2e}{\beta}\right)^{\beta\delta s/2} \left(\frac{\beta\delta s}{M}\right)^{\beta\delta s/2}$$

$$\leq \sum_{\delta} \sum_{\substack{|S|=s \\ \mathrm{d}(S)=\delta s}} \left(2(N^{-C\zeta_0})^{1-\beta} \left(\frac{2e\delta s}{N} \right)^{\beta} \right)^{\delta s/2}. \tag{4.66}$$

We first consider the case where $3 \leq \delta \leq A = N^{\zeta_0}$. Let $\theta_{\delta,s}$ be the proportion of sets of size s that have $\mathrm{d}(S) = \delta s$. In which case, (4.66) becomes

$$\begin{aligned} \Pi_{\beta,s} &\leq_b \sum_{\delta} \theta_{\delta,s} \binom{N}{s} \left(2eN^{-C(1-\beta)\zeta_0} \left(\frac{2eAs}{N} \right)^{\beta} \right)^{\delta s/2} \\ &\leq \sum_{\delta} \theta_{\delta,s} \left(2e^2 N^{-C(1-\beta)\zeta_0/2} \left(\frac{s}{N} \right)^{\beta/2-1/\delta} A^{\beta/2} \right)^{\delta s}. \end{aligned} \tag{4.67}$$

At this point we observe that by assumption, we have $\beta \mathrm{d}(S)/2 \geq |S|$ and so

$$\frac{\beta\delta}{2} \geq 1. \tag{4.68}$$

Now because $\delta \geq 3$ and $\sum_{\sigma} \theta_{\delta,s} = 1$, we have

$$\begin{aligned} \Pi_{\beta,s} &\leq_b \sum_{\delta} \theta_{\delta,s} \left(2e^2 A^{\beta/2} \left(\frac{s}{N} \right)^{1/24} \right)^{\delta s} \leq \left(\frac{s}{N} \right)^{s/16} \qquad \text{if } \beta \geq 3/4. \\ \Pi_{\beta,s} &\leq_b \sum_{\delta} \theta_{\delta,s} \left(2e^2 A^{\beta/2} N^{-C\zeta_0/4} \right)^{\delta s} \leq N^{-3C\zeta_0 s/8} \qquad \text{if } \beta \leq 3/4 \text{ and } C \geq 2. \end{aligned} \tag{4.69}$$

Now the number of choices for β can be bounded by $\mathrm{d}(S)$ and we bound this by $N^{\zeta_0} s$. This gives, for this case,

$$\sum_{\beta,s} \Pi_{\beta,s} \leq \sum_{s=1}^{s_0} N^{\zeta_0} s \left(\frac{s}{N} \right)^{s/16} + \sum_{s=1}^{s_0} N^{\zeta_0} s N^{-3C\zeta_0 s/8} = o(1),$$

if $C \geq 3$.

We now consider those S for which $\mathrm{d}(S) \geq A|S|$. Going back to (4.66) we see that for these we have

$$\begin{aligned} \Pi_{\beta,s} &\leq_b \sum_{\delta} \theta_{\delta,s} \binom{N}{s} \left(2eN^{-C(1-\beta)\zeta_0/2} \left(\frac{2eN^{\zeta_0} s}{N} \right)^{\beta} \right)^{As/2} \\ &\leq \sum_{\delta} \theta_{\delta,s} \left(4e^{1+2/A} N^{-C(1-\beta)\zeta_0} \left(\frac{s}{N} \right)^{\beta-2/A} \right)^{As/2} \end{aligned}$$

This yields

$$\begin{aligned} \Pi_{\beta,s} &\leq \left(\frac{s}{N} \right)^{As/5} \qquad \text{if } \beta \geq 1/2. \\ \Pi_{\beta,s} &\leq \left(4e^{1+o(1)} N^{-C\zeta_0/2} \right)^{As/2} \qquad \text{if } \beta \leq 1/2. \end{aligned} \tag{4.70}$$

and we can easily see from this that $\sum_{\beta,s} \Pi_{\beta,s} = o(1)$ in this case too, for $C \geq 3$. Thus w.h.p.

$$e(S : V_\lambda) = \mathrm{d}(S) - 2e(S) - e(S : V_\sigma) = \mathrm{d}(S) - \beta \mathrm{d}(S) - e(S : V_\sigma) \geq (1-\beta)\mathrm{d}(S)/2.$$

(b) Now consider sets with $e(S) = |S| - 1$ and use Lemma 4.5. If $|S| > s_1$ then either $\mathrm{d}(S) > s \ln N$ or $e(S : V_\sigma) \leq \lceil s/500 \rceil$. The former implies that

$$\frac{e(S : V_\lambda)}{\mathrm{d}(S)} \geq \frac{\mathrm{d}(S) - 2(|S| - 1) - |S|}{\mathrm{d}(S)} = 1 - o(1)$$

and the latter implies that

$$\frac{e(S : V_\lambda)}{\mathrm{d}(S)} \geq \frac{\mathrm{d}(S) - 2(|S| - 1) - \lceil |S|/500 \rceil}{\mathrm{d}(S)} > \frac{249}{250}.$$

If $|S| \leq s_1$ then and since $\mathrm{d}(S) \geq 3|S|$,

$$\frac{e(S : V_\lambda)}{\mathrm{d}(S)} \geq \frac{\mathrm{d}(S) - 2(|S| - 1) - |S|}{\mathrm{d}(S)} \geq \frac{2}{3|S|} \geq \frac{2}{3s_1}.$$

□

We verify next that if $S \subseteq V_0$ and $|S|$ is too close to N then $\kappa(S)$ will exceed $\kappa(G)/2$. Suppose then that $|S| \geq (1 - \eta)N$ where $2a_0\eta^{1/3} = \varepsilon_1$ of Lemma 4.6. It follows from Lemma 4.7 that $\mathrm{d}_{K_F}(V_K \setminus S) \leq 2a_0\eta^{1/3}N = \varepsilon_1 N$. It then follows from Lemma 4.6 that

$$\sum_{\substack{e \in E_K \\ e \cap S = \emptyset}} \ell_e \leq \frac{\varepsilon_1^{1/2} M}{\xi} \text{ and hence } \sum_{\substack{e \in E_K \\ e \cap S \neq \emptyset}} \ell_e \geq 2m - \frac{2\varepsilon_1^{1/2} M}{\xi} \geq (2 - 3\varepsilon_1^{1/2})m.$$

It follows from this and Lemma 4.8 that

$$\kappa(S) \geq \left(1 - \frac{3\varepsilon_1^{1/2}}{2}\right) \kappa(G_0). \tag{4.71}$$

It is shown in [1] that if $S \subseteq V_K$, then in K_F we have

$$e(S : V_K \setminus S) \geq \mathrm{d}(S)/50 \text{ for all sets } S \text{ with } \mathrm{d}(S) \leq M. \tag{4.72}$$

Now suppose that $S \subseteq V_0$ and $\kappa(S) \leq \kappa(G_0)/2$. It follows from (4.71) that $|S| \leq (1 - \eta)N$. This implies that $\mathrm{d}_{K_F}(S) \leq 2M - 3\eta N$.

If $\mathrm{d}_{K_F}(S) \leq M$ then (4.72) implies that $e(S : \bar{S}) \geq \mathrm{d}(S)/50$.

If $\mathrm{d}_{K_F}(S) > M$ then $3\eta N \leq \mathrm{d}_{K_F}(\bar{S}) \leq M$ and hence $e(S : \bar{S}) \geq 3\eta N/50 \geq (3\eta/50a_0)\mathrm{d}(S)$.

It follows that if $\kappa(S) \leq \kappa(G_0)/2$ then

$$e_{\tilde{G}}(S : V_\lambda) \geq \begin{cases} \frac{2d(S)}{3s_1} & |S| \leq s_0 \\ \frac{3\eta}{50a_0}\mathrm{d}(S) - \mathrm{d}(V_\sigma) \geq \frac{3\eta}{50a_0}\mathrm{d}(S) - \frac{6N^{1+\zeta_0}}{\omega^{1/3}} \geq \frac{2\eta}{50a_0}\mathrm{d}(S) & s_0 < |S| \leq (1 - \eta)N \end{cases} \tag{4.73}$$

Now every heavy edge of G_0 has weight at least $1/2$. Applying the argument for (2.13) we see that (4.73) implies that

$$\Phi(G_0) = \min_{\substack{S \subseteq V_0 \\ \kappa(S) \leq \frac{1}{2}\kappa(V_0)}} \Phi_{G_0}(S) = \Omega\left(\frac{1}{\ell_{\max}}\right) \times \min_{\substack{S \subseteq V_K \\ |S| \leq (1-\eta)N}} \frac{e_{\tilde{G}}(S : V_\lambda)}{\mathrm{d}(S)} = \Omega\left(\frac{1}{\omega \ln^2 M}\right).$$

Taking account of the special edges introduced to bypass most of the light edges can only increase the conductance of a set. This is because it won't affect the denominator in the definition of conductance, but it might increase the numerator.

All that is left is to consider the effect of splitting the edge f^* into a path of length two in order to define $G_0^* = G_0^*(f^*)$. The conductance of a connected set S not containing v_1 or v_2 is not affected by this change. If S contains v_1, v_2 then after the split, the numerator remains the same. On the other hand, the denominator can at most double. If S contains one of v_1, v_2 then the numerator still remains the same and again the denominator can at most double.

Thus $\Phi(G) = \Omega(\Phi(G_0))$. Equation (4.64) now follows from $T_{\text{MIX}}(G) = O(\Phi^{-2} \ln M)$.

We then have

$$T_{\text{MIX}}(G)\pi_G(v_{f^*}) = O\left(\frac{\omega^2 \ln^5 M}{\omega M}\right) = o(1). \tag{4.74}$$

Checking (3.5) for G:

Let $f^* = (v_1, v_2)$ as before. Suppose that v_1 is one of the vertices that are placed on a K_F edge $f = (w_1, w_2)$. We allow $v_1 = w_1$ here. We now remind the reader that w.h.p. all K_F-neighborhoods up to depth $2L_0$ contain at most one cycle, see Lemma 2.5(b). Let X be the set of kernel vertices that are within kernel distance L_0 of f in K_F. Let Λ_f be the sub-graph of G obtained as follows: Let H be the subgraph of the kernel induced by X. This definition includes f as an edge of H. If H contains no members of $V'_\sigma = V_\sigma \setminus \{v_1, v_2\}$ then we do nothing. Otherwise, let T be a component of the subgraph of H induced by V'_σ and let $L = \{v_0, v'_0, v_1, \ldots, v_s\} \subseteq N_1$ be the neighbours of T in V^*_λ where v_0, v'_0 are the vertices in L that are closest to $\{w_1, w_2\}$. Here $v_0 = v'_0$ is allowed and this is indeed occurs in the majority of cases w.h.p. Note also that by the construction of V_σ, each $v_i, i \geq 1$ has one neighbour in T. We replace T by special edges $(v_0, v_i), (v'_0, v_i), (v_i, v_0), (v'_i, v_0), i = 1, 2, \ldots, s$. If T contains a vertex w that is at distance L_0 from $\{w_1, w_2\}$ then we remove T completely.

Next add vertices of degree two to the non-special edges of H as in the construction of the 2-core. We obtain Λ_f by contracting paths as in the construction of G_0. Vertices of X that are at maximum kernel distance from f in K_F are said to be at the frontier of Λ_f. Denote these vertices by Φ_f.

We now follow the argument in Section 4.1.1 between "Let us make Φ_f into..." and Lemma 4.1, the proof of which requires some minor tinkering:

Lemma 4.13. *Fix $w \in \Phi_f$. Then*

$$\mathbb{P}(\mathcal{W}_w^{G_0^*} \textit{ visits } f \textit{ within time } T_{\text{MIX}}) = O(N^{-\delta_0/2}) = o(1).$$

Proof. Let P be one of the at most two paths P, P' from w to w_1 in K_F; then $P = P'$ whenever w_1 is locally tree like. Let $e_1, e_2, \ldots, e_{L_0}$ be the edges of P. Assume first that neither of these paths contain a member of V_σ. We will correct for this later. In this case we can follow the argument of Lemma 4.1 until the end.

Suppose now that the paths contain members of V_σ. It is still true that there are only one or two paths from boundary vertex w to w_1 or w_2. The only change needed for the analysis is to note that after contracting special edges these K_F paths can shrink in length to $9L_0/10$. Here we use Lemma 4.4. This changes 2^{L_0-2} in (4.11) to $2^{9L_0/10-2}$ and allows the proof to go through. □

The remainder of the verification follows as in Section 4.1.1.

4.3.1 Analysis of a random walk on G

This is similar to the analysis of Section 4.1.2 and may seem a bit repetitive. We will first argue that

$$\text{the edge cover-time of } G \text{ is w.h.p. at most } \frac{\omega^2 M \ln^2 M}{8+o(1)}. \tag{4.75}$$

After this we have to deal with the time spent crossing edges with at least one endpoint in V_σ. This will be done in Section 4.3.4.

We have a fixed vertex $u \in V_\lambda$ and an edge f^* and we will estimate an upper bound for $\mathbb{P}(\mathcal{A}_t(v_{f^*}))$ using Lemma 3.1. For this we need a good upper bound on $R_{v_{f^*}}$. Let $f = (w_1, w_2)$ be the edge of K_F containing f^*. Recall the definition of Λ_f in Section 4.3 where we were checking (3.5). If f is locally tree like let T_1, T_2 be the trees in G_0 rooted at w_1, w_2 obtained by deleting the edges of Λ_f that are derived from the edge f of K_F. If f is not locally tree like then we can remove an edge of the unique cycle C in Λ_f not incident with v_{f^*} from Λ_f and obtain trees T_1, T_2 in this way. Removing such an edge can only increase resistance and R_f.

We write $R_{v_{f^*}} = R'_{v_{f^*}} + R''_{v_{f^*}}$ where $R'_{v_{f^*}}$ is the expected number of returns to v_{f^*} within time T_{MIX} before the first visit to Φ_f and $R''_{v_{f^*}}$ is the expected number of visits after the first such visit.

$$R'_{v_{f^*}} = 2\alpha R_P \tag{4.76}$$

where R_P is the effective resistance as defined in Section 4.1.2, but associated to the weighted network $\mathcal{N}$. Here α is the weight of the edge f that we split.

We first assume that Λ_f contains no vertices in V_σ and then in the final paragraph of Section 4.3.2 we show what adjustments are needed for this case.

We will show in Section 4.3.3 that

$$R''_{v_{f^*}} = o(R'_{v_{f^*}}). \tag{4.77}$$

We first prune away edges of the trees T_1, T_2 tree-like neighbourhoods to make the branching factor of the associated trees at most two. Of course, in tree like neighborhoods we can say exactly two. This only increases the effective resistance and $R_{v_{f^*}}$. Let R_1, R_2 be the resistances of the pruned trees and let $R = R_1 + R_2$.

We have

$$\frac{1}{R_P} = \frac{1}{\alpha^{-1} + \ell_1/\ell^* + R_1} + \frac{1}{\alpha^{-1} + \ell_2/\ell^* + R_2}. \tag{4.78}$$

Here ℓ_i/ℓ^* is the total resistance of the G edges in the path from v_i to w_i derived from f. If v_1 is a vertex of K_F then we can dispense with ℓ_2, R_2.

Note that, with $\ell = \ell_1 + \ell_2$,

$$\frac{1}{\alpha^{-1} + \ell_1/\ell^* + R_1} + \frac{1}{\alpha^{-1} + \ell_2/\ell^* + R_2} \geq \frac{4}{4 + \ell/\ell^* + R} \tag{4.79}$$

(which follows from $\alpha \geq 1/2$ and the arithmetic-harmonic mean inequality).

Let $\mathcal{E}_{\max}$ be as defined before (4.19) and note that given $\mathcal{E}_{\max}$ we have $\varepsilon = O\left(\frac{3^{2L_0} \ln M}{\xi(M+\nu_2)}\right) = o(1)$, where ε is defined in Part (a) of Lemma 2.4. We re-write (4.19) as

$$\mathbb{P}(R_1 \geq \rho_1, R_2 \geq \rho_2, L = (\ell_1+\ell_2)/\ell^* = \ell/\ell^*) \leq (1+\varepsilon)\widehat{\mathbb{P}}(R_1 \geq \rho_1)\widehat{\mathbb{P}}(R_2 \geq \rho_2)\widehat{\mathbb{P}}(\ell_1+\ell_2 = l). \tag{4.80}$$

Note next, that with $\ell=\ell_1+\ell_2$, and given α and that $\xi=o(1)$,

$$\widehat{\mathbb{P}}(L=(\ell_1+\ell_2)/\ell^*=\ell/\ell^*\mid\mathcal{E}_{\max})\leq\xi(1-\xi)^{\ell-1}\leq_b\xi e^{-L/\omega}.$$

We will show in Section 4.3.2 that for $\rho=M^{o(1)}$ we have

$$\widehat{\mathbb{P}}(R_1\geq\rho\mid\mathcal{E}_{\max})\leq_b 3^{L_0}e^{-\rho/\omega} \tag{4.81}$$

This is a simpler expression than (4.20) because here we have $\xi=o(1)$.

Let Z_{L,ρ_1,ρ_2} be the random variable that is equal to the number of vertices of G_0 with parameters L,ρ_1,ρ_2. Then we have

$$\mathbb{E}(Z_{L,\rho_1,\rho_2})\leq_b\omega M\times L\ell^*\times\xi e^{-L/\omega}\times 3^{L_0}e^{-R/\omega}=3^{L_0}\omega MLe^{-(L+\rho)/\omega}, \tag{4.82}$$

where $\rho=\rho_1+\rho_2$. (The factor $\ell_e=L\ell^*$ comes form the number of choices of edge to split in path P_e).

Using Lemma 3.1 and (4.79) we see that

$$\begin{aligned}\mathbb{E}(\Psi(E(G_0),t))\leq_b 3^{L_0}\omega\xi M\sum_{s\geq t}\ell^*\int_L dL\int_{\rho_1,\rho_2}d_{\rho_1}d_{\rho_2}Le^{-(L+\rho)/\omega}\times\\ \left(\exp\left\{-(1+o(1))\frac{s}{2\omega M}\cdot\frac{4}{4+L+\rho}\right\}+O(T_{\text{MIX}}^2\pi_{\max}e^{-\lambda t/2})\right).\end{aligned} \tag{4.83}$$

where $\pi_{\max}=\max\{\pi_v:v\in V\}$.

Some explanation: The first line is direct from (4.82). Then $\frac{2\alpha}{2\omega M}$ is asymptotic to the steady state for v_{f^*} and there is a $\frac{1}{2\alpha}$ factor from (4.76). So $\frac{\pi_{v_{f^*}}}{R_{v_{f^*}}}$ is asymptotic to $\frac{2\alpha}{2\omega M}\cdot\frac{1}{2\alpha}\cdot\frac{4}{4+L+\rho}=\frac{1}{2\omega M}\cdot\frac{4}{4+L+\rho}$. This is to be compared with the expression in (4.4). Here we are summing our estimate for $\mathbb{P}(\mathcal{A}_s(f))$ over edges f of weight α. Recall that $\mathcal{A}_s(f)$ is the event that we have not crossed edge f in the time interval $[T_{\text{MIX}},s]$.

Notice that the sum over $v\in V$ can be taken care of by the fact that we weight the contributions involving v by π_v. Remember that here v represents the vertex reached by $\mathcal{W}^{G_0}$ at time T_{MIX}.

Ignoring a negligible term we have

$$\begin{aligned}&\mathbb{E}(\Psi(E(G_0)),t))\\ &\leq_b 3^{L_0}\omega\xi M\sum_{s\geq t}\ell^*\int_L dL\int_{\rho_1,\rho_2}d_{\rho_1}d_{\rho_2}L\exp\left\{-(1+o(1))\left(\frac{L+\rho}{\omega}+\frac{2s}{\omega M(4+L+\rho)}\right)\right\}\\ &\leq_b 3^{L_0}\omega\xi M\ell^*\int_L dL\int_{\rho_1,\rho_2}d_{\rho_1}d_{\rho_2}L\frac{\exp\left\{-(1+o(1))\left(\frac{L+\rho}{\omega}+\frac{2t}{\omega M(L+\rho)}\right)\right\}}{1-\exp\left\{-\frac{2+o(1)}{\omega M(L+\rho)}\right\}}.\end{aligned} \tag{4.84}$$

Note now that in the current case, $\xi=o(1)$ and so our estimate for T_{COV} is $\sim C\omega^2M\ln^2M$ where $C\geq 1/8$. So, the contribution from ℓ,ρ such that $L+\rho\leq\omega\ln M/100$ is negligible. As are the contributions from $L+\rho\geq 5\omega\ln M$.

Ignoring negligible values we obtain a bound by further replacing the denominator in (4.84) by $\Omega(1/\omega^2 M \ln M)$. Thus,

$$\mathbb{E}(\Psi(E(G_0),t)) \leq_b 3^{L_0}\omega^3\xi\ell^* M^2 \ln M \int_{L\leq 5\omega\ln M}\int_{\rho\leq 5\omega^2\ln M} L\exp\left\{-\frac{L+\rho}{\omega}-\frac{2t}{\omega M(L+\rho)}\right\}$$
$$\leq_b 3^{L_0}\omega^4 M^2\ln M\times(\omega\ln M)^3\exp\left\{-\sqrt{\frac{8t}{\omega^2 M}}\right\}. \tag{4.85}$$

Putting $t\sim\frac{1}{8}\omega^2 M\ln^2 M$ we claim that the RHS of (4.85) is $o(t)$. Indeed, to see this note that $3^{L_0}\omega^4 M^2\ln M\times(\omega\ln M)^3 = M^{2+\eta}$ for some $\eta=o(1)$, where $M^\eta\to\infty$. Therefore, if we take $t=\frac{1+3\eta}{8}\omega^2 M\ln^2 M$ then the RHS of (4.85) is $\leq_b M^{2+\eta}\times M^{-(1+3\eta)^{1/2}}=o(M)$.

We now consider the contribution of $O(T_{\text{MIX}}^2\pi_{\max}e^{-\lambda T_{\text{cov}}/2})$ to $\mathbb{E}(\Psi(E(G_0),t))$. We bound this by

$$\leq_b (\omega^2\ln^5 M)^2\times\frac{1}{\omega M}\times\exp\left\{-\Omega\left(\frac{\omega^2 M\ln^2 M}{\omega^2\ln^5 M}\right)\right\}=o(1).$$

Summarising, if

$$t\geq\frac{1+o(1)}{8}\omega^2 M\ln^2 M \tag{4.86}$$

then

$$\mathbb{E}(\Psi(E(G_0),t))=o(t)$$

and then the Markov inequality implies that w.h.p.

$$\Psi(E(G_0),t)=o(t).$$

4.3.2 Estimating R_P

We first assume that Λ_f contains no vertices from V_σ.

We follow the argument in Section 4.1.3 down to (4.30), (4.31) which we replace by

$$\widehat{\mathbb{P}}(\ell_1/\ell^*=\rho)=\xi(1-\xi)^{\rho\ell^*-1} \tag{4.87}$$

and

$$\widehat{\mathbb{P}}(\ell_1/\ell^*\geq\rho)=(1-\xi)^{\rho\ell^*}. \tag{4.88}$$

Let the level of a tree like T_1 be the depth of the tree in K_F from which it is derived. Let R_k be the (random) resistance of a tree of level k. Putting $R_0=0$ we get from (4.29), (4.87) and (4.88) that

$$\widehat{\mathbb{P}}(R_1\geq\rho)\leq 2(1-\xi)^{3\rho\ell^*}. \tag{4.89}$$

Assume next that for $a_k=(2.5)^k$, $k=o(\ln M)$ and for integer $1\leq\rho\leq M^{o(1)}$,

$$\widehat{\mathbb{P}}(R_k\geq\rho)\leq a_k(1-\xi)^{2\rho\ell^*} \tag{4.90}$$

for $t\geq 1$. This is true for $k=1$ and $a_1=2+o(1)$. Using (4.29) and arguing as in Section 4.1.3 we get

$$\widehat{\mathbb{P}}(R_{k+1}\geq\rho)\leq 2\left(\sum_{s=1}^{2\rho\ell^*-1}\widehat{\mathbb{P}}(\ell_1=s)\widehat{\mathbb{P}}(R_k\geq 2\rho-s)+\widehat{\mathbb{P}}(\ell_1\geq 2\rho\ell^*)\right) \tag{4.91}$$

$$
\begin{aligned}
&\leq 2\left(\sum_{s=1}^{2\rho\ell^*-1}\xi(1-\xi)^{s\ell^*-1}\times a_k(1-\xi)^{2(2\rho-s)\ell^*}+(1-\xi)^{2\rho\ell^*}\right)\\
&=2\left((1+o(1))a_k\xi(1-\xi)^{4\rho\ell^*}\sum_{s=1}^{2\rho\ell^*-1}(1-\xi)^{-s\ell^*}+(1-\xi)^{2\rho\ell^*}\right) \qquad (4.92)\\
&\leq (2+o(1))(a_k+1)(1-\xi)^{2\rho\ell^*}.\\
&\leq a_{k+1}(1-\xi)^{2\rho\ell^*}.
\end{aligned}
$$

This verifies the inductive step for (4.90) and (4.81) follows. Remember that $(1-\xi)^{2\rho\ell^*}\leq e^{-2\rho\ell^*\xi}=e^{-2\rho/\omega}$.

For the non locally tree like case we now argue as in Section 4.1.3 down to (4.36) and obtain

$$
\begin{aligned}
\widehat{\mathbb{P}}(R\geq\rho) &\leq 2\left(\sum_{s=1}^{2\rho\ell^*-1}(1-\xi)^{s\ell^*/2}(2.5)^k(1-\xi)^{2(2\rho-s)\ell^*}+(1-\xi)^{2\rho\ell^*}\right)\\
&\leq 2\left((2.5)^k(1-\xi)^{\rho\ell^*}(\xi\ell^*)^{-1}+(1-\xi)^{2\rho\ell^*}\right)\\
&\leq_b \omega(2.5)^{k+1}(1-\xi)^{\rho\ell^*}.
\end{aligned}
$$

There is enough slack in (4.81) to absorb the ω factor when $k=L_0$.

Now suppose that Λ_f contains vertices from V_σ. When we encounter a component T of $V_\sigma\cap\Lambda_f$ we replace it $\mathcal{N}$ by edges (v_0,v_i) (or (v_0',v_i)) and these edges will have been given the same resistance distribution as other edges of Λ_f, conditioned on being heavy. This happens with probability $1-o(1)$ and the net result is to replace the factor 2 in (4.91) by $2+o(1)$. This will not significantly affect the rest of the calculation here.

4.3.3 Estimating $R''_{v_{f^*}}$

It follows from Lemma 4.13 that

$$
R''_{v_{f^*}}\leq n^{-\delta_0/6}(R'_{v_{f^*}}+R''_{v_{f^*}})
$$

and hence

$$
R''_{v_{f^*}}\leq n^{-\delta_0/7}R'_{v_{f^*}}. \qquad (4.93)
$$

4.3.4 Completing the proof of upper bound in Case (c) of Theorem 1

We are almost ready to apply (4.63). We have estimated the cover time, but we have ignored some of the time. Specifically, let

$$
E_1=\bigcup_{\substack{e\in E_K\\ e\cap V_\sigma\neq\emptyset}}P_e.
$$

We have not accounted for the time that $\mathcal{W}^{G_F}$ spends covering E_1.

For this we can apply a theorem of Gillman [19]: Let $G=(V,E)$ be an edge weighted graph and for $x\in V$ let $N_{\mathbf{q}}=\left|\left|\frac{\mathbf{q}}{\sqrt{\pi}}\right|\right|_2$ where $\pi(x),x\in V$ is the steady state distribution for the associated random walk and $q(x), x \in V$ is any initial distribution for the starting point of the walk. Let θ denote the spectral gap for the associated probability transition matrix.

Theorem 4.14. *Let $A \subseteq V$ and let Z_t be the number of visits to A in t steps. Then, for any $\gamma \geq 0$,*

$$\mathbb{P}(Z_t - t\pi(A) \geq \gamma) \leq (1 + \gamma\theta/10t) N_{\mathbf{q}} e^{-\gamma^2\theta/20t}.$$

We apply this theorem to the random walk $\mathcal{W}^{G_F}$. Let $A = E_1$ and $\gamma = M/\xi^2$. It follows from Lemmas 2.4 (Part (c)) and 4.3 that w.h.p.

$$\pi(A) = O\left(\frac{\omega^{-1/3} M \times \xi^{-1} \ln M}{M + \nu_2}\right) = O(\omega^{-1/3} \ln M).$$

It follows from Lemma 2.7 that $\theta = \Omega(\xi^2 / \ln^2 M)$. Now let $t = M \ln^2 M/\xi^2$. Then with $\mathbf{q}$ of the form $(0, 0, \ldots, 1, 0, \ldots, 0)$ we have

$$\mathbb{P}(Z_t \geq t\pi(A) + \gamma) = O(m^{1/2} e^{-\Omega(M/\ln^4 M)}) = o(1).$$

This completes the proof of Case (c2).

4.4 Case (c3): $\nu_2 \geq e^{\omega}$

In this case we can use the fact that w.h.p. $\ell_e \in [\ell_{\min}, \ell_{\max}]$ for $e \in E_K$ to (i) partition all induced paths of G_F into sub-paths of length $\sim \mu = me^{-\omega/2}$, (ii) replace these sub-paths by edges to create a graph Γ and then (iii) apply the Case (c) reasoning to Γ and then scale up by μ^2 to get the claimed upper bound.

The proof of the upper bound for Case (c) of Theorem 1 is now complete.

4.5 Case (b): $\nu_2 = M^{\alpha}$, $0 < \alpha < 1$

Our argument for this case will not be so detailed as for the previous cases. It is closer in spirit to that of the previous papers of the first two authors.

Note that in this case

$$1 - \xi \leq \frac{1}{M^{1-\alpha}}.$$

So,

Lemma 4.15. *Let $\theta > 0$ be an arbitrarily small positive constant. Then w.h.p. $\ell_e \leq \ell_\alpha = \lceil 1/(1-\alpha) + 1 + \theta \rceil$ for $e \in E(K_F)$.*

Proof. Going back to (2.6) we have

$$\mathbb{P}(\exists e : \ell_e \geq \ell_\alpha) \leq M \sum_{s \geq \ell_\alpha} M^{-(1-\alpha)(s-1)} \left(1 + \frac{3}{M + M^{\alpha}}\right)^{s-1} = o(1). \qquad \square$$

The next thing to observe in this case that there will be very few vertices of degree two close to any vertex of K_F. Suppose that $d_n = \Delta$. We choose $\delta_0 \leq 1/100$ such that $\Delta^{L_0} \leq M^{(1-\alpha)/2}$. Let $E_{v,s}$ be the set of edges of K_F that are within distance s of vertex $v \in V(K_F)$.

Lemma 4.16. *W.h.p., for all $v \in V(K_F)$,*

$$\sum_{e \in E_{v,L_0}} \ell_e \leq |E_{v,L_0}| + 2\ell_\alpha.$$

Proof. Let $h_v = |E_{v,L_0}| \leq 2M^{(1-\alpha)/2}$. Then we have

$$\begin{aligned}
\mathbb{P}\left((\sum_{e \in E_{v,L_0}} \ell_e \geq h_v + 2\ell_\alpha \right) &\leq o(1) + \sum_{v \in V(K_F)} \sum_{s \geq h_v + 2\ell_\alpha} \sum_{\substack{z_e, e \in E_{v,L_0} \\ \sum_e z_e = s}} M^{-(1-\alpha)(s-h_v)} \left(1 + \frac{3}{M + M^\alpha}\right)^{s-h_v} \\
&\leq o(1) + M \sum_{s \geq h_v + 2\ell_\alpha} \binom{s-1}{h_v - 1} M^{-(s-h_v)(1-\alpha+o(1))} \\
&\leq_b o(1) + M \sum_{s \geq h_v + 2\ell_\alpha} \left(\frac{se}{s - h_v} \cdot \frac{1}{M^{1-\alpha+o(1)}}\right)^{s-h_v} \\
&\leq o(1) + M \sum_{s \geq h_v + 2\ell_\alpha} \left(\frac{eh_v}{2\ell_\alpha} \cdot \frac{1}{M^{1-\alpha+o(1)}}\right)^{s-h_v} \\
&\leq o(1) + M \sum_{s \geq h_v + 2\ell_\alpha} M^{-(s-h_v)(1-\alpha+o(1))/2} \\
&= o(1).
\end{aligned}$$

□

It is not difficult to show that the conditions of Lemma 3.1 hold w.h.p. and so it is a matter of estimating the R_v's. This involves estimating the effective resistances R'_v so that we can use (4.16). The inequalities

$$\begin{aligned}
1 + \frac{1}{\frac{1}{R-1} + \frac{1}{S}} &\geq \frac{1}{\frac{1}{R} + \frac{1}{S}} \\
\frac{1}{R+1} + \frac{1}{S-1} &\leq \frac{1}{R} + \frac{1}{S} \text{ for positive integers } R < S
\end{aligned}$$

imply the following:

(i) If $v \in V_K$ and if we assume $k = O(1)$ vertices of degree two within distance L_0 of v then we get the maximum effective resistance in (4.16) by distributing these degree two vertices equitably on the edges incident with v.

(ii) If $\mathrm{d}(v) = 2$ then we get the maximum resistance when v is in the middle of the path P_e that it lies.

There are now three cases to consider:

(1) If $k = 0$ and v is locally tree like, then the resistance satisfies

$$R'_v \leq \rho_d = \frac{d-1}{d(d-2)}, \tag{4.94}$$

where d is the minimum degree in K_F. The value $\frac{d-1}{d(d-2)}$ is the resistance $R_{d,\infty}$ of an infinite d-regular tree T_∞. Trimming the tree at depth L_0 explains the inequality. We obtain the resistance of T_∞ by first computing the resistance ρ of an infinite tree with branching factor $d-1$. This satisfies the recurrence $\frac{1}{\rho}=\frac{d-1}{1+\rho}$ giving $\rho=\frac{1}{d-2}$. The resistance $R_{d,\infty}$ then satisfies $\frac{1}{R_{d,\infty}}=\frac{d}{1+\rho}$, giving $R_{d,\infty}=(1+\rho)/d$.

If on the other hand, $k=pd+q$ where $0\le q<d$ then

$$\begin{aligned}\frac{1}{R'_v} &\ge \left(\frac{d-q}{p+\frac{1}{d-2}}+\frac{q}{p+1+\frac{1}{d-2}}\right)\\ &= \frac{d}{p+1+\frac{1}{d-2}}+\frac{d-q}{\left(p+\frac{1}{d-2}\right)\left(p+1+\frac{1}{d-2}\right)}\\ &= \frac{d}{\frac{k}{d}+\frac{1}{d-2}}+\frac{d-q}{\left(p+\frac{1}{d-2}\right)\left(p+1+\frac{1}{d-2}\right)}-\frac{d-q}{\left(\frac{k}{d}+\frac{1}{d-2}\right)\left(p+1+\frac{1}{d-2}\right)}\\ &\ge \frac{d}{\frac{k}{d}+\frac{1}{d-2}}.\end{aligned}$$

The case (4.94) is equivalent to $p=q=0$.

Next observe that the number of vertices with this value of k is $O(M^{1-(1-\alpha)k})$ w.h.p. Thus the main contribution from these vertices to $\Psi(V,t)$ can be bounded by

$$\le_b \sum_{s\ge t} M\exp\left\{-(1+o(1))\frac{d}{2M}\cdot\frac{s}{d\rho}\right\}+ \sum_{s\ge t}\sum_{k\ge 1} M^{1-(1-\alpha)k}\exp\left\{-(1+o(1))\frac{s}{2M}\cdot\frac{d}{\frac{k}{d}+\frac{1}{d-2}}\right\} \tag{4.95}$$

(2) If $v\in P_e$, e is locally tree like and v is the middle of $k\ge 1$ vertices of degree two, then

$$\frac{1}{R'_v}\ge\left(\frac{1}{\lfloor (k+1)/2\rfloor+\frac{1}{d-2}}+\frac{1}{\lceil (k+1)/2\rceil+\frac{1}{d-2}}\right). \tag{4.96}$$

Observe that once again the number of vertices with this value k is $O(M^{1-(1-\alpha)k})$ w.h.p. Thus the main contribution from these vertices to $\Psi(V,t)$ can be bounded by

$$\le_b \sum_{s\ge t}\sum_{k\ge 1} M^{1-(1-\alpha)k}\exp\left\{-(1+o(1))\frac{s}{2M}\left(\frac{1}{\lfloor (k+1)/2\rfloor+\frac{1}{d-2}}+\frac{1}{\lceil (k+1)/2\rceil+\frac{1}{d-2}}\right)\right\} \tag{4.97}$$

Comparing (4.95) and (4.97) we see that the latter dominates, except possibly for the first term corresponding to (4.94). As in [1], this first term forces $T_{\text{COV}}\ge(1+o(1))\frac{2\rho_d}{d}M\ln M$. The other terms in (4.95) force

$$\min_k\left\{(1-\alpha)k\ln M+\frac{T_{\text{COV}}}{2M}\left(\frac{1}{\lfloor (k+1)/2\rfloor+\frac{1}{d-2}}+\frac{1}{\lceil (k+1)/2\rceil+\frac{1}{d-2}}\right)\right\}\ge(1+o(1))\ln M.$$

(3) Non locally tree like edges and vertices: This follows from two easily proven facts: (i) There are $M^{o(1)}$ such vertices and edges, (ii) the resistance R'_v in all such cases is $O(1/(1-\alpha))$. This means that all such vertices will w.h.p. have been visited after $o(M \ln M)$ steps.

This completes the upper bound for Case (b) of Theorem 1.

4.6 Case (a): $\nu_2 = M^{o(1)}$

This is essentially treated in [1]. W.h.p. every K_F neighbourhood up to depth L_0 attracts at most one vertex of degree two when edges are split. Furthermore all but an $M^{-(1-o(1)}$ fraction are free of vertices of degree two. It is easy therefore to amend the proof in [1] to handle this.

5 Lower Bounds

5.1 Case (a): $\nu_2 = M^{o(1)}$

This is essentially treated in [1].

5.2 Case (b): $\nu_2 = M^{\alpha}$, $0 < \alpha < 1$

This can be treated via the second moment method as described in [7]. We give a bare outline of the approach. Let

$$\psi_{\alpha,d} = \max\left\{\frac{2(d-1)}{d(d-2)}, \phi_{a,d}\right\},$$

set $t = (1 - o(1))\psi_{\alpha,d} M \ln M$ and suppose for example that $\psi_{\alpha,d} = \frac{2(d-1)}{d(d-2)}$. This is true for α small and d large. We then let S denote the set of vertices that (i) are locally tree like, (ii) have no degree two vertices added to their L_0-neighbourhood and (iii) have only degree d vertices in their L_0-neighbourhood. We find that $|S| = \Omega(n^{1-o(1)})$ w.h.p. and we greedily choose a sub-set S_1 of S so that (i) if $v, w \in S_1$ then $\text{dist}(v, w) > 2L_0$ and (ii) $|S_1| = n^{1-o(1)}$. Let S^* denote the set of vertices in S_1 that remain unvisited at time t. We choose the $o(1)$ term in the definition of t so that $\mathbb{E}(|S^*|) \to \infty$. We will then argue that if $v, w \in S_1$ then

$$\mathbb{P}(\mathcal{A}_t(v) \cap \mathcal{A}_t(w)) \sim \mathbb{P}(\mathcal{A}_t(v))\mathbb{P}(\mathcal{A}_t(w)). \tag{5.1}$$

This means, via the Chebyshev inequality, that w.h.p. $S^* \neq \emptyset$, giving the lower bound. To prove (5.1) we consider a new graph G' where we identify v, w to make a vertex Υ of degree $2d$. We then apply Lemma 3.1 to G' to estimate $\mathbb{P}(\mathcal{A}_t(\Upsilon))$. Observe that up until the walk visits Υ in G', its moved can be coupled with moves in G. Also, v has steady state probability approximately equal to that of v, w combined, but $R_\Upsilon \sim R_v \sim R_w$ and (5.1) follows.

5.3 Case (c): $\nu_2 = \Omega(M^{1-o(1)})$

We use the following result of Matthews [23]. For any graph G

$$T_{\text{cov}}(G) \geq \frac{1}{2} \max_{S \subset V} K_S \ln |S|,$$

where

$$K_S = \min_{u,v \in S} K(u,v).$$

Here $K(u,v)$ is commute time between u and v, i.e., the expected time for a walk $\mathcal{W}$ that starts at u to visit v and then return to u. This in turn is given by

$$K(u,v) = 2|E(G)|R_{\text{eff}}(u,v),$$

$E(G)$ is edges of G, and $R_{\text{eff}}(u,v)$ is effective resistance between u and v.

It is now simply a matter of finding a suitable set S.

Fix an integer ℓ and consider

$$S_\ell = \{u : \exists e \in K_F \text{ such that } u \text{ is the middle vertex of } P_e \text{ and } \ell_e \geq \ell\}.$$

Now

$$R_{\text{eff}}(u,v) \geq \ell/2 \text{ for } u,v \in S_\ell.$$

To see this, let P_e, P_f be two paths of length (at least) ℓ and let a,b,c,d be their respective endpoints. Let u,v be the midpoints of P_e, P_f. Let V_λ be the set of vertices not on P_e or P_f. Contract the set $V_\lambda \cup \{a,b,c,d\}$ to a single vertex z. This does not increase the effective resistance between u and v. What results is a graph consisting of two cycles intersecting at z. The effective resistance between u and v is now at least $\ell/4 + \ell/4 = \ell/2$. Here $\ell/4$ is a lower bound on the resistance between u and z etc.

Now $m \geq \nu_2$ and we will choose our ℓ to be $\ell_0 = \frac{\ln M}{-2\ln(1-\xi)}$. It follows from Lemma 2.4 (Part (b)) with $k=1$ that $\mathbb{E}(|S_{\ell_0}|) \sim M(1-\xi)^{\ell_0}$. Lemma 2.4 (Part (b)) with $k=2$ allows us to use the Chebyshev inequality to show that $|S_{\ell_0}| \sim M(1-\xi)^{\ell_0}$ w.h.p. (Here we take $\zeta \leq 2\ell_{\max}$ so that $\frac{\zeta}{\nu_2+M} = O\left(\frac{\ln^2 M}{M}\right) = o(1)$.) Note that $M(1-\xi)^{\ell_0} = M^{1/2} \to \infty$.

Putting this altogether we see that w.h.p.

$$T_{\text{COV}}(G_F) \geq (1-o(1))\nu_2 \times \frac{\ln M}{-4\ln(1-\xi)} \times \frac{\ln M}{2}. \tag{5.2}$$

Since $-\ln(1-\xi) \sim \xi$ for small ξ, this also includes Case (c). This completes the proof of Case (c) of Theorem 1.

Remark 5.1. *Our assumption, $-\ln(1-\xi) = o(\ln M)$ implies that we can ignore the fact that ℓ_0 is an integer. That is, by defining ℓ_0 without $\lceil\cdot\rceil$ we can include the error in the $(1-o(1))$ factor.*

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