GLAUBER DYNAMICS FOR THE MEAN-FIELD POTTS MODEL

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Abstract. We study Glauber dynamics for the mean-field (Curie-Weiss) Potts model with $q \geqslant 3$ states and show that it undergoes a critical slowdown at an inverse-temperature $\beta_s(q)$ strictly lower than the critical $\beta_c(q)$ for uniqueness of the thermodynamic limit. The dynamical critical $\beta_s(q)$ is the spinodal point marking the onset of metastability.

We prove that when $\beta < \beta_s(q)$ the mixing time is asymptotically $C(\beta, q)n \log n$ and the dynamics exhibits the cutoff phenomena, a sharp transition in mixing, with a window of order n. At $\beta = \beta_s(q)$ the dynamics no longer exhibits cutoff and its mixing obeys a power-law of order $n^{4/3}$. For $\beta > \beta_s(q)$ the mixing time is exponentially large in n. Furthermore, as $\beta \uparrow \beta_s$ with n, the mixing time interpolates smoothly from subcritical to critical behavior, with the latter reached at a scaling window of $O(n^{-2/3})$ around β_s. These results form the first complete analysis of mixing around the critical dynamical temperature — including the critical power law — for a model with a first order phase transition.

1. Introduction and Results

We study the dynamics of the Potts model on the complete graph (mean-field) known as the *Curie-Weiss Potts* model. For $n \geqslant 1$, $\beta \geqslant 0$, the Curie-Weiss Potts distribution is a probability measure on $\Sigma_n = Q^V$ where $Q = \{1, \dots, q\}$ and $V = \{1, \dots, n\}$, defined by

$$\mu_n(\sigma) = Z_{\beta,n}^{-1} \exp\Big\{(\beta/n) \sum_{u,v \in V} \mathbb{1}_{\sigma(u)=\sigma(v)}\Big\},$$

where $\sigma \in \Sigma_n$ and $Z_{\beta,n}$ is the normalizing constant. When $q = 2$ this is the classic Ising model while in this paper we will focus on the case $q \geqslant 3$ for an integer q (for an extension to non-integer q via the random cluster model, see e.g. [23]). We use the standard notation $\beta_c(q)$ for the (explicitly known) threshold value between the ordered and the disordered phases (see [16]).

Throughout the paper $(\sigma_t)_{t \geqslant 0}$ will denote the discrete time Glauber dynamics for this model, namely, starting from σ_0, at each step we choose a vertex $u \in V$ uniformly and set

$$\sigma_{t+1}(v) = \begin{cases} \sigma_t(v) & \text{if } v \neq u \\ k & \text{with probability } \mu_n\left(\sigma(u) = k \,\middle|\, \sigma(w) = \sigma_t(w)\ \forall w \neq u\right) \text{ if } v = u. \end{cases}$$

We denote by P_n the transition kernel for this Markov process and by $\mathbb{P}_{\sigma_0}$ the underlying probability measure. We will measure the distance between the distribution of the chain at time t and its stationary distribution μ_n via the total-variation norm. Accordingly,

$$d_t^{\sigma_0}(n) = \|\mathbb{P}_{\sigma_0}(\sigma_t \in \cdot) - \mu_n\|_{\mathrm{TV}} \quad \text{and} \quad d_t(n) = \max_{\sigma_0 \in \Sigma_n} d_t^{\sigma_0}(n).$$

For $\epsilon \in (0,1)$, the ϵ-mixing time is the number of steps until the total-variation distance to stationarity is at most ϵ in the worst case, i.e.:

$$t_{\mathrm{MIX}(\epsilon)}(n) = \inf\{t : \ d_t(n) \leqslant \epsilon\}$$

and by convention we set $t_{\mathrm{MIX}}(n) := t_{\mathrm{MIX}(1/4)}(n)$. If for any fixed $\epsilon \in (0,1)$

$$w_\epsilon(n) \triangleq t_{\mathrm{MIX}(\epsilon)}(n) - t_{\mathrm{MIX}(1-\epsilon)}(n) = o(t_{\mathrm{MIX}(1/4)}(n)) \qquad \text{as} \quad n \to \infty,$$

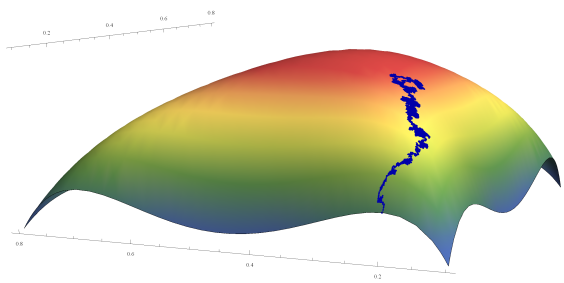


FIGURE 1. Rapid mixing with cutoff in the subcritical regime of $\beta < \beta_s(q)$ for $q = 3$. Blue curve marks the magnetization vector of the Glauber dynamics along time.

we say that the family of Markov chains exhibits the *cutoff phenomenon*, which describes a sharp drop in the total variation distance from close to 1 to close 0 (in an interval of time of smaller order than $t_{\text{MIX}}(n)$ denoted as the *cutoff window*). Observe that cutoff occurs if and only if $t_{\text{MIX}(\delta)}(n)/t_{\text{MIX}(\epsilon)}(n) \to 1$ as $n \to \infty$ for any fixed $\delta, \epsilon \in (0,1)$.

1.1. **Results.** We show that the dynamics for the Curie-Weiss model undergoes a critical slowdown at an inverse-temperature $\beta_s(q) > 0$. This dynamical threshold is given by

$$\beta_s(q) = \sup\left\{\beta \geqslant 0 : \left(1 + (q-1)\mathrm{e}^{2\beta\frac{1-qx}{q-1}}\right)^{-1} - x \neq 0 \text{ for all } x \in (1/q, 1)\right\}. \tag{1.1}$$

Unlike mean-field Ising, for which $\beta_s(2) = \beta_c(2) = 1$, the dynamical transition for $q \geqslant 3$ occurs at a strictly higher temperature than the static phase transition, i.e., $\beta_s(q) < \beta_c(q)$.

Our first result addresses the regime $\beta < \beta_s(q)$, where rapid mixing occurs within $O(n \log n)$ steps and the dynamics exhibits cutoff with a window of size $O(n)$ (see Fig. 1).

Theorem 1. *Let $q \geqslant 3$ be an integer. If $\beta < \beta_s(q)$ then the Glauber dynamics for the q-state Curie-Weiss Potts model exhibits cutoff at mixing time*

$$t_{\text{MIX}}(n) = \alpha_1(\beta, q) n \log n \tag{1.2}$$

with cutoff window $w_\epsilon(n) = O_\epsilon(n)$ where $\alpha_1(\beta, q) = [2(1 - 2\beta/q)]^{-1}$.

We proceed to analyze the order of the mixing time as $\beta(n) \to \beta_s(q)$ as $n \to \infty$,

$$\beta(n) = \beta_s(q) - \xi(n) \tag{1.3}$$

where $\xi(n) \to 0$ as $n \to \infty$. The asymptotics of the mixing time will, of course, depend on how fast ξ decays, but it turns out that cutoff is observed only iff the decay is slow enough. This is captured in the following theorem.

Theorem 2. *Let $q \geqslant 3$ be an integer. With $\beta(n)$ given as in equation (1.3) we have:*

(1) If $\lim_{n\to\infty} n^{2/3}\xi(n) = \infty$ then the Glauber dynamics has cutoff with mixing time and cutoff window given by

$$t_{\text{MIX}}(n) = \alpha_1(\beta(n), q) n \log n + \alpha_2(q) n/\sqrt{\xi(n)}\,,$$
$$w_\epsilon(n) = O_\epsilon\big(n + \sqrt{n/\xi(n)^{5/2}}\,\big)\,, \tag{1.4}$$

where $\alpha_2(q)$ is a positive constant and α_1 is the constant defined in Theorem 1.

(2) If $0 \leqslant \liminf_{n\to\infty} n^{2/3}\xi(n) \leqslant \limsup_{n\to\infty} n^{2/3}\xi(n) < \infty$ then the dynamics does not exhibit cutoff and has mixing time

$$t_{\text{MIX}(\epsilon)}(n) = \Theta_\epsilon\big(n^{4/3}\big)\,. \tag{1.5}$$

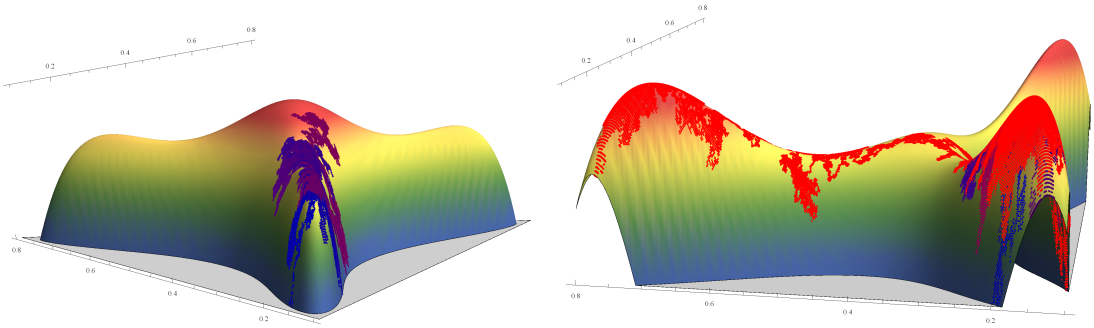


FIGURE 2. Slow mixing without cutoff in the supercritical regime $\beta > \beta_s$ for $q = 3$. On left $\beta_s < \beta < \beta_c$ and on right $\beta > \beta_c$. Curve color marks time from blue to red.

Part (2) of Theorem 2 in particular applies at criticality $\beta = \beta_s(q)$ where the mixing time is of order $n^{4/3}$ with a scaling window of order $n^{-2/3}$ (in contrast, the mixing time for the critical Ising model is of order $n^{3/2}$ with a window of $\sqrt{n}$).

Finally, above $\beta_s(q)$ the mixing time is exponentially large in n, as depicted in Fig. 2.

Theorem 3. *Let $q \geqslant 3$ be an integer, and fix $\beta > \beta_s(q)$. For every $0 < \epsilon < 1$ there exist $C_1, C_2 > 0$ such that for all n,*

$$t_{\mathrm{MIX}(\epsilon)}(n) \geqslant C_1 \exp(C_2 n)\,.$$

Combined these results give a complete analysis of the mixing time of Glauber dynamics for the Curie-Weiss Potts model.

The slowdown in the mixing of the dynamics occurring as soon as $\beta \geqslant \beta_s(q)$ (be it power-law at $\beta_s(q)$ or exponential mixing above this point) is due to the existence of states from which the Markov chain takes a long time to escape. However, in the range $\beta \in [\beta_s(q), \beta_c(q))$ the subset of initial configurations from which mixing is slow is exponentially small in probability. One can then ask instead about the mixing time from typical starting locations, known as *essential mixing*. Define the mixing time started from a subset of initial configurations $\widetilde{\Sigma}_n \subseteq \Sigma_n$ via $d_t^{\widetilde{\Sigma}_n}(n) = \max_{\sigma \in \widetilde{\Sigma}_n} d_t^{\sigma}(n)$ as well as

$$t_{\mathrm{MIX}(\epsilon)}^{\widetilde{\Sigma}_n}(n) = \inf\{t : \, d_t^{\widetilde{\Sigma}_n}(n) \leqslant \epsilon\} \quad \text{and} \quad w_\epsilon^{\widetilde{\Sigma}_n}(n) = t_{\mathrm{MIX}(\epsilon)}^{\widetilde{\Sigma}_n}(n) - t_{\mathrm{MIX}(1-\epsilon)}^{\widetilde{\Sigma}_n}(n).$$

With these definitions we have the following result, showing that the subcritical mixing time behavior from Theorem 1 extends all the way to $\beta < \beta_s(q)$ once one excludes a subset of initial configurations with a total mass that is exponentially small in n.

Theorem 4. *Let $q \geqslant 3$ be an integer and let $\beta < \beta_c(q)$. There exist constants $C_1, C_2 > 0$ and subsets $\widetilde{\Sigma}_n \subseteq \Sigma_n$ such that the Glauber dynamics has cutoff with mixing time and cutoff window given by*

$$t_{\mathrm{MIX}}^{\widetilde{\Sigma}_n}(n) = \alpha_1(\beta, q) n \log n \quad ; \qquad w_\epsilon^{\widetilde{\Sigma}_n}(n) = O_\epsilon(n)\,,$$

where $\mu_n(\Sigma_n \setminus \widetilde{\Sigma}_n) \leqslant C_1 e^{-C_2 n}$ and α_1 is the constant in Theorem 1.

1.2. **Related work.** Through several decades of work by mathematicians, physicists and computer scientists a general picture of how the mixing time varies with the temperature has been developed. It is believed that in a wide class of models and geometries the mixing time undergoes the following "critical slowdown". For some critical inverse-temperature β_0 and a geometric parameter $L(n)$, where n is the size of the system, we should have:

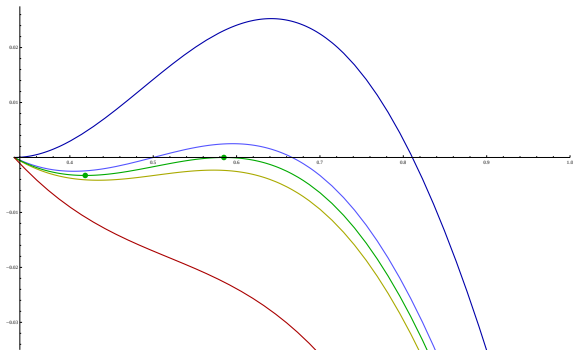


FIGURE 3. Minimal drift towards $1/q$ of a single coordinate of S_t as a function of its value for the Curie-Weiss Potts model with $q = 3$ and different values of β. Two lowest curves correspond to $\beta < \beta_s$, green middle curve has $\beta = \beta_s$ (with points marking its two local extrema), second from top curve has $\beta_s < \beta < \beta_c$ and top curve has $\beta = \beta_c$.

- High temperature ($0 \leqslant \beta < \beta_d$): mixing time of order $n \log n$ with cutoff.
- Critical temperature ($\beta = \beta_d$): mixing time of order $nL(n)^z$ for some fixed $z > 0$.
- Low temperature ($\beta > \beta_d$): mixing time of order $\exp(\tau_\beta L(n))$ for some fixed $\tau_\beta > 0$.

For a more comprehensive description of critical slowdown see [19, 30, 33]. It should be noted that to demonstrate the above phenomenon in full, one needs to derive precise estimates on the mixing time up to the critical temperature, which can be quite challenging.

Perhaps the most studied model in this context is Ising. For the complete graph, a comprehensive treatment is given in [17, 18, 28], where critical slowdown (as described above) around the uniqueness threshold β_c is established in full. In this setting, finer statements about the asymptotics of the mixing time can be made. For instance, in [17] the case where β approaches β_c with the size of the system is analyzed (in Theorem 2 here we consider this case as well). The same picture, yet with the notable exclusion of a cutoff proof at high temperatures, is also known on the d-regular tree where [3] established the high and low temperature regimes and recently [19] proved polynomial mixing at criticality.

From a mathematical physics point of view, the most interesting underlying graph to consider is the lattice $\mathbb{Z}^d$. For $d = 2$ the full critical slowdown is now known: for a box with n vertices the mixing time is $O(n \log n)$ throughout the high temperature regime [34, 35] whereas it is $\exp((\tau_\beta + o(1))n)$ throughout the low temperature regime [13, 14, 39] with τ_β being the surface tension. The $d = 2$ picture was very recently completed by two of the authors establishing cutoff in the high temperature regime [32] and polynomial mixing at the critical temperature [30]. For a more comprehensive survey of recent literature for Ising on the lattice see [30].

Turning back to the Potts model, understanding the kinetic picture here is of interest, not just as an extension of the results for Ising, but as an example of a model with a first order phase transition. Unlike in Ising, the free energy in the Potts model on various graphs and values of q undergoes a first order phase transition as the temperature is varied. This is certainly true for all $q \geqslant 3$ in the mean-field approximation, i.e. on the complete graph as treated here, but also known to be the case on $\mathbb{Z}^d$ for $d \geqslant 2$ and $q > Q(d)$ for some $Q(d) < \infty$ [23] (although most values of $Q(d)$ are not known rigorously, it was shown that $Q(2) = 4$ [2] and $Q(d) < 3$ for all d large enough [6]).

A first order phase transition has direct implications on the dynamics of the system. For one, the coexistence of phases at criticality, implies slow mixing. This is because

getting from one phase to another requires passing through a large free energy barrier, i.e. states which are exponentially unlikely. Indeed, in [10, 11] the mixing time for Potts on a box with n vertices in $\mathbb{Z}^d$ for any fixed $d \geqslant 2$ and sufficiently large q is shown to be exponential in the surface area of the box for any β larger or (notably) equal to the uniqueness threshold $\beta_c(d, q)$. This should be compared with the aforementioned polynomial mixing of Glauber dynamics for Ising at criticality. In fact, coexistence of the ordered and disordered phases also accounts for the slow mixing of the Swendsen-Wang dynamics at the critical temperature. This is shown in [10, 11] for $\mathbb{Z}^d$ under a similar range of d and q and in [22] for the complete graph. Other dynamics also exhibit slow mixing at criticality [4].

First order phase transitions are expected to lead to metastability type phenomena on the lattice in some instances. There has been extensive work on this topic (see [5, 12] and the references therein) yet the picture remains incomplete. It is expected that the transition to equilibrium will be carried through a nucleation process, which has an $O(1)$ lifetime and therefore does not affect the order of the mixing time in contrast to the mean-field case. This is affirmed, for instance, in Ising where $O(n \log n)$ mixing time is known for low enough temperatures under an (arbitrarily small) non-zero external field, despite the first order phase transition (in the field) around 0. For related works see e.g. [7, 8, 15, 36, 37] as well as [33] and the references there.

Similarly, the Potts model on the lattice should feature rapid mixing of $O(n \log n)$ throughout the sub-critical regime $\beta < \beta_c(d, q)$ due to the vanishing surface-area-to-volume ratio. Thus, contrary to the critical slowdown picture predicted for Ising, whenever there is a first order phase transition it should be accompanied by a sharp transition from fast mixing at $\beta < \beta_c$ to an exponentially slow mixing at β_c in lieu of a critical power law. However, fast mixing of the Potts model on $\mathbb{Z}^d$ for $\beta < \beta_c$ is not rigorously known except at very high temperatures (where it follows from standard arguments [33]). For sufficiently high temperatures, cutoff was very recently shown in [31].

On the complete graph however, metastability is apparent. In the absence of geometry, the order parameter sufficiently characterizes the state of the system and thus the dynamics and its stationary distribution are described effectively by the free energy of the system as a function of the order parameter. While coexistence of phases implies that at criticality the free energy is minimized at more than one value of the order parameter (corresponding to each phase), continuity entails that some of these global minima will turn into local minima just below or above criticality. These local minimizers correspond exactly to the metastable states and the value (or curve) of the thermodynamic parameter (e.g. temperature) beyond which these local minima cease to appear is called *spinodal*.

Consider the model at hand, namely Potts on the complete graph (for general background on the Potts model, see e.g. [9, 21, 23]). The order parameter here, analogous to the magnetization in the Ising model, is the *vector of proportions* of each color $s \in \mathcal{S} = \{x \in \mathbb{R}_+^q : \|x\|_1 = 1\}$. It is well known that there exists $\beta_c(q)$, below which the Potts distribution μ_n is supported almost entirely on configurations with roughly equal (about $1/q$) proportions of each color and above which μ_n is supported almost entirely on configurations where one of the q colors is dominant. In the former case we say that in equilibrium the system is in the disordered phase, while in the latter case we say that in equilibrium the system is in one of the q ordered phases, corresponding to the q-colors.

Up to relabeling of the vertices configurations are essentially described by the proportions vector s and as such, on a logarithmic scale, the Potts distribution can be read from the graph of the free energy as a function of s. As depicted in Figure 4 (showing $q = 3$, the

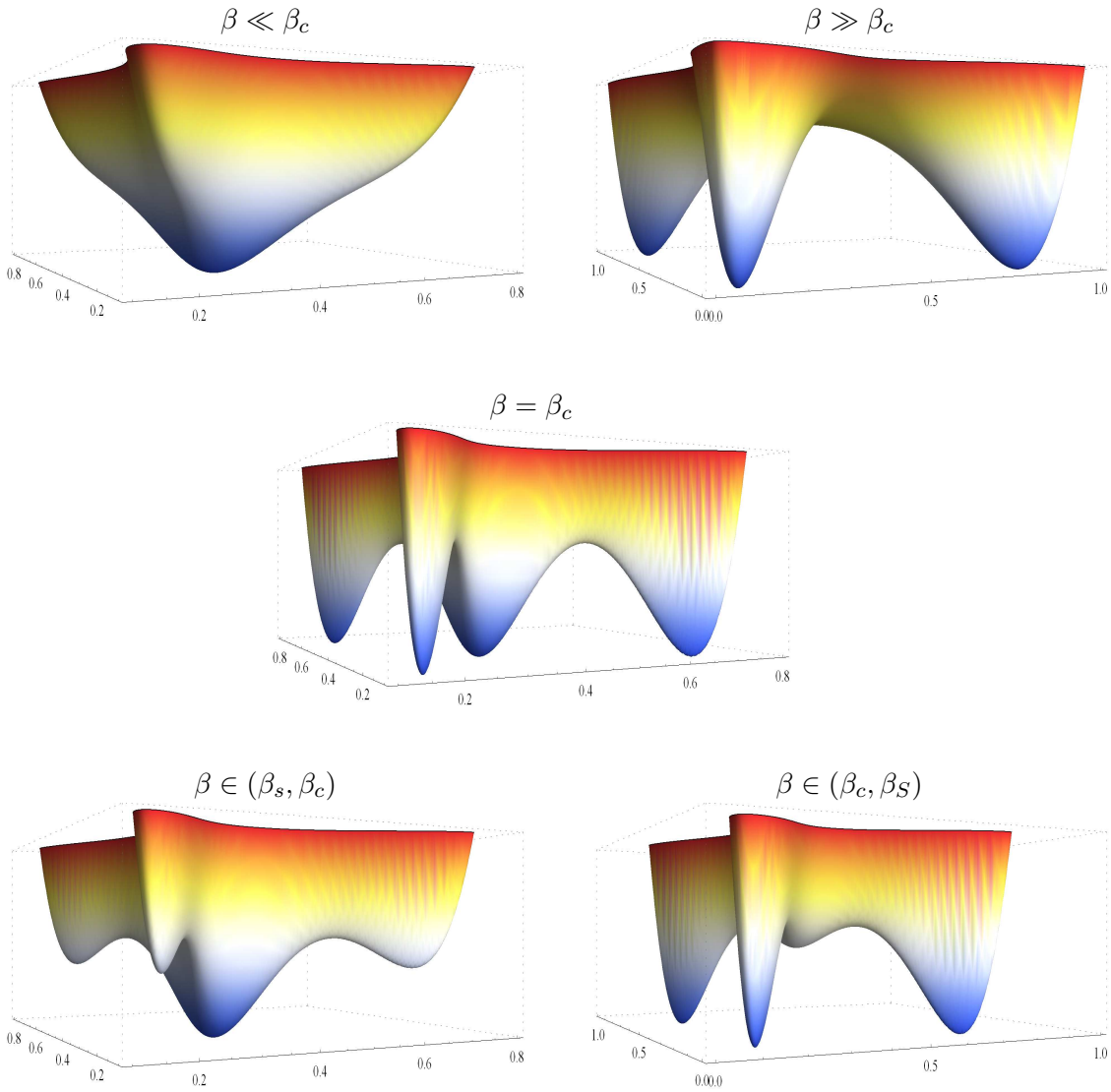
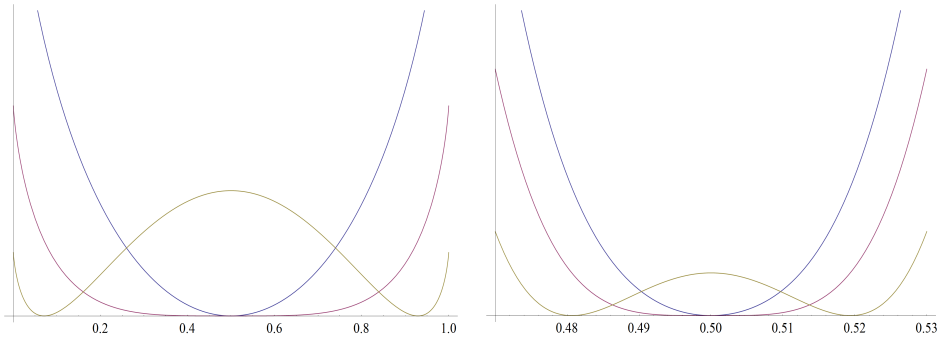


FIGURE 4. The free energy as a function of the proportions vector s for the Potts model on the complete graph with $q = 3$ and large n. The simplex $\mathcal{S} = \{s \in \mathbb{R}^3_+ : \|s\|_1 = 1\}$ is mapped into the XY plane via $(s^1, s^2, s^3) \mapsto (s^1, s^2, 1 - s^1 - s^2)$.

situation is qualitatively the same for all $q > 2$), when $\beta < \beta_c$ the free energy has a single global minimum at the center, corresponding to the disordered phase, while for $\beta > \beta_c$ there are q "on-axis" global minima, corresponding to the q ordered phases obtainable from one another through a permutation of the coordinates. At β_c, coexistence of the ordered and disordered phases is evident in the presence of $q+1$ global minima of the free energy. For more details see, e.g., [16].

Below β_c but sufficiently close to it, the free energy, globally minimized only at the center, has q local minima in place of the q global minima which corresponded to the ordered phases at criticality. Once β is too small, these local minima disappear. The threshold value for the appearance of these local minima is the spinodal inverse temperature β_s (there is a similar behavior above β_c marked by a second spinodal temperature β_S, as illustrated by Figure 4, but we do not address this regime in the paper).



$\beta < \beta_c(2)$, $\beta = \beta_c(2)$, $\beta > \beta_c(2)$ $\beta < \beta_c(2)$, $\beta = \beta_c(2)$, $\beta > \beta_c(2)$

FIGURE 5. The free energy as a function of the magnetization m for the Ising model on the complete graph for large n. No phase coexistence at β_c and the global maximizers for $\beta > \beta_c$ are seen to emerge continuously from $m = 0.5$.

Once the system starts from an initial configuration whose proportions vector is close to a local minimizer, the system will spend a time which is exponential in n near this minimizer before escaping to the global minimizer and reaching equilibrium. This is because away from a local minimizer, energy increases locally exponentially (in n), i.e. there is an energy barrier of an exponential order to cross. Thus, as $n \to \infty$ the system will spend an unbounded amount of time at a non-equilibrium state, which will be seemingly stable. In terms of the mixing time of the dynamics, as the definition involves the worst case initial configuration, metastable states will result in exponentially slow mixing.

Our result is a rigorous affirmation of this picture. Although the definition of β_s in (1.1) seems different than the one given above for the spinodal inverse temperature, it can be shown, in fact, that this is indeed the threshold value of β for the emergence of local minima below β_c. Theorem 3 then asserts that above β_s mixing is exponentially slow while Theorem 1 shows that below β_s mixing is still fast. The set of configurations whose exclusion in Theorem 4 leads to fast mixing all the way up to (but below) β_c are precisely the ones from which the process will get stuck in a metastable state. Indeed as the free energy of such initial configurations is higher than that of configurations near the globally minimizing stable state, such configurations will have a probability which is exponentially small in the size of the system.

Furthermore, the transition from fast to slow mixing passes through polynomial mixing which occurs at β_s and in its vicinity (Theorem 2). This in fact establishes that the aforementioned critical slowdown phenomenon occurs here as well, albeit at the spinodal rather than the uniqueness threshold. We predict that this should be the case for the dynamical behavior on other mean-field geometries such as an Erdős-Rényi random graph or a random regular graph.

For a (non-rigorous) treatment of metastability and its effect on the rate of convergence to equilibrium in other mean-field models with a first order phase transition, see for example [24, 25]. A rigorous analysis of such a system (the Blume-Capel model), below and above criticality, was recently carried out in [26]. It is illuminating to contrast the graph of the free energy as a function of the proportions vector in the Potts model to that of the free energy as a function of the magnetization in Curie-Weiss Ising, given in Figure 5. The second order phase transition and lack of phase coexistence at the critical temperature, implies the absence of a local minima at any value below or above β_c. As a consequence there is no spinodal temperature and mixing is fast throughout the whole range $\beta < \beta_c$.

1.3. **Proof Ideas.** As discussed before, up to a permutation of the vertices, configurations can be described by their proportions vector. Formally, for a configuration $\sigma \in \Sigma_n$, we denote by $S(\sigma)$ the q-dimensional vector $(S^1(\sigma), \dots S^q(\sigma))$, where

$$S^k(\sigma) = \frac{1}{n} \sum_{v \in V} \mathbb{1}_{\{\sigma(v)=k\}} \,.$$

Note that $S(\sigma) \in \mathcal{S}$ where $\mathcal{S} = \{x \in \mathbb{R}_+^q : \|x\|_1 = 1\}$. Now it is not difficult to see that $S_t = S(\sigma_t)$ is itself a Markov process with state space $\mathcal{S}$ and stationary distribution $\pi_n = \mu_n \circ S^{-1}$, the distribution of $S(\sigma)$ under μ_n. We shall refer to this process as the *proportions chain*. Figures 1 and 2 show a realization of the proportion chains superimposed on the free energy graph plotted upside down for better visibility. 3 different values of β, corresponding to 3 different regimes are exhibited. The color of the curve, representing time, shows the temporal evolution of the proportion chain. Notice how local minima (shown as local maxima) "trap" the chain for a long time.

As a projection of the chain, S_t mixes at least as fast as σ_t. Moreover, when starting in one of the q configurations where all sites have the same color, a symmetry argument reveals that the mixing time of S_t is equal to that of σ_t. One therefore must control the effect of starting from a initial state which is not monochrome. Using a coupling argument we show that the difference in the mixing times is of the same order as the cutoff window for S_t and can thus be absorbed into our error estimates. It will then suffice to analyze the proportions chain, which is of a lesser complexity than the original process. In particular the state space of S_t has a fixed $q-1$ dimension, independently of n, and its transition probabilities can be easily calculated.

Next, we show that when $\beta < \beta_c(q)$ most of the mass of the stationary distribution π_n is concentrated on *balanced* states whose distance from the "equi-proportionality" vector $(1/q, \dots, 1/q)$ is $O(1/\sqrt{n})$. A simple coupling argument then shows that the Markov chain is mixed soon after arriving at such a state. Thus the main effort in the proof becomes finding sharp estimates on the time is takes for S_t to reach a balanced state from a worst-case initial configuration, in different regimes of β.

It turns out that these hitting times are determined by the function $D_\beta(x)$, which is defined as (up to a multiplication by $1/n$) the drift of one coordinate of S_t when that coordinate has value x in the worst case, i.e. the minimum drift towards $1/q$, where the minimum is taken over all possible values for the remaining coordinates. An explicit formula for $D_\beta(x)$ can be obtained (3.2). Its graph is plotted in Figure 3 for $x \in [1/q, 1]$ and different values of β. For $\beta \ll \beta_c$, this drift is strictly negative in $(1/q, 1]$ and thus each coordinate quickly (in $O(n \log n)$ time) gets to within $1/\sqrt{n}$ of $1/q$. On the other hand, the function $D_\beta(x)$ is monotone increasing in β and therefore for sufficiently large β, it will no longer be negative throughout $(1/q, 1]$ - there will be an interval in $(1/q, 1]$ where it is positive. Such an interval will take an exponential amount of time to traverse and this will lead to an exponential mixing time. The smallest β for which this happens is, by definition, β_s. This β_s in turn coincides with the inverse temperature at which local minima begin to appear in the free energy as a function of s. In fact, to show exponentially slow mixing, we use standard conductance arguments, which in face of local minima in the free energy give exponential mixing quite automatically.

The most delicate analysis is in the critical regime where β is near or equal to β_s. In this case the x-axis is tangential to the graph of $D_\beta(x)$ at its peak (the green curve in Figure 3) and the challenge is in finding the asymptotics of the passage time through the tangential point on the x-axis (left green dot in the figure). As the drift there is 0, locally

around this point, a coordinate of S_t behaves as a random walk and Doob's decomposition of a suitably chosen function of the coordinate yields the right passage time estimates.

1.4. **Organization.** Section 2 sets notation and contains some useful facts on the Curie-Weiss Potts model, as well as tools (and a few non-standard variations on them) needed in the analysis of mixing time. In Section 3 we derive basic properties of the proportions chain that will be repeatedly used in the remainder of the proof. In Section 4 we analyze the case $\beta < \beta_s(q)$ and prove Theorem 1 while Section 5 treats the case $\beta > \beta_s(q)$ and establishes Theorem 3. The near critical regime is analyzed in Section 6, which includes the proof of Theorem 2. The final section, Section 7, gives the proof of Theorem 4.

2. Preliminaries

2.1. **Notation.** We let $[a, b]$ denote the set $\{a, \dots, b\}$ for $a, b \in \mathbb{Z}$. We use the same notation for vector and scalar valued variables. For an m-dimensional vector s, we denote by s^k its k-th component and for $I = (i_1, \dots, i_k) \subseteq [1, m]$, $s^I = (s^{i_1}, \dots, s^{i_k})$. Matrix-valued variables will appear in bold. We let $\mathbf{W}^{m,k}$ denote the (m, k) element of $\mathbf{W}$ and let $\mathbf{W}^m$ denote is its m-th row.

We write e_i for the unit vector in the i-th direction and $\widehat{\mathsf{e}} = (1/q, 1/q, \dots, 1/q) \in \mathbb{R}^q$ for the *equiproportionality* vector. For $s \in \mathbb{R}^q$, we denote $\widehat{s} = s - \widehat{e}$.

Most of our vectors will live on the simplex $\mathcal{S} = \{x \in \mathbb{R}^q_+ : \|x\|_1 = 1\}$ or even $\mathcal{S}_n = \mathcal{S} \cap \frac{1}{n}\mathbb{Z}^q$. Occasionally we would like to further limit this set and for $\rho > 0$ we define

$$\begin{array}{lll} \mathcal{S}^\rho = \{s \in \mathcal{S} : \|\widehat{s}\|_\infty < \rho\}, & \mathcal{S}^\rho_n = \mathcal{S}^\rho \cap \frac{1}{n}\mathbb{Z}^q, & \Sigma^\rho_n = S^{-1}(\mathcal{S}^\rho_n) \\ \mathcal{S}^{\rho+} = \{s \in \mathcal{S} : \ s^k < 1/q + \rho \ \ \forall k \in [1, q]\}, & \mathcal{S}^{\rho+}_n = \mathcal{S}^{\rho+} \cap \frac{1}{n}\mathbb{Z}^q, & \Sigma^{\rho+}_n = S^{-1}(\mathcal{S}^{\rho+}_n). \end{array}$$

Note that $\mathcal{S}^{\rho+} \subseteq \mathcal{S}^{(q-1)\rho}$ and similar relations hold for $\mathcal{S}_n$ and Σ_n.

Vectors in $\mathcal{S}$ will often be viewed also as distributions on $[1, q]$. A coupling of ν, $\widetilde{\nu} \in \mathcal{S}$ is the joint distribution of two random variables X, $\widetilde{X}$, defined on the same probability space and marginally distributed according to ν, $\widetilde{\nu}$. If $\mathbb{P}^\star$ is the underlying probability measure then we always have $\|\nu - \widetilde{\nu}\|_{\mathrm{TV}} \leqslant \mathbb{P}^\star(X \neq \widetilde{X})$. We shall call this coupling a *best coupling* if it satisfies $\|\nu - \widetilde{\nu}\|_{\mathrm{TV}} = \mathbb{P}^\star(X \neq \widetilde{X})$. Such a coupling always exists.

In the course of the proofs, we introduce various couplings of two copies of the Glauber dynamics $(\sigma_t)_t$. For the second copy we shall use the notation $\widetilde{\sigma}_t$ and $\widetilde{S}_t = S(\widetilde{\sigma}_t)$. Couplings will be identified by their underlying probability measure, for which we will use the symbol $\mathbb{P}$ with a superscript that changes from coupling to coupling, e.g. $\mathbb{P}^{BC}$. A subscript will indicate initial state or states, e.g. $\mathbb{P}^{BC}_{\sigma_0, \widetilde{\sigma}_0}$. The expectation and variance, $\mathbb{E}[\cdot]$ and $\mathbb{V}\mathrm{ar}(\cdot)$ resp., will be decorated in the same way as the underlying measure with respect to which they are defined. The σ-algebra $\mathcal{F}_t$ will always include all the randomness up-to time t. For example, with a single chain $(\sigma_t)_t$ this is the σ-algebra generated by $\{\sigma_s : \ s \leqslant t\}$, for a coupling of two chains $(\sigma_t)_t$, $(\widetilde{\sigma}_t)_t$, it is the one generated by $\{\sigma_s, \widetilde{\sigma}_s : \ s \leqslant t\}$, etc.

2.2. **Large Deviations Results for the Curie-Weiss Potts Distribution.** In this subsection we recall several results concerning the concentration of the proportions vector measures π_n. See, e.g., [16, 20] for proofs of these results.

It is a consequence of Sanov's Theorem together with an application of Varadhan's Lemma that the sequence $(\pi_n)_{n \geqslant 1}$ satisfies a large deviation principle (LDP) on $\mathcal{S}$ with rate function

$$I_{\beta,q}(s) = \sum_{k=1}^{q} s^k \log(q s^k) - \beta \|s\|_2^2 - C, \tag{2.1}$$

where C is chosen so that $\min_{s\in\mathcal{S}} I_{\beta,q}(s) = 0$. The minimizing set $\varepsilon_{\beta,q} = \{s : I_{\beta,q}(s) = 0\}$ which is the support of the weak limit $\mathfrak{p}_{\beta,q}$ of $(\pi_n)_{n\geqslant 1}$ is then

$$\varepsilon_{\beta,q} = \begin{cases} \{\widehat{\mathsf{e}}\} & \text{if } \beta < \beta_c(q) \\ \left\{\widehat{\mathsf{e}}, \mathrm{T}^1\check{s}_{\beta_c,q}, \mathrm{T}^2\check{s}_{\beta_c,q}, \ldots, \mathrm{T}^q\check{s}_{\beta_c,q}\right\} & \text{if } \beta = \beta_c(q) \\ \left\{\mathrm{T}^1\check{s}_{\beta,q}, \mathrm{T}^2\check{s}_{\beta,q}, \ldots, \mathrm{T}^q\check{s}_{\beta,q}\right\} & \text{if } \beta > \beta_c(q) \end{cases}, \tag{2.2}$$

where

$$\beta_c(q) = \frac{(q-1)\log(q-1)}{q-2} \tag{2.3}$$

and

$$\check{s}_{\beta,q} = \left(\check{s}^1_{\beta,q}, \frac{1-\check{s}^1_{\beta,q}}{q-1}, \ldots, \frac{1-\check{s}^1_{\beta,q}}{q-1}\right). \tag{2.4}$$

The function $\beta \mapsto \check{s}^1_{\beta,q}$ is continuous and increasing on $[\beta_c(q), \infty)$ and $\mathrm{T}^k : \mathcal{S} \to \mathcal{S}$ interchanges the 1-*st* and k-th coordinates. Furthermore, the value of $\check{s}_{\beta,q}$ for all β, q is known in implicit form and in particular for $\beta = \beta_c(q)$, we have

$$\check{s}_{\beta_c(q),q} = \left(1 - \tfrac{1}{q}, \tfrac{1}{q(q-1)}, \ldots, \tfrac{1}{q(q-1)}\right). \tag{2.5}$$

This is true for all $q \geqslant 3$. For $q = 2$, (2.1),(2.2), (2.4), (2.5) still hold, but the critical inverse-temperature is now

$$\beta_c(2) = 1. \tag{2.6}$$

It is here that a fundamental difference between $q = 2$ and $q > 2$ can be observed. If $q = 2$ then $\check{s}_{\beta_c(2),2} = \widehat{\mathsf{e}}$, in which case $\beta \mapsto \mathfrak{p}_{\beta,2}$ is continuous for all $\beta \geqslant 0$. On the other hand, if $q \geqslant 3$ we have $\check{s}_{\beta_c(q),q} \neq \widehat{\mathsf{e}}$ and $\beta \mapsto \mathfrak{p}_{\beta,q}$ is discontinuous at $\beta_c(q)$. Thus, as it is recorded in the Physics literature, the system exhibits a first order phase transition if $q \geqslant 3$, but only a second order phase transition if $q = 2$.

2.3. **Hitting Time Estimates for General Supermartingales.** We will require some standard hitting time estimates for supermartingales and related processes.

Lemma 2.1. *For $x_0 \in \mathbb{R}$, let $(X_t)_{t\geqslant 0}$ be a discrete time process, adapted to $(\mathcal{F}_t)_{t\geqslant 0}$ which satisfies*

*(1) $\exists\delta \geqslant 0 :\quad \mathbb{E}_{x_0}\left[X_{t+1} - X_t \middle| \mathcal{F}_t\right] \leqslant -\delta$ **on** $\{X_t \geqslant 0\}$ for all $t \geqslant 0$.*
(2) $\exists R > 0 :\quad |X_{t+1} - X_t| \leqslant R, \forall t \geqslant 0$.
(3) $X_0 = x_0$.

where $\mathbb{P}_{x_0}$ is the underlying probability measure. Let $\tau_x^- = \inf\{t : X_t \leqslant x\}$ and $\tau_x^+ = \inf\{t : X_t \geqslant x\}$. The following holds.

(1) If $\delta > 0$ then for any $t_1 \geqslant 0$:

$$\mathbb{P}_{x_0}\left(\tau_0^- > t_1\right) \leqslant \exp\left\{-\frac{(\delta t_1 - x_0)^2}{8t_1R^2}\right\}. \tag{2.7}$$

(2) If $x_0 \leqslant 0$, then for any $x_1 > 0$ and $t_2 \geqslant 0$,

$$\mathbb{P}_{x_0}\left(\tau_{x_1}^+ \leqslant t_2\right) \leqslant 2\exp\left\{-\frac{(x_1 - R)^2}{8t_2R^2}\right\}. \tag{2.8}$$

(3) If $x_0 \leqslant 0$, $\delta > 0$, then for any $x_1 > 0$ and $t_3 \geqslant 0$:

$$\mathbb{P}_{x_0}\left(\tau_{x_1}^+ \leqslant t_3\right) \leqslant t_3^2\exp\left\{-\frac{(x_1 - R)\delta^2}{8R^3}\right\}. \tag{2.9}$$

Proof. Starting with (1), if $x_0 < 0$, there is nothing to prove. Otherwise, let Z_t be a supermartingale independent of X_t, which starts from 0, has drift $-\delta$ and steps which are bounded by R. Set $Y_t = X_{t\wedge\tau_0^-} + Z_{\left(t-\tau_0^-\right)^+}$ and write $Y_t = M_t + A_t$ as its Doob Decomposition, with M_t a martingale, A_t a predictable processes and $A_0 = x_0$. Clearly $A_t \leqslant x_0 - \delta t$ and $|M_{t+1} - M_t| \leqslant 2R$, $\mathbb{P}_{x_0}$-a.s. The Hoeffding-Azuma inequality implies:

$$\begin{aligned}
\mathbb{P}_{x_0}\left(\tau_0^- > t_1\right) &\leqslant \mathbb{P}_{x_0}\left(Y_{t_1} > 0\right) \\
&\leqslant \mathbb{P}_{x_0}\left(M_{t_1} > \delta t_1 - x_0\right) \\
&\leqslant \exp\left\{-\frac{\left(\delta t_1 - x_0\right)^2}{8t_1R^2}\right\}
\end{aligned}$$

as desired.

Now set $Y_t = X_{t\wedge\tau_{x_1}^+}$ and observe that $(Y_t)_{t\,\geqslant\,0}$ satisfies conditions 1–3, and consequently it is enough to prove (2.8), with $\mathbb{P}_{x_0}\left(Y_{t_2} > x_1\right)$ as the LHS. Therefore, for all t, let $W_{t+1} = \mathbb{E}_{x_0}\left[Y_{t+1} - Y_t \middle| \mathcal{F}_t\right]$ and $Z_{t+1} = Y_{t+1} - Y_t - W_{t+1}$. Then clearly, W_{t+1}, Z_t are $\mathcal{F}_t$-measurable, $W_{t+1} \leqslant 0$ on $\{Y_t \geqslant 0\}$ and $\mathbb{E}_{x_0}\left[Z_{t+1}|\mathcal{F}_t\right] = 0$. This is a Doob-type decomposition. Now, define M_t for all t inductively as follows.

$$M_0 = 0\ ; \quad M_{t+1} = M_t + \operatorname{sign}(M_t)Z_{t+1}$$

and set $N_t = |M_t|$. We claim the following:

(1) $(M_t)_{t\in\mathbb{N}}$ is an $(\mathcal{F}_t)_{t\,\geqslant\,0}$-adapted martingale.
(2) $N_0 = 0\ ; \quad N_{t+1} = \left|N_t + Z_{t+1}\right|, \forall t.$
(3) $Y_t \leqslant N_t + R.$

The first two assertions follow straightforwardly from the construction. The third one, can be proven by induction, since it clearly holds for $t = 0$ and assuming $Y_{t-1} \leqslant N_{t-1} + R$, if $Y_{t-1} \geqslant 0$, then

$$\begin{aligned}
Y_t &= Y_{t-1} + \left(Y_t - Y_{t-1}\right) \\
&\leqslant N_{t-1} + R + Z_t \leqslant N_t + R,
\end{aligned}$$

and if $Y_{t-1} < 0$, then

$$Y_t \leqslant R \leqslant N_t + R.$$

Finally, by the Hoeffding-Azuma inequality applied to M_t we have

$$\begin{aligned}
\mathbb{P}_{x_0}\left(Y_{t_2} > x_1\right) &\leqslant \mathbb{P}_{x_0}\left(|M_{t_2}| > x_1 - R\right) \\
&\leqslant 2\exp\left(-\frac{(x_1 - R)^2}{8t_2R^2}\right).
\end{aligned}$$

This shows (2.8).

For part (3), let $\tau_0^-(s) = \inf\{t \geqslant s : X_t \leqslant 0\}$ and $Y_t(s)$, $M_t(s)$ and $A_t(s)$ defined as in the proof of part (1) only with $\tau_0^-(s)$ replacing τ_0^-. Then,

$$\begin{aligned}
\mathbb{P}_{x_0}\left(\tau_{x_1}^+ \leqslant t_3\right) &\leqslant \sum_{0 \leqslant s_1 < s_2 \leqslant t_3} \mathbb{P}_{x_0}\left(X_{s_1} \leqslant R,\, X_{s_2} \geqslant x_1,\, X_t > 0\ \forall s_1 \leqslant t \leqslant s_2\right) \\
&\leqslant \sum_{0 \leqslant s_1 < s_2 \leqslant t_3} \mathbb{P}_{x_0}\left(Y_{s_2}(s_1) \geqslant x_1, Y_{s_1}(s_1) \leqslant R\right) \\
&\leqslant \sum_{0 \leqslant s_1 < s_2 \leqslant t_3} \mathbb{P}_{x_0}\left(M_{s_2}(s_1) - M_{s_1}(s_1) \geqslant \delta(s_2 - s_1) + (x_1 - R)^+\right) \\
&\leqslant t_3^2 \exp\left\{-\frac{\delta^2 \frac{x_1 - R}{R}}{8R^2}\right\}
\end{aligned}$$

where the last inequality follows from Hoeffding-Azuma inequality and since the summands are non zero only when $0 \leqslant s_1 < s_2 \leqslant t_3$ and $(s_2 - s_1)R \geqslant x_1 - R$. □

Lemma 2.2. *Let $(X_t)_{t\in\mathbb{N}}$ be a process adapted to $(\mathcal{F}_t)_{t\in\mathbb{N}}$ and satisfying the following conditions for some $a > 2\delta \geqslant 0$:*

(1) $X_{t+1} - X_t \in \{-1, 0, 1\}$.
(2) $\mathbb{E}[X_{t+1} - X_t | \mathcal{F}_t] \geqslant -\delta$.
(3) $\mathbb{V}\mathrm{ar}(X_{t+1}|\mathcal{F}_t) \geqslant a$
(4) $X_0 \geqslant 0$

Let $\tau_r^+ = \inf\{t : X_t \geqslant r\}$. Then

$$\mathbb{P}(\tau_r^+ \leqslant t) \geqslant C_1 \exp\{-C_2(r/\sqrt{t} + \delta\sqrt{t})^2\} + O(t^{-1/2}) \tag{2.10}$$

where C_1, C_2 are positive constants which depends only on a.

Proof. It is easy to verify that conditions 1–3 imply

$$\mathbb{P}(X_{t+1} \neq X_t|\mathcal{F}_t) \geqslant a \quad ; \quad \mathbb{P}(X_{t+1} - X_t = -1|\mathcal{F}_t,\, X_{t+1} \neq X_t) \leqslant \tfrac{1}{2} + \tfrac{\delta}{2a}\,. \tag{2.11}$$

Now, let $T_0 = 0$ and $T_k = \inf\{t > T_{k-1} :\ X_t \neq X_{T_{k-1}}\}$, $Y_k = X_{T_k}$ for $k \geqslant 1$. Also, let $N_k = T_k - T_{k-1}$ and $Z_k = Y_k - Y_{k-1}$. From (2.11), it is not hard to see that we can couple $(N_k)_k$, $(Z_k)_k$ with two i.i.d sequences $(\widetilde{N}_k)_k$,$(\widetilde{Z}_k)_k$ such that $\widetilde{N}_k \geqslant N_k$, $\widetilde{Z}_k \leqslant Z_k$ a.s. and

$$\widetilde{N}_k \sim \mathrm{Geom}(a) \quad ; \quad \widetilde{Z}_k = \begin{cases} +1 & \text{w.p } 1/2 - \delta/2a \\ -1 & \text{w.p } 1/2 + \delta/2a \end{cases}.$$

Consequently, with $\widetilde{Y}_k = \sum_{m \leqslant k} \widetilde{Z}_m$ and $\widetilde{T}_k = \sum_{m \leqslant k} \widetilde{N}_m$ we have:

$$\begin{aligned}
\mathbb{P}(\tau_r^+ \leqslant t) &\geqslant \mathbb{P}(Y_{ta/2} \geqslant r,\, T_{ta/2} \leqslant t) \\
&\geqslant \mathbb{P}(\widetilde{Y}_{ta/2} \geqslant r) - \mathbb{P}(\widetilde{T}_{ta/2} > t) \\
&\geqslant C_1 \exp\left\{-C_2 \frac{(r + \delta t)^2}{t}\right\} + O(t^{-1/2}) + \exp\{-C_3 t\}\,,
\end{aligned}$$

where the last inequality follows by the local CLT for $\widetilde{Y}_k$, which is a nearest neighbor random walk whose steps have mean $-\delta/a$ and variance $1 - (\delta/a)^2$ and also by Cramer's theroem for $\widetilde{T}_k$. All constants depend only on a. □

Finally, we will also make use of the following result from [29]:

Lemma 2.3 ([29, Proposition 17.20])**.** *Let $(Z_t)_{t \geqslant 0}$ be a non-negative supermartingale adapted to $(\mathcal{G}_t)_{t \geqslant 0}$ and N a stopping time. Suppose that:*

(1) $Z_0 = z_0$
(2) $|Z_{t+1} - Z_t| \leqslant B$
(3) $\exists \sigma > 0$ *such that* $\mathbb{V}\mathrm{ar}\,(Z_{t+1}|\mathcal{G}_t) > \sigma^2$ *on the event* $\{N > t\}$.

If $u > 4B^2/(3\sigma^2)$, *then:*

$$\mathbb{P}_{z_0}\,(N > u) \leqslant \frac{4z_0}{\sigma\sqrt{u}}.$$

2.4. **Variance Lemma.** The following is a straight-forward extension of [28, Lemma 2.6] to vector valued Markov processes. We include a proof here for completeness.

Lemma 2.4. *Let* (Z_t) *be a Markov chain taking values in* $\mathbb{R}^d$ *and with transition matrix* P. *Write* $\mathbb{P}_{z_0}$, $\mathbb{E}_{z_0}$ *for its probability measure and expectation respectively, when* $Z_0 = z_0$. *Suppose that there is some* $0 < \eta < 1$ *such that for all pairs of starting states* $(z_0, \widetilde{z}_0)$,

$$\|\mathbb{E}_{z_0} Z_t - \mathbb{E}_{\widetilde{z}_0} Z_t\|_2 \leqslant \eta^t \|z_0 - \widetilde{z}_0\|_2 \tag{2.12}$$

Then $v_t \triangleq \sup_{z_0} \mathbb{V}\mathrm{ar}_{z_0}\,(Z_t) = \sup_{z_0} \mathbb{E}_{z_0}\|Z_t - \mathbb{E}_{z_0} Z_t\|_2^2$ *satisfies:*

$$v_t \leqslant v_1 \min\left\{t, \left(1-\eta^2\right)^{-1}\right\}. \tag{2.13}$$

Proof. Let Z_t and Z_t^* be independent copies of the chain with the same starting state z_0. By the assumption (2.12), we obtain that

$$\|\mathbb{E}_{z_0}[Z_t \mid Z_1 = z_1] - \mathbb{E}_{z_0}[Z_t^* \mid Z_1^* = z_1^*]\|_2 = \|\mathbb{E}_{z_1}[Z_{t-1}] - \mathbb{E}_{z_1^*}[Z_{t-1}^*]\|_2 \leqslant \eta^{t-1}\|z_1 - z_1^*\|_2\,.$$

Hence, we see that

$$\mathbb{V}\mathrm{ar}_{z_0}(\mathbb{E}_{z_0}[Z_t \mid Z_1]) = \frac{1}{2}\mathbb{E}_{z_0}\|(\mathbb{E}_{Z_1}[Z_{t-1}] - \mathbb{E}_{Z_1^*}[Z_{t-1}^*])\|_2^2 \leqslant \frac{\eta^{2(t-1)}}{2}\mathbb{E}_{z_0}\|Z_1 - Z_1^*\|_2^2 \leqslant \eta^{2(t-1)} v_1.$$

Combined with the total variance formula, it follows that

$$v_t \leqslant \sup_{z_0}\{\mathbb{E}_{z_0}[\mathbb{V}\mathrm{ar}_{z_0}(Z_t \mid Z_1)] + \mathbb{V}\mathrm{ar}_{z_0}(\mathbb{E}_{z_0}[Z_t \mid Z_1])\} \leqslant v_{t-1} + \eta^{2(t-1)} v_1\,,$$

which then gives that $v_t = \sum_{i=1}^t (v_i - v_{i-1}) \leqslant \sum_{i=1}^t \eta^{2(t-1)} v_1$, implying the desired upper bound immediately. □

2.5. **Bottleneck Ratio.** Let P be an irreducible, aperiodic transition kernel for a Markov chain on S with stationary measure π. The *bottleneck ratio* of a set $A \subseteq S$ is:

$$\Phi(A) = \frac{\sum_{x\in A, y\notin A} \pi(x)P(x,y)}{\sum_{x\in A}\pi(x)} \leqslant \frac{\pi(\partial_P A)}{\pi(A)},$$

where $\partial_P A = \{x \in A \;:\; P(x,y) > 0 \text{ for some } y \notin A\}$. The *bottleneck ratio* of the chain is

$$\Phi_* = \min_{A:\pi(A)\leqslant \frac{1}{2}} \Phi(A)\,. \tag{2.14}$$

The following result, due to [1,27,38] in several similar forms (see, e.g., [29, Theorem 7.3]) relates the bottleneck ratio with the mixing time of the chain.

Theorem 2.5. *If* Φ_* *is the bottleneck ratio defined in* (2.14) *then* $t_{\mathrm{MIX}(1/4)} \geqslant \frac{1}{4\Phi_*}$.

3. Drift Analysis for the Proportions Chain

In this section we prove various results concerning the drift of the process S_t. We analyze both the one coordinate process S_t^1 and the distance-to-equiproportionality $\|S_t - \widehat{\mathsf{e}}\|_2$. In the course of this analysis, we also define two couplings which will be of independent use later on and prove a uniform bound on the variance of S_t.

3.1. **The Drift of One Proportion Coordinate.** From symmetry, it is enough to analyze the drift of S_t^1. For $\beta \geqslant 0$, define $g_\beta : \mathcal{S} \to \mathcal{S}$ as

$$g_\beta(s) = (g_\beta^1(s), \ldots, g_\beta^q(s)) \quad ; \quad g_\beta^k(s) = \frac{e^{2\beta s^k}}{\sum_{j=1}^q e^{2\beta s^j}} \,.$$

We can express the drift of S_t^1 as follows:

$$\begin{aligned} \mathbb{E}\left[S_{t+1}^1 - S_t^1 \middle| \mathcal{F}_t\right] &= \frac{1}{n}\Big[-S_t^1 + \sum_{k=1}^q g_\beta^1\left(S_t - \tfrac{1}{n}\mathsf{e}_k\right) S_t^k\Big] \\ &= \frac{1}{n}\left[-S_t^1 + g_\beta^1\left(S_t\right)\right] + O\left(n^{-2}\right) \\ &= \frac{1}{n} d_\beta(S_t) + O\left(n^{-2}\right) \end{aligned} \tag{3.1}$$

with

$$d_\beta(s) \triangleq -s^1 + g_\beta^1(s). \tag{3.2}$$

The function $d_\beta : \mathcal{S} \to \mathbb{R}$ thus describes (up to a constant factor of n^{-1} and an error term) the *drift* of the first coordinate given the current proportions vector. It turns out the rapid mixing hinges on whether $d_\beta(s)$ is strictly negative whenever $s^1 > 1/q$ (and for any values for the remaining coordinates of s). Accordingly we define $D_\beta : [0,1] \to \mathbb{R}$ as

$$D_\beta(x) \triangleq \max_{\substack{s \in \mathcal{S} \\ s^1 = x}} d_\beta(s) = d_\beta\big(x, \tfrac{1-x}{q-1}, \ldots, \tfrac{1-x}{q-1}\big) \tag{3.3}$$

$$= -x + \frac{\exp(2\beta x)}{\exp(2\beta x) + (q-1)\exp(2\beta \frac{1-x}{q-1})} \tag{3.4}$$

and check when $D_\beta(x)$ is strictly negative for all $x \in (1/q, 1]$. We will see in Proposition 3.1 below that this happens if and only if $\beta < \beta_s(q)$, where $\beta_s(q) > 0$ is defined in (1.1).

For $\beta \geqslant 0$, define $s^*(\beta)$ and $s^\sharp(\beta)$ as:

$$\begin{aligned} s^*(\beta) &\triangleq \sup\left\{s \in [1/q, 1) : \tfrac{d}{ds} D_\beta(s) = 0\right\} \\ s^\sharp(\beta) &\triangleq \inf\left\{s \in (1/q, 1) : D_\beta(s) \geqslant 0\right\}, \end{aligned}$$

with the inf or sup being 1, if the respective sets are empty. We may now state:

Proposition 3.1. *For all $q \geqslant 3$ the following holds:*

1. *We have that $D_\beta(s)$ is increasing in β if $s \in [1/q, 1]$ and for all $\beta \geqslant 0$, that $D_\beta(1/q) = 0$ and that $D_\beta(1) < 0$.*
2. *That $s^\sharp(\beta) > \frac{1}{q}$ if $\beta < q/2$.*
3. *The following statements are equivalent if $\beta < q/2$:*
   - *(a) $s^\sharp(\beta) = 1$*
   - *(b) $D_\beta(s)$ has no roots in $(1/q, 1]$.*
   - *(c) $D_\beta(s^*(\beta)) < 0$.*
4. *All the statements in part (3) hold if and only if $\beta < \beta_s(q)$.*

Proof. We start by proving part (1). It is clear that

$$D_\beta(s) = -s + \frac{1}{1 + (q-1)\exp(2\beta \frac{1-qs}{q-1})} \,,$$

and hence is strictly increasing in β if $1/q < s \leqslant 1$. Furthermore, we have that

$$D_\beta(1/q) = -\frac{1}{q} + \frac{\mathrm{e}^{2\beta/q}}{\mathrm{e}^{2\beta/q} + (q-1)\mathrm{e}^{2\beta/q}} = -\frac{1}{q} + \frac{1}{q} = 0\,,$$

as well as

$$D_\beta(1) = -1 + \frac{1}{1 + (q-1)\mathrm{e}^{-\beta}} < 0\,.$$

For part (2), taking the derivative of $D_\beta(s)$ with respect to s evaluated at $1/q$, one obtains $\frac{d}{ds}D_\beta(s)\big|_{s=1/q} = -1 + \frac{2\beta}{q}$ which is negative if $\beta < q/2$. Together with $D_\beta(1/q) = 0$, this completes the proof.

For part 3, we first show that there exists at most two points in $[1/q, 1]$ such that $\frac{d}{ds}D_\beta(s) = 0$. To see this, we compute the first derivative and obtain that

$$\frac{d}{ds}D_\beta(s) = -1 + \frac{2q\beta\exp(2\beta\frac{1-qs}{q-1})}{[1 + (q-1)\exp(2\beta\frac{1-qs}{q-1})]^2} = -1 + \frac{2q\beta}{q-1}h\big((q-1)\mathrm{e}^{2\beta\frac{1-qs}{q-1}}\big)\,,$$

where $h(x) = \frac{x}{(1+x)^2}$. Obviously, there are at most two zeros for $-1 + \frac{2q\beta}{q-1}h(x)$ and since $(q-1)\exp(2\beta\frac{1-qs}{q-1})$ is a strictly monotone in s, we conclude that there are at most two points such that $\frac{d}{ds}D_\beta(s)$ vanishes.

Notice also that $\frac{d}{ds}D_\beta(1/q) < 0$ provided that $\beta < q/2$ and hence $D_\beta(1/q + \xi) < 0$ for all $\xi \leqslant \xi_0$, where ξ_0 is a sufficiently small positive number. We are now ready to derive the equivalence stated in the proposition. Observing that $D_\beta(s)$ is a smooth function and $D_\beta(1) < 0$, we deduce that (3a) $\Rightarrow$ (3b) $\Rightarrow$ (3c). It remains to prove that (3c) $\Rightarrow$ (3a). Suppose now that (3c) holds and there exists $s_0 \in (1/q, 1]$ such that $D_\beta(s_0) \geqslant 0$. Recalling that $D_\beta(1/q + \xi) < 0$ and $D_\beta(1) < 0$, we deduce the following:

- If $s_0 < s^*$, we will then have at least two zeros in $(1/q, s^*)$ for $\frac{d}{ds}D_\beta(s)$.
- If $s_0 > s^*$, we will then have at least one zero in $(s^*, 1)$ for $\frac{d}{ds}D_\beta(s)$.

We see that the first case contradicts with the fact that there can be at most two zeros for $\frac{d}{ds}D_\beta(s)$ and the second case contradicts our definition of s^*. Altogether, we established that (3c) $\Rightarrow$ (3a).

As for the last part, continuity and part (1) imply that (1.1) is equivalent to

$$\beta_s = \sup\{\beta \geqslant 0 :\ D_\beta(s) < 0 \text{ for all } s \in (1/q, 1]\}\,.$$

Since $D_\beta(s)$ is increasing in β for all $s \in [1/q, 1]$, it follows that for all $\beta < \beta_s(q)$, we indeed have $D_\beta(s) < 0$ for all $s \in (1/q, 1]$. On the other hand, by the continuity of the function $D_\beta(s)$ and our definition of β_s, we know that there exists $s_M \in (1/q, 1]$ such that $D_{\beta_s}(s_M) = 0$. Now, using the result of part (1) we conclude that $D_\beta(s_M) > 0$ for any $\beta > \beta_s$, completing the proof. □

In the following proposition we discuss the relation between $\beta_s(q)$ and $\beta_c(q)$.

Proposition 3.2. *For $q \geqslant 3$ we have that $0 < \beta_s(q) < \beta_c(q) < q$ while $\beta_s(2) = \beta_c(2) = 1$.*

Proof. As recalled in the Subsection 2.2,

$$\beta_c(2) = 1 \qquad \text{and} \qquad \beta_c(q) = \frac{(q-1)\log(q-1)}{q-2}\,;\ q \geqslant 3.$$

In the $q = 2$ case, it is easy to verify that $\frac{d}{ds}D_\beta(s) < 0$ for all $s \in [\frac{1}{2}, 1]$ if $\beta < 1$ and $\frac{d}{ds}D_\beta(\frac{1}{2}) > 0$ if $\beta > 1$. Since in addition $D_\beta(\frac{1}{2}) = 0$, $D_\beta(1) < 0$, we obtain $\beta_s(2) = 1$.

For $q \geqslant 3$ we have $\beta_c(q) < q/2$ and therefore

$$D_{\beta_c}\left(\frac{q-1}{q}\right) = -\frac{q-1}{q} + \frac{1}{1+(q-1)\exp\left(2\beta_c \frac{1-q\frac{q-1}{q}}{q-1}\right)} = 0$$

and

$$\begin{aligned}\frac{d}{ds}D_{\beta_c}(s)\Big|_{s=\frac{q-1}{q}} &= -1 + \frac{2\beta_c q \exp\left(2\beta_c \frac{1-q\frac{q-1}{q}}{q-1}\right)}{\left(1+(q-1)\exp\left(2\beta_c \frac{1-q\frac{q-1}{q}}{q-1}\right)\right)^2} \\ &= \frac{2(q-1)\log(q-1) - q(q-2)}{q(q-2)}.\end{aligned}$$

Now if $\phi(q) = 2(q-1)\log(q-1) - q(q-2)$ then $\phi(2) = \phi'(2) = 0$ and $\phi''(s) = -2 + \frac{2}{q-1}$ which is negative when $q > 2$. It follows that $\phi(q)$ is negative when $q > 2$ and hence for $q \geqslant 3$,

$$\frac{d}{ds}D_{\beta_c}(s)\Big|_{s=\frac{q-1}{q}} < 0.$$

So for small enough $\epsilon > 0$ and $s \in (\frac{q-1}{q} - \epsilon, \frac{q-1}{q})$ we have that $D_{\beta_c}(s) > 0$ and hence $\sup_s D_{\beta_c}(s) > 0$. By the smoothness of $D_\beta(s)$ this implies that there exists $\beta < \beta_c$ such that $\sup_s D_\beta(s) > 0$ which establishes that $\beta_s < \beta_c$. □

We will make repeated use of the the following proposition throughout the paper.

Proposition 3.3.

(1) Assume $\beta < q/2$. For all $0 < \rho_0 < \rho$ small enough, there exists $\gamma > 0$ and $C, c > 0$ such that for all n with $t = e^{\gamma n}$ we have

$$\mathbb{P}_{\sigma_0}\left(\exists 0 \leqslant s \leqslant t : \sigma_s \notin \Sigma_n^{\rho+}\right) \leqslant Ce^{-cn} \tag{3.5}$$

for all $\sigma_0 \in \Sigma_n^{\rho_0^+}$.

(2) Assume $\beta < q/2$. For all $r_0 > 0$, $\gamma > 0$ there exists $C, c > 0$ such that for all n and $r > r_0$ with $t = \gamma n$, $\rho_0 = \frac{r_0}{\sqrt{n}}$ and $\rho = \frac{r}{\sqrt{n}}$ we have

$$\mathbb{P}_{\sigma_0}\left(\exists 0 \leqslant s \leqslant t : \sigma_s \notin \Sigma_n^{\rho+}\right) \leqslant Ce^{-cr^2} \tag{3.6}$$

for all $\sigma_0 \in \Sigma_n^{\rho_0+}$.

(3) Assume $\beta < \beta_s(q)$. For all $\rho > 0$ there exists $\gamma > 0$ and $C, c > 0$, such that for all n, with $t = \gamma n$ we have

$$\mathbb{P}_{\sigma_0}\left(\sigma_t \notin \Sigma_n^{\rho+}\right) \leqslant Ce^{-cn} \tag{3.7}$$

for all $\sigma_0 \in \Sigma_n$.

Proof. Consider the process $(S_t^1 - s_0^1 : t \geqslant 0)$ until the first time it is to the right of $1/q + \rho$. If n is large enough and, in Case (1), if ρ is small enough, Proposition 3.1 Part (1) and Eq. (3.1) imply that this process satisfies the conditions of Lemma 2.1. Parts (1) and (2) of the proposition then follow from Parts (3) and (2) of the lemma and summing over all coordinates.

As for part (3), consider this time $(S_t^1 - 1/q + \rho/2 : t \geqslant 0)$. Since $\beta < \beta_s(q)$, it follows from Proposition 3.1 (Parts (1) and (3)) that this process also satisfies the conditions of

Lemma 2.1, with $\delta > C'n^{-1}$, for some positive constant C'. Now set $\gamma = 2/C'$ and apply part (1) of the Lemma to conclude that except with probability exponentially small in n, $S_t^1 \leqslant 1/q + \rho/2$ for some $t \leqslant \gamma n$. Once this happens, by Lemma 2.1 Part (3), as in the proof of (3.5), we have $S_{\gamma n}^1 \leqslant 1/q + \rho$ again except with probability tending exponentially fast to zero with n. It remains to use union bound to complete the proof. □

3.2. **Bounded Dynamics.** The *bounded dynamics* is a process that evolves like σ_t, only that $S(\sigma_t)$ is forced to stay close to $\widehat{\mathsf{e}}$ by rejecting transitions which violate this condition. Formally, fix $\rho > 0$ and let $(\sigma_t)_{t \geqslant 0}$ be a Markov chain on $\Sigma_n^{\rho+}$, which evolves as follows. Start from some $\sigma_0 \in \Sigma_n^{\rho+}$ and at step $t+1$:

- Draw $\widetilde{\sigma}_{t+1}$ according to $P_n(\sigma_t, \cdot)$, where P_n is the original transition kernel.
- If $\widetilde{\sigma}_{t+1} \in \Sigma_n^{\rho+}$ set $\sigma_{t+1} = \widetilde{\sigma}_{t+1}$ and otherwise set $\sigma_{t+1} = \sigma_t$.

We shall denote by $\mathbb{P}^\rho$ the underlying probability measure.

The unbounded and ρ-bounded dynamics admit a natural coupling, under which the two processes start from the same configuration and evolve together until time $\tau = \inf\{t \geqslant 0 : S_t \notin \mathcal{S}^{\rho+}\}$, where S_t is the unbounded process. This leads to the following two immediate observations which will be useful later.

(1) For any integer t and bounded function $f : (\Sigma_n)^{t+1} \mapsto \mathbb{R}$:
$$|\mathbb{E} f(\sigma_{[0,t]}) - \mathbb{E}^\rho f(\sigma_{[0,t]})| \leqslant 2\|f\|_\infty \mathbb{P}(\tau \leqslant t). \tag{3.8}$$
(2) In particular for any set $A \subseteq (\Sigma_n)^{t+1}$:
$$|\mathbb{P}(\sigma_{[0,t]} \in A) - \mathbb{P}^\rho(\sigma_{[0,t]} \in A)| \leqslant 2\mathbb{P}(\tau \leqslant t) \tag{3.9}$$

3.3. **Synchronized Coupling.** The *synchronized coupling* is a (Markov) coupling of two ρ-bounded dynamics in which the two chains "synchronize" their steps as much as possible. Formally, define $(\sigma_t)_{t \geqslant 0}$, $(\widetilde{\sigma}_t)_{t \geqslant 0}$ on the same probability space such that starting from σ_0, $\widetilde{\sigma}_0$, at time $t+1$:

(1) Choose colors I_{t+1}, $\widetilde{I}_{t+1}$ according to an optimal coupling of S_t, $\widetilde{S}_t$.
(2) Choose colors J_{t+1}, $\widetilde{J}_{t+1}$, according to an optimal coupling of $g_\beta\left(S_t - n^{-1}\mathsf{e}_{I_{t+1}}\right)$, $g_\beta\left(\widetilde{S}_t - n^{-1}\mathsf{e}_{\widetilde{I}_{t+1}}\right)$.
(3) Change a uniformly chosen vertex of color I_{t+1} in σ_t to have color J_{t+1} in σ_{t+1}, but only if $\sigma_{t+1} \in \Sigma_n^{\rho+}$.
(4) Change a uniformly chosen vertex of color $\widetilde{I}_{t+1}$ in $\widetilde{\sigma}_t$ to have color $\widetilde{J}_{t+1}$ in $\widetilde{\sigma}_{t+1}$, but only if $\widetilde{\sigma}_{t+1} \in \Sigma_n^{\rho+}$.

We shall write $\mathbb{P}^{SC,\rho}_{\sigma_0,\widetilde{\sigma}_0}$ for the underlying measure and omit ρ if it is large enough for the dynamics not to be bounded.

The following shows that this coupling contracts the $\|\cdot\|_1$ distance of the proportions vector.

Lemma 3.4. *There exists $C(\beta, q) > 0$ such that for any $\rho > 0$, uniformly in $\sigma_0, \widetilde{\sigma}_0 \in \Sigma_n^{\rho+}$ as $n \to \infty$*
$$\mathbb{E}^{SC,\rho}_{\sigma_0,\widetilde{\sigma}_0} \|S_t - \widetilde{S}_t\|_1 \leqslant \left(p + \frac{C\rho}{n}\right)^t \|s_0 - \widetilde{s}_0\|_1 , \tag{3.10}$$
where
$$p = p(n, \beta, q) = 1 - \frac{1 - 2\beta/q}{n} . \tag{3.11}$$

Proof. For $s, \widetilde{s} \in \mathcal{S}_n$ by a Taylor expansion of g_β around s, then another expansion for ∇g_β around $\widehat{\mathsf{e}}$ one has:

$$\|g_\beta(s) - g_\beta(\widetilde{s})\|_1 = \frac{2\beta}{q}\|s - \widetilde{s}\|_1 \left(1 + O\left(\|s - \widehat{\mathsf{e}}\|_1 + \|\widetilde{s} - \widehat{\mathsf{e}}\|_1\right)\right) ,$$

where we use the easily verified:

$$\partial_j g_\beta^k = \begin{cases} -2\beta g_\beta^j g_\beta^k & k \neq j\,, \\ -2\beta \left(g_\beta^k\right)^2 + 2\beta g_\beta^k & k = j\,. \end{cases} \tag{3.12}$$

Now under the bounded dynamics, $I_{t+1} \neq \widetilde{I}_{t+1}$ implies that $S_t^{I_{t+1}} > \widetilde{S}_t^{I_{t+1}}$ and $\widetilde{S}_t^{\widetilde{I}_{t+1}} > S_t^{\widetilde{I}_{t+1}}$ while $J_{t+1} \neq \widetilde{J}_{t+1}$ implies that $\left(S_t - n^{-1}\mathsf{e}_{I_{t+1}}\right)^{J_{t+1}} > \left(\widetilde{S}_t - n^{-1}\mathsf{e}_{\widetilde{I}_{t+1}}\right)^{J_{t+1}}$ and $\left(\widetilde{S}_t - n^{-1}\mathsf{e}_{\widetilde{I}_{t+1}}\right)^{\widetilde{J}_{t+1}} > \left(S_t - n^{-1}\mathsf{e}_{I_{t+1}}\right)^{\widetilde{J}_{t+1}}$. It follows by the definition of the coupling that

$$\|S_{t+1} - \widetilde{S}_{t+1}\|_1 - \|S_t - \widetilde{S}_t\|_1 = -\frac{2}{n}\left[\mathbb{I}_{\{I_{t+1} \neq \widetilde{I}_{t+1}\}} - \mathbb{I}_{\{J_{t+1} \neq \widetilde{J}_{t+1}\}}\right].$$

Recalling that for $s \in \mathcal{S}_n$, $\|s\|_{\mathrm{TV}} = \frac{1}{2}\|s\|_1$ and that under the best coupling of distributions s, $\widetilde{s}$ the probability of disagreement is $\|s - \widetilde{s}\|_{\mathrm{TV}}$, we have:

$$\begin{aligned} &\mathbb{E}^{SC,\rho}_{\sigma_0,\widetilde{\sigma}_0}\|S_1 - \widetilde{S}_1\|_1 \\ &\quad\leqslant\quad \|s_0 - \widetilde{s}_0\|_1 + \frac{2}{n}\left(-\tfrac{1}{2}\|s_0 - \widetilde{s}_0\|_1 + \tfrac{1}{2}\|g_\beta(s_0) - g_\beta(\widetilde{s}_0)\|_1 + O(n^{-1})\right) \\ &\quad\leqslant\quad \|s_0 - \widetilde{s}_0\|_1 \left(1 - \frac{1 - 2\beta/q + O(\rho)}{n} + O(n^{-2})\right) . \end{aligned}$$

The result follows by iteration. □

3.4. **Uniform Variance Bound.**

Lemma 3.5. *Assume $\beta < q/2$. There exists $\rho_0 = \rho_0(\beta, q)$ such that if $\rho \leqslant \rho_0$*

$$\mathbb{V}\mathrm{ar}^\rho_{\sigma_0}(S_t) = O\left(n^{-1}\right) , \tag{3.13}$$

uniformly in $\sigma_0 \in \Sigma_n^{\rho+}$ and $t \geqslant 0$, and there exists $\gamma_0 > 0$ such that

$$\mathbb{V}\mathrm{ar}_{\sigma_0}(S_t) = O\left(n^{-1}\right) , \tag{3.14}$$

uniformly in $\sigma_0 \in \Sigma_n^{\rho_0+}$ and $t \leqslant e^{\gamma_0 n}$.

Proof. Equation (3.13) will follow directly from Lemma 2.4 applied to S_t under the ρ-bounded dynamics. Indeed, Lemma 3.4 gives a stronger version of Condition 2.12 with $\eta = p + \rho O(n^{-2})$. Now, if $\beta < q/2$ and ρ is small enough, we have $\eta \leqslant 1 - \frac{1 - \frac{2\beta}{q}}{2n}$ for large enough n. Then (3.13) follows from (2.13) since $\mathbb{V}\mathrm{ar}^\rho_{\sigma_0} S_1 = O\left(n^{-2}\right)$.

For (3.14), find $\rho' < \rho$ and use (3.5) and (3.8) to conclude that for all $\sigma_0 \in \Sigma_n^{\rho'+}$ and $t \leqslant e^{\gamma n}$ for some $\gamma = \gamma(\rho, \rho')$:

$$\mathbb{V}\mathrm{ar}_{\sigma_0}(S_t) = \mathbb{V}\mathrm{ar}^\rho_{\sigma_0}(S_t) + o(n^{-1}) = O\left(n^{-1}\right) .$$ □

Corollary 3.6. *For $\beta < \beta_c(q)$, we have $\mathbb{V}\mathrm{ar}_{\mu_n}(S) = O(n^{-1})$.*

Proof. Fix $\rho < \rho_0$, where ρ_0 is given in Lemma 3.5 and notice that the bounded dynamics is reversible with respect to the Potts measure μ_n restricted to $\Sigma_n^{\rho+}$. Therefore the bounded dynamics has $\mu_n^\rho(\cdot) = \mu_n(\cdot|\sigma \in \Sigma_n^{\rho+})$ as its stationary measure. From the large deviation analysis in Subsection 2.2 it is straightforward to conclude that if $\beta < \beta_c(q)$

$$\mu_n(S \notin \mathcal{S}^{\rho+}) \leqslant \mathrm{e}^{-Cn}\,,$$

for some $C > 0$ and n large enough, depending on ρ and β. Therefore, we have $\|\mu_n - \mu_n^\rho\|_{\mathrm{TV}} \leqslant \mathrm{e}^{-Cn}$. Since μ_n, μ_n^ρ live on a compact space, this gives $\mathbb{V}\mathrm{ar}_{\mu_n}(S) \leqslant \mathbb{V}\mathrm{ar}_{\mu_n^\rho}(S) + \mathrm{e}^{-Cn}$. Since $\mathbb{P}_{\sigma_0}^\rho(\sigma_t \in \cdot)$ converges to μ_n^ρ as $t \to \infty$ for any fixed $\sigma_0 \in \Sigma_n^{\rho+}$, Lemma 3.5 can be extended to σ_0 chosen from μ_n^ρ. This completes the proof. □

3.5. **The Drift of the Distance to Equiproportionality.** Here we show that $\widehat{S}_t$ has drift towards 0. Write $S_{t+1} = S_t + \xi_{t+1}$ where we have that for i, $j = 1$, ..., q,

$$\mathbb{P}\left(\xi_{t+1} = \tfrac{1}{n}\left(\mathsf{e}_j - \mathsf{e}_i\right)\right) = S_t^i g_\beta^j\left(S_t - \tfrac{1}{n}\mathsf{e}_i\right) = S_t^i g_\beta^j\left(S_t\right) + O\left(n^{-1}\right)\,.$$

Then,

$$\begin{aligned}
\mathbb{E}\left[\|S_{t+1} - \widehat{\mathsf{e}}\|_2^2 \,|\, S_t\right] &= \mathbb{E}\left[\|S_t - \widehat{\mathsf{e}} + \xi_{t+1}\|_2^2 \,|\, S_t\right] \\
&= \|S_t - \widehat{\mathsf{e}}\|_2^2 + \mathbb{E}\left[\|\xi_{t+1}\|_2^2 |\, S_t\right] + 2\left\langle \mathbb{E}\left[\xi_{t+1}|\, S_t\right], S_t\right\rangle \\
&= \|\widehat{S}_t\|_2^2 + \frac{2}{n^2}\left(1 - h(S_t)\right) + \frac{2}{n}\sum_{j=1}^q \frac{e^{2\beta S_t^j}}{\sum_{k=1}^q e^{2\beta S_t^k}} S_t^j - \frac{2}{n}\|S_t\|_2^2 + O\left(n^{-2}\right) \\
&= \|\widehat{S}_t\|_2^2\left(1 - \tfrac{2}{n}\right) + \left(2h(S_t) - \tfrac{2}{q}\right)n^{-1} + \left(2 - 2h(S_t)\right)n^{-2} + O\left(n^{-2}\right)\,,
\end{aligned}$$

where $h(s) \triangleq \sum_{k=1}^q g_\beta^k(s)s^k$. Notice that since $g_\beta^k(s) = g_\beta^k(s - \widehat{\mathsf{e}})$ we have that $h(s) = 1/q + h(\widehat{s})$ and its gradient and Hessian are:

$$\mathrm{D}_1 h(0) = \frac{1}{q} \quad \text{and} \quad \mathrm{D}_2 h(0) = \frac{4\beta}{q}P\,.$$

where P is a projection matrix onto $(\widehat{\mathsf{e}})^\perp$. Therefore, we may write

$$h(s) = \frac{1}{q} + \frac{2\beta}{q}\|\widehat{s}\|_2^2 + O\left(\|\widehat{s}\|_2^3\right)\,.$$

This gives

$$\begin{aligned}
\mathbb{E}\left[\|\widehat{S}_{t+1}\|_2^2 | \mathcal{F}_t\right] &= \|\widehat{S}_t\|_2^2\left(1 - \frac{2\left(1 - 2\beta/q\right)}{n}\right) + n^{-1}O\left(\|\widehat{S}_t\|_2^3\right) + O\left(n^{-2}\right) \\
&= \|\widehat{S}_t\|_2^2 p^2 + n^{-1}O\left(\|\widehat{S}_t\|_2^3\right) + O\left(n^{-2}\right)\,. \qquad (3.15)
\end{aligned}$$

where p is defined in (3.11).

3.6. **Contraction for the Distance to Equiproportionality.** Fix $\beta < q/2$ and $\rho < \rho_0$ where ρ_0 is given in Lemma 3.5. For what follows, assume that $\sigma_0 \in \Sigma_n^{\rho+}$ and $t \leqslant e^{\gamma_0 n}$ where γ_0 is also given. Then, taking expectation in equation (3.15), we get:

$$\mathbb{E}_{\sigma_0}\|\widehat{S}_{t+1}\|_2 = p^2\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2 + \left(\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^3\right)O\left(n^{-1}\right) + O\left(n^{-2}\right)\,. \qquad (3.16)$$

Now by Taylor expansion of $s \mapsto \|s\|_2^3$ around $\mathbb{E}_{\sigma_0}\widehat{S}_t$ in view of (3.14),

$$\begin{aligned}\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^3 &= \|\mathbb{E}_{\sigma_0}\widehat{S}_t\|_2^3 + \mathbb{E}_{\sigma_0}\left\langle \mathrm{D}_1\|\cdot\|_2^3\left(\mathbb{E}_{\sigma_0}\widehat{S}_t\right), \widehat{S}_t - \mathbb{E}_{\sigma_0}\widehat{S}_t\right\rangle + O\left(\mathbb{E}_{\sigma_0}\|\widehat{S}_t - \mathbb{E}_{\sigma_0}\widehat{S}_t\|_2^2\right)\\ &= \|\mathbb{E}_{\sigma_0}\widehat{S}_t\|_2^3 + O\left(n^{-1}\right) = \left(\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2 + O\left(n^{-1}\right)\right)^{3/2} + O\left(n^{-1}\right)\\ &= \left(\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2\right)^{3/2} + O\left(n^{-1}\right).\end{aligned}$$

Then

$$\mathbb{E}_{\sigma_0}\left[\|\widehat{S}_{t+1}\|_2^2\right] = p^2\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2 + \left(\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2\right)^{3/2} O\left(n^{-1}\right) + O\left(n^{-2}\right). \tag{3.17}$$

This will in turn imply:

Proposition 3.7. *Fix $\beta < q/2$. There exist $\rho_0 = \rho_0(\beta, q) > 0$ and $C = C(\beta, q) > 0$ such that if $\rho \leqslant \rho_0$ there exists $\gamma(\rho) > 0$ such that:*

$$\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2 = p^{2t}(\|\widehat{s}_0\|_2^2 + C\rho^3) + O\left(n^{-1}\right), \tag{3.18}$$

uniformly in $\sigma_0 \in \Sigma_n^{\rho+}$ and $t \leqslant e^{\gamma(\rho)n}$, where $p = p(n, \beta, q)$ is defined in 3.11.

Proof. Set $\lambda_t \triangleq \mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2$. It follows from (3.17) and (3.5) that for any $\rho < \overline{\rho} < \rho_0$, where ρ_0 is given in Lemma 3.5, there exists $\gamma = \gamma(\rho, \overline{\rho}) > 0$ such that uniformly in $\sigma_0 \in \Sigma_n^{\rho+}$ and $t \leqslant e^{\gamma n}$:

$$\lambda_{t+1} \leqslant \lambda_t\left(p^2 + \overline{\rho}O\left(n^{-1}\right)\right) + O\left(n^{-2}\right) = \lambda_t\left(p + \overline{\rho}O\left(n^{-1}\right)\right)^2 + O\left(n^{-2}\right).$$

We now use the following fact, which can be easily verified. If $(\lambda_t)_{t\geqslant 0}$ is a sequence satisfying:

$$\lambda_{t+1} = p\lambda_t + ar^t + b,$$

for some $p \neq r$, $p \neq 1$, a and b, then:

$$\lambda_t = \lambda_0 p^t + a\frac{p^t - r^t}{p - r} + \frac{b}{1-p}\left(1 - p^t\right). \tag{3.19}$$

Apply this (with $a = 0$) and use the monotonicity in λ_t of the right hand side above (at least if n is large enough), to conclude:

$$\lambda_t \leqslant \lambda_0\left(p + \overline{\rho}O(n^{-1})\right)^{2t} + \tfrac{1}{1-(p+\overline{\rho}O(n^{-1}))^2}O\left(n^{-2}\right) \leqslant C\overline{\rho}^2(p + \overline{\rho}O(n^{-1}))^{2t} + O\left(n^{-1}\right).$$

Plugging this a priori bound back into (3.17) we see that:

$$\lambda_{t+1} = p^2\lambda_t + \overline{\rho}^3\left(p + \overline{\rho}O(n^{-1})\right)^{3t} O\left(n^{-1}\right) + O\left(n^{-2}\right).$$

Using (3.19) again and choose $\overline{\rho}$ small enough to obtain

$$\begin{aligned}\lambda_t &= \lambda_0 p^{2t} + \frac{p^{2t} - \left(p + \overline{\rho}O(n^{-1})\right)^{3t}}{p^2 - (p + \overline{\rho}O(n^{-1}))^3}\overline{\rho}^3 O\left(n^{-1}\right) + \tfrac{1}{1-p^2}O\left(n^{-2}\right)\\ &= (\lambda_0 + O(\overline{\rho}^3))p^{2t} + O\left(n^{-1}\right).\end{aligned}$$

as desired. □

4. Mixing in the Subcritical Regime

In this section we prove Theorem 1. Recall that $\alpha_1 = \alpha_1(\beta, q) = \frac{1}{2(1-2\beta/q)}$ and set

$$t^{\alpha_1}(n) = \alpha_1 n\log n \quad ; \quad t_\gamma^{\alpha_1}(n) = \alpha_1 n\log n + \gamma n. \tag{4.1}$$

4.1. **Proof of Lower Bound in Theorem 1.**

Proof. The analysis in this subsection pertains to all $\beta < \beta_c(q)$. Fix $0 < \rho_2 < \rho_1 < \rho_0$, where ρ_0 is given in Proposition 3.7 and let $\sigma_0 \in \Sigma_n$ be such that $\rho_2 < \|\widehat{s}_0\|_2 < \rho_1$. Then if $t < t_\gamma^{\alpha_1}(n)$ and ρ_1 is small enough, Proposition 3.7 implies

$$\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2^2 \geqslant \frac{\rho_2^2}{2}\big(1 - \tfrac{1-2\beta/q}{n}\big)^{2t_\gamma^{\alpha_1}(n)} + O\left(n^{-1}\right) \geqslant \frac{1}{n}e^{-(1-2\beta/q)\gamma}\,,$$

for sufficiently large $-\gamma$ depending on ρ_2 and large enough n. Combined with the uniform variance bound given in Lemma 3.5, it follows that for large enough n

$$\mathbb{E}_{\sigma_0}\|\widehat{S}_t\|_2 \geqslant \frac{\mathrm{e}^{-(1-2\beta/q)\gamma/2}}{\sqrt{n}}\,.$$

Applying Chebyshev's inequality and using Lemma 3.5 again, we conclude that uniformly in all $r > 0$, $t \leqslant t_\gamma^{\alpha_1}(n)$ and $\sigma_0 \in \Sigma_n^{\rho_1+} \setminus \Sigma_n^{\rho_2+}$

$$\mathbb{P}_{\sigma_0}\left(\|\widehat{S}_t\|_2 < \frac{r}{\sqrt{n}}\right) \leqslant \frac{\mathbb{V}\mathrm{ar}_{\sigma_0}(\widehat{S}_t)}{\big(\frac{\mathrm{e}^{-(1-2\beta/q)\gamma/2}}{\sqrt{n}} - \frac{r}{\sqrt{n}}\big)^2} = O\Big((\mathrm{e}^{-(1-2\beta/q)\gamma/2} - r)^{-2}\Big)\,. \tag{4.2}$$

In particular, this implies

$$\lim_{\gamma\to-\infty}\limsup_{n\to\infty}\mathbb{P}_{\sigma_0}\left(\|\widehat{S}_{t_\gamma^{\alpha_1}(n)}\|_2 < \frac{r}{\sqrt{n}}\right) = 0. \tag{4.3}$$

On the other hand, $\mathbb{E}_{\mu_n}S_t = \widehat{\mathsf{e}}$ and from Corollary 3.6 it follows that $\mathbb{V}\mathrm{ar}_{\mu_n}S_t = O\left(n^{-1}\right)$ for $\beta < \beta_c(q)$. Therefore another application of Chebyshev's inequality yields that

$$\mu_n\left(\|\widehat{s}_t\|_2 < \frac{r}{\sqrt{n}}\right) \geqslant 1 - \frac{O(1)}{r^2}\,, \tag{4.4}$$

for all $t \geqslant 0$. Altogether, we have that for any $r > 0$,

$$\lim_{\gamma\to-\infty}\liminf_{n\to\infty} d_{t_\gamma^{\alpha_1}(n)}(n) \geqslant 1 - \frac{O(1)}{r^2}$$

and it remains to send $r \to \infty$. □

In the remainder of the section we prove the upper bound on the mixing time when $\beta < \beta_s(q)$. The proof is based on upper bounding the coalescence time of two coupled dynamics, one starting from any configuration in Σ_n and the other starting from the stationary distribution μ_n. This coupling will be done in several stages with different couplings from one stage to the next. In what follows, $(\sigma_t)_{t\geqslant 0}$ and $(\widetilde{\sigma})_{t\geqslant 0}$ will denote the two coupled processes.

4.2. $O(n^{-1/2})$ **from Coalescence.** We now show that with arbitrarily high probability, S_t gets $O(n^{-1/2})$-close to $\widehat{\mathsf{e}}$ in $O(n\log n)$ steps, if initially its distance is at most ρ, where ρ is small enough. More precisely,

Lemma 4.1. *Fix $\beta < q/2$. Then for all $r > 0$:*

$$\mathbb{P}_{\sigma_0}\left(S_{t^{\alpha_1}(n)} \notin \mathcal{S}^{\frac{r}{\sqrt{n}}}\right) = O(r^{-1})\,,$$

uniformly in $\sigma_0 \in \Sigma_n^{\rho_0+}$ where $\rho_0 = \rho_0(\beta, q)$ is defined in Proposition 3.7 and $t^{\alpha_1}(n)$ is defined in (4.1).

Proof. This follows immediately from Proposition 3.7 and a first moment argument:

$$\begin{aligned}\mathbb{P}_{\sigma_0}\big(S_{t^{\alpha_1}(n)} \notin \mathcal{S}^{\frac{r}{\sqrt{n}}}\big) &\leqslant \mathbb{P}_{\sigma_0}\big(\|\widehat{S}_{t^{\alpha_1}(n)}\|_2 \geqslant rn^{-\frac{1}{2}}\big) \\ &\leqslant \frac{\mathbb{E}_{\sigma_0}\|\widehat{S}_{t^{\alpha_1}(n)}\|_2}{rn^{-1/2}} \leqslant \frac{\big(\mathbb{E}_{\sigma_0}\|\widehat{S}_{t^{\alpha_1}(n)}\|_2^2\big)^{1/2}}{rn^{-1/2}} = O\big(\tfrac{1}{r}\big)\,. \qquad \square\end{aligned}$$

4.3. $O(n^{-1})$ **from Coalescence.** To get the correct order of the mixing time it is not sufficient to simply use the drift to couple the chains as the drift is very weak when S_t is close to $\widehat{\mathsf{e}}$. As such, in this section we define a different coupling of the dynamics which will bring σ_t and $\widetilde{\sigma}_t$ to distance $O(n^{-1})$ apart in linear time. This will be achieved one coordinate after the other. We begin by giving a general definition of, what we call, a semi-independent coupling and then use it to define the coupling of the dynamics.

Let ν, $\widetilde{\nu}$ be two positive distributions on $\Omega_m = [1, m]$ and fix a non-empty $A \subseteq \Omega_m$, where m is some positive integer. We shall write $\nu|_A$ for the conditional distribution given A, i.e. $\nu|_A(x) = \nu(x)/\nu(A)$, for $x \in A$. The *A-semi-independent coupling* of ν, $\widetilde{\nu}$ is a coupling of two random variables X and $\widetilde{X}$ with underlying measure $\mathbb{P}^\star$, constructed according to the following procedure:

(1) Choose $U \in [0, 1]$ uniformly.
(2) If $U \leqslant \min\{\nu(A), \widetilde{\nu}(A)\}$, draw X and $\widetilde{X}$ using a best coupling of $(\nu|_A, \widetilde{\nu}|_A)$.
(3) Otherwise, independently:
  (a) Draw X according to $\nu|_A$ if $U < \nu(A)$ and according to $\nu|_{A^c}$ if $U \geqslant \nu(A)$.
  (b) Draw $\widetilde{X}$ according to $\widetilde{\nu}|_A$ if $U < \widetilde{\nu}(A)$ and according to $\widetilde{\nu}|_{A^c}$ if $U \geqslant \widetilde{\nu}(A)$.

Clearly a Ω_m-semi-independent coupling is a best coupling and for $A = \emptyset$, we define $\emptyset$-semi-independent coupling to be the standard independent coupling. The following proposition states a few properties of this coupling, which will be useful for the sequel.

Proposition 4.2. *The following holds for the A-semi-independent coupling of $(\nu, \widetilde{\nu})$:*

(1) X, $\widetilde{X}$ are distributed according to ν, $\widetilde{\nu}$ respectively.
(2) $\mathbb{P}^\star\left(\cup_{x\in A}\{X = x\}\Delta\{\widetilde{X} = x\}\right) \leqslant \frac{3}{2}\sum_{x\in A}|\nu(x) - \widetilde{\nu}(x)|$
(3) $\forall x \notin A,\ \mathbb{P}^\star\left(X = x,\ \widetilde{X} \neq x\right) \geqslant \nu(x)\widetilde{\nu}(A^c \setminus \{x\})$ *and*
$\forall x \notin A,\ \mathbb{P}^\star\left(\widetilde{X} = x,\ X \neq x\right) \geqslant \widetilde{\nu}(x)\nu(A^c \setminus \{x\})$.

Proof. Part one of the Lemma is immediate. Part (2) follows from a straight forward calculation:

$$\begin{aligned}\mathbb{P}^\star\left(\cup_{x\in A}\{X = x\}\Delta\{\widetilde{X} = x\}\right) &\leqslant \mathbb{P}^\star\left(U \leqslant \nu(A)\wedge\widetilde{\nu}(A)\right)\mathbb{P}^\star\left(X \neq \widetilde{X}\,|U \leqslant \nu(A)\wedge\widetilde{\nu}(A)\right) + \\ &\qquad \mathbb{P}^\star\left(\nu(A)\wedge\widetilde{\nu}(A) < U < \nu(A)\vee\widetilde{\nu}(A)\right) \\ &\leqslant (\nu(A)\wedge\widetilde{\nu}(A))\tfrac{1}{2}\sum_{x\in A}|(\nu|_A(x) - \widetilde{\nu}|_A(x))| + |\nu(A) - \widetilde{\nu}(A)| \\ &\leqslant \tfrac{3}{2}\sum_{x\in A}|\nu(x) - \widetilde{\nu}(x)|\,.\end{aligned}$$

As for part (3), we have:

$$\begin{aligned}\mathbb{P}^\star\left(\{X = x\}\setminus\{\widetilde{X} = x\}\right) &\geqslant \mathbb{P}^\star\left(U > \nu(A)\vee\widetilde{\nu}(A)\right)\nu|_{A^c}(x)(1 - \widetilde{\nu}|_{A^c}(x)) \\ &\geqslant \nu(x)\widetilde{\nu}(A^c\setminus\{x\})\end{aligned}$$

and similarly for $\mathbb{P}^\star(\widetilde{X} = x, X \neq x)$. $\square$

We are now ready to define the coupling of σ_t, $\widetilde{\sigma}_t$ for this section. Fix $\sigma_0, \widetilde{\sigma}_0 \in \Sigma_n$ and $y_1, \dots y_{q-1} > 0$. The *coordinate-wise coupling* with parameters $y_1, \dots, y_{q-1}$ and starting configurations σ_0, $\widetilde{\sigma}_0$ is defined as follows.

(1) Set $T^{(0)} = 0$, $k = 1$.
(2) As long as $k \leqslant q-1$:
  (a) As long as $\left|S_t^k - \widetilde{S}_t^k\right| > \frac{y_k}{n}$:
    (i) Draw I_{t+1}, $\widetilde{I}_{t+1}$, using a $\{1, \dots, k-1\}$-semi-independent coupling of S_t, $\widetilde{S}_t$.
    (ii) Draw J_{t+1}, $\widetilde{J}_{t+1}$, using a $\{1, \dots, k-1\}$-semi-independent coupling of $g_\beta\left(S_t - \frac{1}{n}\mathsf{e}_{I_{t+1}}\right)$, $g_\beta\left(\widetilde{S}_t - \frac{1}{n}\mathsf{e}_{\widetilde{I}_{t+1}}\right)$.
    (iii) Change a uniformly chosen vertex of color I_{t+1} in σ_t to have color J_{t+1} in σ_{t+1}.
    (iv) Change a uniformly chosen vertex of color $\widetilde{I}_{t+1}$ in $\widetilde{\sigma}_t$ to have color $\widetilde{J}_{t+1}$ in $\widetilde{\sigma}_{t+1}$.
    (v) Set $t = t+1$.
  (b) When $\left|S_t^k - \widetilde{S}_t^k\right| \leqslant \frac{y_k}{n}$ set $T^{(k)} = t - T^{(k-1)}$ and $k = k+1$.
(3) Set $T^{CC} = \sum_{k=1}^{q-1} T^{(k)}$.

We shall use $\mathbb{P}^{CC}_{\sigma_0,\widetilde{\sigma}_0}$ to denote the probability measure for this coupling and $\mathbb{P}^{CC(m)}_{\sigma_0,\widetilde{\sigma}_0}$ for the same coupling, only with $k = m$ instead of $k = 1$ in step (1), i.e. starting from the m-th stage. Notice that, in principle, the stopping condition at stage k, may never get satisfied, in which case we stay at that stage forever and $T^{(k)} = T^{CC} = \infty$.

For $u, r > 0$, define

$$\mathcal{H}^k_{u,r} = \left\{(\sigma, \widetilde{\sigma}) \in \Sigma_n \times \Sigma_n : \ \|s^{[1,k]} - \widetilde{s}^{[1,k]}\|_1 < \frac{u}{n}, \max\left(\|s - \widehat{\mathsf{e}}\|_2, \|\widetilde{s} - \widehat{\mathsf{e}}\|_2\right) < \frac{r}{\sqrt{n}}\right\} .$$

where above $(s, \widetilde{s}) = (S(\sigma), S(\widetilde{\sigma}))$. Finally, set $\mathcal{H}_{u,r} \triangleq \mathcal{H}^q_{u,r}$. The following lemma will be the main ingredient in an inductive proof for an upper bound on T^{CC}:

Lemma 4.3. *Fix $\beta < q/2$. Let $k \in [1, q-1]$. For all $u_{k-1}, r_{k-1}, \epsilon > 0$, there exist $y_k, u_k, r_k, \gamma_k > 0$, such that if $(\sigma_0, \widetilde{\sigma}_0) \in \mathcal{H}^k_{u_{k-1},r_{k-1}}$ then*

$$\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(T^{(k)} < \gamma_k n, \ (\sigma_{T^{(k)}}, \widetilde{\sigma}_{T^{(k)}}) \in \mathcal{H}^k_{u_k,r_k}\right) \geqslant 1 - \epsilon . \tag{4.5}$$

Proof. Recall the expression for the drift of one coordinate (3.1). Near $\widehat{\mathsf{e}}$, this becomes by Taylor expansion for any $i \in [1, q]$:

$$\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left[S^i_{t+1} - S^i_t \middle| \mathcal{F}_t\right] = \frac{1}{n}\left\{-\left(1 - \frac{2\beta}{q}\right)\widehat{S}^i_t + O\left(\|\widehat{S}_t\|_2^2\right)\right\} + O\left(n^{-2}\right) .$$

and if $W_t = S_t - \widetilde{S}_t$ then

$$\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}[W^k_{t+1} - W^k_t \mid S_t, \widetilde{S}_t] = \frac{1}{n}\left\{-\left(1 - \frac{2\beta}{q}\right)W^k_t + O\left(\|S_t - \widehat{\mathsf{e}}\|_2^2 + \|\widetilde{S}_t - \widehat{\mathsf{e}}\|_2^2\right)\right\} + O\left(\frac{1}{n^2}\right) . \tag{4.6}$$

Now, for some $r_k > 0$ to be chosen later, let $\tau^{(k)} = \inf\left\{t : \ \|S_t - \widehat{\mathsf{e}}\|_2 \vee \|\widetilde{S}_t - \widehat{\mathsf{e}}\|_2 \geqslant \frac{r_k}{\sqrt{n}}\right\}$. Then, from (4.6) it follows that there exists $y_k > 0$ such that $\overline{W}^k_t \triangleq \left|W^k_{t\wedge\tau^{(k)}\wedge T^{(k)}}\right|$ is a

supermartingale. Clearly $\left|\overline{W}^k_{t+1} - \overline{W}^k_t\right| < \frac{2}{n}$. Also, from Proposition 4.2, if $t < \tau^{(k)} \wedge T^{(k)}$:

$$\begin{aligned}
\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(\overline{W}^k_{t+1} \neq \overline{W}^k_t \middle| \mathcal{F}_t\right) &\geqslant \mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(I_{t+1} = k,\, J_{t+1} \neq k,\, \widetilde{I}_{t+1} \neq k \middle| \mathcal{F}_t\right) \\
&\geqslant S^k_t \widetilde{S}^{k+1}_t \left(1 - g^k_\beta\left(S_t - \tfrac{1}{n}\mathsf{e}_k\right)\right) \\
&= \frac{q-1}{q^3} + O\left(n^{-\frac{1}{2}}\right),
\end{aligned}$$

which implies that $\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left[\left(\overline{W}^k_{t+1} - \overline{W}^k_t\right)^2 \middle| \mathcal{F}_t\right] \geqslant \frac{1}{2}q^{-2}n^{-2} + O\left(n^{-5/2}\right)$. On the other hand, in view of (4.6), $\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left[\overline{W}^k_{t+1} - \overline{W}^k_t \middle| \mathcal{F}_t\right] = O(n^{-3/2})$. Combining the two bounds, we infer that there exists $C > 0$, which doesn't depend on r_k or y_k, such that on $\left\{t < \tau^{(k)} \wedge T^{(k)}\right\}$

$$\mathbb{V}\mathrm{ar}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(\overline{W}^k_{t+1} \middle| \mathcal{F}_t\right) \geqslant Cn^{-2}, \tag{4.7}$$

for n sufficiently large. We now apply Lemma 2.3 with $Z_t = \overline{W}^k_t$, $z_0 = \frac{2r_{k-1}}{\sqrt{n}}$ and $N = \tau^{(k)} \wedge T^{(k)}$. This gives for $\gamma_k > 0$:

$$\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(T^{(k)} \wedge \tau^{(k)} > \gamma_k n\right) \leqslant \frac{Cr_{k-1}}{\sqrt{\gamma_k}},$$

whence we may choose $\gamma_k = \gamma_k(r_{k-1}, \epsilon)$ independently of r_k, y_k but sufficiently large, such that

$$\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(T^{(k)} \wedge \tau^{(k)} > \gamma_k n\right) \leqslant \frac{\epsilon}{3}. \tag{4.8}$$

This gives an upper bound on $T^{(k)}$, since by Proposition 3.3 Part (2) we may choose r_k large enough such that:

$$\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(\tau^{(k)} < \gamma_k n\right) \leqslant \frac{\epsilon}{3}. \tag{4.9}$$

It remains to ensure that we do not increase the distances in the first $k-1$ coordinates, by too much. Proposition 4.2 implies that for any t:

$$\begin{aligned}
&\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(\overline{W}^{[1,k-1]}_{t+1} \neq \overline{W}^{[1,k-1]}_t \middle| \mathcal{F}_t\right) \\
&\leqslant \sum_{l \leqslant k-1} \mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(\left(\{I_{t+1} = l\} \,\Delta\, \left\{\widetilde{I}_{t+1} = l\right\}\right) \cup \left(\{J_{t+1} = l\} \,\Delta\, \left\{\widetilde{J}_{t+1} = l\right\}\right) \middle| \mathcal{F}_t\right) \\
&\leqslant \frac{3}{2}\sum_{l \leqslant k-1}\left(|\overline{W}^l_t| + |g^l_\beta(S_t) - g^l_\beta(\widetilde{S}_t)|\right) + O(n^{-1}) \\
&\leqslant C\|\overline{W}^{[1,k-1]}_t\|_1 + r_k^2 O\left(n^{-1}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left[\|\overline{W}^{[1,k-1]}_{t+1}\|_1 \middle| \mathcal{F}_t\right] &\leqslant \|\overline{W}^{[1,k-1]}_t\|_1 + \frac{C_1}{n}\|\overline{W}^{[1,k-1]}_t\|_1 + r_k^2 O\left(n^{-2}\right) \\
&\leqslant \|\overline{W}^{[1,k-1]}_t\|_1\left(1 + \frac{C_1}{n}\right) + r_k^2 O\left(n^{-2}\right).
\end{aligned}$$

Taking expectation of both sides and using the assumption on $\|\overline{W}_0^{[1,k-1]}\|_1$, we have

$$\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\|\overline{W}^{[1,k-1]}_{\gamma_k n}\|_1 \leqslant \|\overline{W}^{[1,k-1]}_0\|_1 \left(1+\frac{C_1}{n}\right)^{\gamma_k n} \leqslant C_1 \frac{u_{k-1}}{n} e^{C_2\gamma_k}.$$

Hence by Markov's inequality, there exists $u_k > 0$ such that

$$\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left(\|\overline{W}^{[1,k-1]}_{\gamma_k n}\|_1 > \frac{u_k - y_k}{n}\right) \leqslant \frac{\epsilon}{3}.$$

Combined with (4.8) and (4.9), the proof is complete. □

Corollary 4.4. *Fix $\beta < q/2$. For any $\epsilon, r > 0$, there exist $\gamma, u, r' > 0$ and $y_1, \dots y_{q-1} > 0$ such that for $\sigma_0, \widetilde{\sigma}_0 \in \Sigma_n^{\frac{r}{\sqrt{n}}}$.*

$$\mathbb{P}^{CC}_{\sigma_0,\widetilde{\sigma}_0}\left(T^{CC} < \gamma n,\ (\sigma_{T^{CC}}, \widetilde{\sigma}_{T^{CC}}) \in \mathcal{H}_{u,r'}\right) \geqslant 1-\epsilon.$$

Proof. Starting from $r_0 = u_0 = r$ and applying Lemma 4.3 inductively, we obtain for some $(y_k, u_k, r_k, \gamma_k)_{k\in[1,q-1]}$:

$$\mathbb{P}^{CC}_{\sigma_0,\widetilde{\sigma}_0}(T^{(k)} < \gamma_k n, (\sigma_{T^{(k)}}, \widetilde{\sigma}_{T^{(k)}}) \in \mathcal{H}^k_{u_k,r_k}\ \forall k \in [1, q-1]) \geqslant 1-\epsilon\,.$$

It remains to set $\gamma = \sum_{k=1}^{q-1}\gamma_k$, $r' = r_{q-1}$ and $u = 2u_{q-1}$. □

4.4. **Coalescence of Proportions Vector Chains.** The next lemma completes the coupling of the proportions chains.

Lemma 4.5. *Fix $\beta < q/2$. For all $r, u, \epsilon > 0$ there exists $\gamma > 0$ such that if $\sigma_0, \widetilde{\sigma}_0 \in \Sigma_n$ satisfy $(\sigma_0, \widetilde{\sigma}_0) \in \mathcal{H}_{u,r}$ and $t \geqslant \gamma n$, then*

$$\mathbb{P}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left(S_t = \widetilde{S}_t\right) \geqslant 1-\epsilon\,.$$

where under $\mathbb{P}^{SC}_{\sigma_0,\widetilde{\sigma}_0}$, the processes $(\sigma_t)_{t\geqslant 0}$, $(\widetilde{\sigma}_t)_{t\geqslant 0}$ evolve according to the synchronized coupling, as defined in Subsection 3.3.

Proof. If ρ is small enough, it follows from Lemma 3.4 that

$$\mathbb{E}^{\mathrm{SC},\rho}_{\sigma_0,\widetilde{\sigma}_0}\|S_t - \widetilde{S}_t\|_1 \leqslant \left(1 - \frac{1-2\beta/q}{2n}\right)^t \frac{u}{n}\,,$$

Combined with Proposition 3.3 Part (1), this implies that there exists $\gamma = \gamma(u)$ such that $\mathbb{E}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\|S_t - \widetilde{S}_t\|_1 \leqslant \frac{\epsilon}{n}$ for $t \geqslant \gamma n$. Then by Markov's Inequality:

$$\mathbb{P}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left(S_t \neq \widetilde{S}_t\right) = \mathbb{P}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left(\|S_t - \widetilde{S}_t\|_1 > \tfrac{1}{n}\right) \leqslant \epsilon\,.$$ □

4.5. **Basket-wise Proportions Coalescence.** The next coupling will allow us to turn a well-mixed proportions chains into a well-mixed configurations chain. Let $\mathfrak{B} = (\mathcal{B}_m)_{m=1}^q$ be a partition of $[1, n]$. We shall refer to $\mathcal{B}_m$ as a *basket* and call $\mathfrak{B}$ a λ-partition if $|\mathcal{B}_m| > \lambda n$ for all m. Given $\sigma \in \Sigma_n$, let $\mathbf{S}(\sigma)$ denote a $q \times q$ matrix whose (m, k) entry is equal to the proportion in σ of color k in basket m, namely

$$\mathbf{S}^{m,k}(\sigma) = \frac{1}{|\mathcal{B}_m|}\sum_{v\in\mathcal{B}_m} \mathbb{1}_{\{\sigma(v)=k\}}\,.$$

$\mathbf{S}$ is an element of $\mathbb{S} \triangleq \prod_{m=1}^q \mathcal{S}$ and we define $\mathbb{S}^\rho = \prod_{m=1}^q \mathcal{S}^\rho$ and $\mathbb{S}^{\rho+} = \prod_{m=1}^q \mathcal{S}^{\rho+}$. We also let $\mathcal{B}_{[m_0,m_1]} = \cup_{m_0 \leqslant m \leqslant m_1}\mathcal{B}_m$ and as before use $\mathbf{S}_t$ as a shorthand for $\mathbf{S}(\sigma_t)$. The following is an analogue of Lemma 4.1 for the basket proportions matrix.

Lemma 4.6. *Let $\mathfrak{B}$ be a λ-partition for some $\lambda > 0$. If either of the following holds:*

(1) $\sigma_0 \in \Sigma_n^{\rho_0}$ and $t^{\alpha_1}(n) \leqslant t \leqslant e^{\gamma(\rho_0)n}$ where ρ_0, $\gamma(\rho_0)$ are given in Proposition 3.7 and $t^{\alpha_1}(n)$ is defined in (4.1).

(2) $\mathbf{S}_0 \in \mathbb{S}^{\frac{r_0}{\sqrt{n}}}$ and $t \leqslant \gamma_0 n$ for some $r_0, \gamma_0 > 0$,

then

$$\mathbb{P}_{\sigma_0}\left(\mathbf{S}_t \notin \mathbb{S}^{\frac{r}{\sqrt{n}}}\right) = O(r^{-2})\,.$$

where the $O(r^{-2})$ term is as $r \to \infty$, uniformly in n.

In order to prove Lemma 4.6, we use the following proposition to bound the second moment of the basket proportions matrix.

Proposition 4.7. *If $\mathfrak{B}$ is a λ-partition for $\lambda > 0$ and $m, k \in [1, q]$, then*

$$\mathbb{E}\left[(\mathbf{S}_{t+1}^{m,k} - S_{t+1}^k)^2 \mid \mathcal{F}_t\right] = \left(1 - \frac{2}{n}\right)(\mathbf{S}_t^{m,k} - S_t^k)^2 + O\left(\frac{1}{n^2}\right)\,. \tag{4.10}$$

Proof. Let $\lambda_0 = |\mathcal{B}_m|/n \geqslant \lambda > 0$ and set $\mathbf{Q}_t^{m,k} = \mathbf{S}_t^{m,k} - S_t^k$. Then:

$$\begin{aligned}
&\mathbb{E}\left[(\mathbf{Q}_{t+1}^{m,k})^2 - (\mathbf{Q}_t^{m,k})^2) \mid \mathcal{F}_t\right] \\
&= p_1((\mathbf{Q}_t^{m,k} + \tfrac{1}{n})^2 - (\mathbf{Q}_t^{m,k})^2) + p_2((\mathbf{Q}_t^{m,k} - \tfrac{1}{n})^2 - (\mathbf{Q}_t^{m,k})^2) + \\
&\qquad p_3((\mathbf{Q}_t^{m,k} - \tfrac{1}{\lambda_0 n} + \tfrac{1}{n})^2 - (\mathbf{Q}_t^{m,k})^2) + p_4((\mathbf{Q}_t^{m,k} + \tfrac{1}{\lambda_0 n} - \tfrac{1}{n})^2 - (\mathbf{Q}_t^{m,k})^2) \\
&= \frac{2}{n}\mathbf{Q}_t^{m,k}\left((p_1 - p_2) + \left(1 - \tfrac{1}{\lambda_0}\right)(p_3 - p_4)\right) + O\left(\tfrac{1}{n^2}\right)\,,
\end{aligned} \tag{4.11}$$

where (denoting by V_{t+1} the chosen vertex at step $t+1$):

$$\begin{aligned}
p_1 &= \mathbb{P}(V_{t+1} \notin \mathcal{B}_m,\, \sigma_t(V_{t+1}) = k,\, \sigma_{t+1}(V_{t+1}) \neq k|\mathcal{F}_t) = (S_t^1 - \lambda_0 \mathbf{S}_t^{m,k})(1 - g_\beta^k(S_t)) + O(n^{-1}), \\
p_2 &= \mathbb{P}(V_{t+1} \notin \mathcal{B}_m,\, \sigma_t(V_{t+1}) \neq k,\, \sigma_{t+1}(V_{t+1}) = k|\mathcal{F}_t) = (1 - S_t^1 - \lambda_0 + \lambda_0 \mathbf{S}_t^{m,k}) g_\beta^k(S_t) + O(n^{-1}), \\
p_3 &= \mathbb{P}(V_{t+1} \in \mathcal{B}_m,\, \sigma_t(V_{t+1}) = k,\, \sigma_{t+1}(V_{t+1}) \neq k|\mathcal{F}_t) = \lambda_0 \mathbf{S}_t^{m,k}(1 - g_\beta^k(S_t) + O(n^{-1}), \\
p_4 &= \mathbb{P}(V_{t+1} \in \mathcal{B}_m,\, \sigma_t(V_{t+1}) \neq k,\, \sigma_{t+1}(V_{t+1}) = k|\mathcal{F}_t) = \lambda_0(1 - \mathbf{S}_t^{m,k}) g_\beta^k(S_t) + O(n^{-1}).
\end{aligned}$$

Plugging these into (4.11), we obtain

$$\mathbb{E}[(\mathbf{S}_{t+1}^{m,k} - S_{t+1}^k)^2 \mid \mathcal{F}_t] = (1 - \tfrac{2}{n})(\mathbf{S}_t^{m,k} - S_t^k)^2 + O(\tfrac{1}{n^2})$$

as required. □

Proof of Lemma 4.6. Taking expectation in (4.10) and applying (3.19) one gets:

$$\mathbb{E}_{\sigma_0}\left[\mathbf{S}_t^{m,k} - S_t^k\right]^2 \leqslant \left(1 - \frac{2}{n}\right)^t \mathbb{E}_{\sigma_0}\left[\mathbf{S}_0^{m,k} - S_0^k\right]^2 + O(n^{-1})\,.$$

In both cases (note that $\alpha_1(\beta, q) > \frac{1}{2}$ for all $\beta < q/2$), it implies $\mathbb{E}_{\sigma_0}\left[\mathbf{S}_t^{m,k} - S_t^k\right]^2 = O(n^{-1})$. Summing over all m and k and using Markov's Inequality we get

$$\mathbb{P}_{\sigma_0}\left(\sum_{m \leqslant q} \|\mathbf{S}_t^m - S_t\|_2 > r/(2\sqrt{n})\right) = O(r^{-2}).$$

Now, by Proposition 3.7 Case (1) and Proposition 3.3 Case (2) we also have that

$$\mathbb{P}_{\sigma_0}\left(\|S_t - \widehat{\mathsf{e}}\|_2 > r/(2\sqrt{n})\right) = O(r^{-2}).$$

Combining the two, we complete the proof. □

Suppose now that you have two initial configurations $\sigma_0, \widetilde{\sigma}_0$, such that $s_0 = \widetilde{s}_0$. The following is a coupling under which eventually (with probability 1) also $\mathbf{S}_t = \widetilde{\mathbf{S}}_t$. Equality is achieved one basket at a time, indexed below by m and once the proportions in a basket are equated they remain so. We shall call this coupling *Basket-wise Coupling* and denote by $\mathbb{P}^{BC}$ the underlying probability measure.

(1) Set $t = 0$, $m = 1$.
(2) As long as $m \leqslant q$:
  (a) As long as $\mathbf{S}_t^m \neq \widetilde{\mathbf{S}}_t^m$:
    (i) Choose "old" color I_{t+1} according to distribution $S_t = \widetilde{S}_t$.
    (ii) Choose "new" color J_{t+1} according to distribution $g_\beta\left(S_t - \frac{1}{n}\mathtt{e}_{I_{t+1}}\right) = g_\beta\left(\widetilde{S}_t - \frac{1}{n}\mathtt{e}_{I_{t+1}}\right)$.
    (iii) Choose a vertex V_{t+1} uniformly among all vertices in $[1, n]$ having color I_{t+1} under σ_t.
    (iv) Choose $\widetilde{V}_{t+1}$:
      (A) If $V_{t+1} \in \mathcal{B}_{m_0}$ for $m_0 < m$, choose $\widetilde{V}_{t+1}$ uniformly among all vertices in $\mathcal{B}_{m_0}$ having color I_{t+1} under $\widetilde{\sigma}_t$.
      (B) Otherwise, if $\mathbf{S}_t^{m,I_{t+1}} \neq \widetilde{\mathbf{S}}_t^{m,I_{t+1}}$ and $\mathbf{S}_t^{m,J_{t+1}} \neq \widetilde{\mathbf{S}}_t^{m,J_{t+1}}$, choose $\widetilde{V}_{t+1}$ uniformly among all vertices in $\mathcal{B}_{[m,q]}$ having color I_{t+1} under $\widetilde{\sigma}_t$.
      (C) Otherwise, let $v_1, v_2, \dots$ be an enumeration of the vertices in $\mathcal{B}_{[m,q]}$ having color I_{t+1} under σ_t ordered first by the index of the basket they belong to and then by their index in V and let $\widetilde{v}_1, \widetilde{v}_2, \dots$ be the same for $\widetilde{\sigma}_t$. Then set $\widetilde{V}_{t+1} = \widetilde{v}_i$ where i is such that $V_{t+1} = v_i$.
    (v) Set $\sigma_{t+1}(V_{t+1}) = \widetilde{\sigma}_{t+1}(\widetilde{V}_{t+1}) = J_{t+1}$ and $t = t + 1$.
  (b) Set $m = m + 1$.

The following lemma gives an upper bound for the time of basket-wise proportions coalescence.

Lemma 4.8. *Fix $\beta < q/2$. For any $\lambda > 0$, $r > 0$, $\epsilon > 0$, there exists $\gamma = \gamma(\lambda, r, \epsilon)$, such that for any λ-partition and any $\sigma_0, \widetilde{\sigma_0}$ such that $S_0 = \widetilde{S}_0$ and $\mathbf{S}_0, \widetilde{\mathbf{S}}_0 \in \mathbb{S}^{\frac{r}{\sqrt{n}}}$,*

$$\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\left(\mathbf{S}_{\gamma n} = \widetilde{\mathbf{S}}_{\gamma n}\right) \geqslant 1 - \epsilon\,. \tag{4.12}$$

Proof. From the definition of the coupling, once the proportions of basket m have coalesced they will remain equal forever. It suffices, therefore, to analyze the coalescence time of each basket separately. Note also that the coupling preserves the equality $S_t = \widetilde{S}_t$ for all $t \geqslant 0$.

Define $\mathbf{W}_t = \mathbf{S}_t - \widetilde{\mathbf{S}}_t$, $W_t^m = \|\mathbf{W}_t^m\|_1$ and let $\tau^{(0)} = 0$ and $\tau^{(m)} = \min\{t \geqslant \tau^{(m-1)} : W_t^m = 0\}$ for $m \in [1, q]$. Also set

$$\tau_* = \inf\{t : \mathbf{S}_t \notin \mathbb{S}^\rho \text{ or } \widetilde{\mathbf{S}}_t \notin \mathbb{S}^\rho\}$$

for $\rho > 0$ sufficiently small and $\tau_*^{(m)} = \tau^{(m)} \wedge \tau_*$. We claim that for all m, $(W_t^m)_{t \geqslant 0}$ is a supermartingale between $\tau^{(m-1)}$ and $\tau^{(m)}$ as long as τ_* is not reached. In order to see this, fix m, t and assume $\{\tau^{(m-1)} \leqslant t < \tau^{(m)}, \tau_* > t\}$. Then at step (2(a)iv), according to the coupling, there are 3 cases:

(A) Clearly $\mathbf{S}_{t+1}^m = \mathbf{S}_t^m$ and $\widetilde{\mathbf{S}}_{t+1}^m = \widetilde{\mathbf{S}}_t^m$ and hence $W_{t+1}^m = W_t^m$.

(B) Notice that in this case, we have $\mathbf{W}_t^{m,I_{t+1}}\mathbf{W}_{t+1}^{m,I_{t+1}} \geqslant 0$ and $\mathbf{W}_t^{m,J_{t+1}}\mathbf{W}_{t+1}^{m,J_{t+1}} \geqslant 0$. Therefore,

$$\begin{aligned}
&\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[W_{t+1}^m - W_t^m|\mathcal{F}_t] \\
&= \left|\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\left[\mathbf{W}_{t+1}^{m,I_{t+1}}|\mathcal{F}_t\right]\right| - \left|\mathbf{W}_t^{m,I_{t+1}}\right| + \left|\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\left[\mathbf{W}_{t+1}^{m,J_{t+1}}|\mathcal{F}_t\right]\right| - \left|\mathbf{W}_t^{m,J_{t+1}}\right| \\
&\leqslant \frac{-|\mathbf{W}_t^{m,I_{t+1}}|}{\sum_{m_0 \geqslant m}\mathbf{S}_t^{m_0,I_{t+1}}|\mathcal{B}_{m_0|}} + \frac{|\mathbf{W}_t^{m,I_{t+1}}|}{\sum_{m_0 \geqslant m}\mathbf{S}_t^{m_0,I_{t+1}}|\mathcal{B}_{m_0}|} = 0\,.
\end{aligned}$$

(C) If $V_{t+1}, \widetilde{V}_{t+1} \in \mathcal{B}_m$ or $V_{t+1}, \widetilde{V}_{t+1} \notin \mathcal{B}_m$, then $W_{t+1}^m = W_t^m$, otherwise from the construction we must have:

$$|\mathbf{W}_{t+1}^{m,I_{t+1}}| - |\mathbf{W}_t^{m,I_{t+1}}| = -\tfrac{1}{|\mathcal{B}_m|}\,,$$

as well as

$$|\mathbf{W}_{t+1}^{m,J_{t+1}}| - |\mathbf{W}_t^{m,J_{t+1}}| \leqslant \tfrac{1}{|\mathcal{B}_m|}\,.$$

Summing these two, we obtain a non-positive drift for W_t^m.

Observe that as long as τ_* is not reached, both $\mathbb{V}\mathrm{ar}^{BC}(W_{t+1}^m|\mathcal{F}_t)$ under case (B) and the probability that this case happens are bounded below uniformly in n and t. This gives a uniform lower bound on the variance $\mathbb{V}\mathrm{ar}^{BC}(W_{t+1}^m|\mathcal{F}_t)$. Furthermore, if for some t, we have $\mathbf{S}_t$, $\widetilde{\mathbf{S}}_t \in \mathbb{S}^{\frac{r'}{\sqrt{n}}}$, then in view of Lemma 4.6 after $\gamma' n$ time, we have $\mathbf{S}_{t+\gamma' n}$, $\widetilde{\mathbf{S}}_{t+\gamma' n} \in \mathbb{S}^{\frac{r''}{\sqrt{n}}}$ with probability $1-O((r'')^{-2})$. Therefore, using Lemma 2.3 we may find $\gamma_1,\dots,\gamma_{q-1}$ such that inductively, conditioned on $\tau_*^{(m-1)} \leqslant \gamma_{m-1}n$ with probability at least $1-\epsilon/(2q)$ we have $\tau_*^{(m)} \leqslant \gamma_m n$. This in turn implies that $\tau_*^{(q-1)} \leqslant \gamma n$ with probability at least $1-\epsilon/2$, where $\gamma \triangleq \gamma_{q-1}$.

It remains to bound τ_* below with high probability. Let $B^{m,j} = \bigcup_{t=1}^{\gamma n}\{|\mathbf{S}_t^{m,j} - 1/q| \geqslant \rho\}$ and

$$Y^{m,j} = \left|\{t:\ \left|\mathbf{S}_t^{m,j} - 1/q\right| \geqslant \rho/2,\ 1 \leqslant t \leqslant \gamma n\}\right|.$$

Using Lemma 4.6 we obtain that

$$\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[Y^{m,j}] \leqslant \gamma n O(\tfrac{1}{n}) = O(\gamma)\,.$$

Then as $B^{m,j}$ implies that $Y^{m,j} > \frac{n\lambda\rho}{2}$,

$$\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}(B^{m,j}) \leqslant \mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\left(Y^{m,j} \geqslant \frac{n\lambda\rho}{2}\right) \leqslant \frac{2\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[Y^{m,j}]}{n\lambda\rho} = O(n^{-1})\,.$$

Summing over all m, j and arguing the same for $\widetilde{\mathbf{S}}_t$ we obtain

$$\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}(\tau_* < \gamma n) = O(n^{-1})\,,$$

Finally by a union bound we have $\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}(\tau^{(q)} \leqslant \gamma n) \geqslant 1 - \frac{\epsilon}{2} + O(n^{-1})$ as desired. □

4.6. **The Overall Coupling.** We now describe precisely how the previous couplings are combined together to create the *overall coupling*. This coupling will be the main tool in proving the upper bound. Formally, let $\gamma_1, \gamma_3, \gamma_4, \gamma_5$ and $y_1,\dots,y_{q-1}$ be positive numbers and $\sigma_0 \in \Sigma_n$. The overall coupling with parameters $\gamma_1,\dots,\gamma_5$, $y_1,\dots y_{q-1}$ and initial configuration σ_0 is a coupling of two chains $(\sigma_t)_t$, $(\widetilde{\sigma}_t)_t$ under measure $\mathbb{P}^{OC}_{\sigma_0}$. The initial configuration for $(\sigma_t)_t$ is σ_0, while $\widetilde{\sigma}_0$ is chosen according to μ_n. Then, the two processes evolve as follows.

(1) Run σ_t and $\widetilde{\sigma}_t$ independently until time $t^{(1)}(n) = \gamma_1 n$.
   (1A) Partition the vertex set $[1, n]$ into baskets $\mathfrak{B} = (\mathcal{B}_1, \dots \mathcal{B}_q)$ such that $\mathcal{B}_k = \{v : \sigma_{t^{(1)}(n)}(v) = k\}$ for $k \in [1, q]$.
(2) Run σ_t and $\widetilde{\sigma}_t$ independently (again) until time $t^{(2)}(n) = t^{(1)} + t^{\alpha_1}(n)$ time where $t^{\alpha_1}(n)$ is defined in (4.1).
(3) Run σ_t and $\widetilde{\sigma}_t$ according to the coordinate-wise coupling with parameters $y_1, \dots, y_{q-1}$ until time $t^{(3)}(n) = t^{(2)}(n) + \gamma_3 n$ (unless stopped before).
(4) Run σ_t and $\widetilde{\sigma}_t$ according to the synchronized coupling until time $t^{(4)}(n) = t^{(3)}(n) + \gamma_4 n$ time.
(5) Run σ_t and $\widetilde{\sigma}_t$ according to the basket-wise coupling for $t^{(5)}(n) = t^{(4)}(n) + \gamma_5 n$ time with the baskets above.

4.7. **Proof of Upper Bound in Theorem 1.** We will now use the overall coupling with appropriate parameters to establish the upper bound of the mixing time. Recall that $\beta < \beta_s(q)$. Fix $\epsilon > 0$, pick $\rho > 0$ small enough and let σ_0 be any initial configuration. By Proposition 3.3 Part (3), we can choose γ_1 large enough such that

$$\mathbb{P}^{OC}_{\sigma_0}\left(S_{t^{(1)}(n)} \in \mathcal{S}^{\rho}\right) \geqslant 1 - \epsilon\,. \tag{4.13}$$

Assuming that this event indeed occurred, $\mathfrak{B}$ is a $(\frac{1}{q} - \rho)$-partition and provided that ρ is small enough, the conditions in Lemma 4.1 are satisfied. From the latter we conclude that for some $r > 0$, with probability at least $1 - \epsilon$, $S_{t^{(2)}(n)} \in \mathcal{S}^{\frac{r}{\sqrt{n}}}$. On the other hand, as in (4.4) with probability at least $1 - 2\epsilon$ we also have $\widetilde{S}_{t^{(2)}(n)} \in \mathcal{S}^{\frac{r}{\sqrt{n}}}$ if r is large enough. Then Corollary 4.4 and Lemma 4.5 ensure that there exist $y_1, \dots y_{q-1}$ and γ_3, γ_4, such that $S_{t^{(4)}(n)} = \widetilde{S}_{t^{(4)}(n)}$ with probability at least $1 - 3\epsilon$. From Lemma 4.6 we have that $\mathbf{S}_{t^{(4)}(n)}, \widetilde{\mathbf{S}}_{t^{(4)}(n)} \in \mathbb{S}^{\frac{r'}{\sqrt{n}}}$ with probability at least $1 - 4\epsilon$ for some $r' > 0$. Then, by Lemma 4.8 we may choose γ_5 such that $\mathbf{S}_{t^{(5)}(n)} = \widetilde{\mathbf{S}}_{t^{(5)}(n)}$ with probability at least $1 - 5\epsilon$.

Now, by symmetry, for any $t \geqslant t^{(1)}(n)$ the distribution of σ_t, given $\mathcal{F}_{t^{(1)}(n)}$, is invariant under permutations of the vertices in each basket of $\mathcal{B}$ and the same is clearly true for μ_n. Therefore we conclude that

$$\begin{aligned}
&\|\mathbb{P}^{OC}_{\sigma_0}\left(\sigma_{t^{(5)}(n)} \in \cdot \,\middle|\, \mathcal{F}_{t^{(1)}(n)}, S_{t^{(1)}(n)} \in \mathcal{S}^{\rho}\right) - \mu_n\|_{\mathrm{TV}} \\
&= \|\mathbb{P}^{OC}_{\sigma_0}\left(\mathbf{S}_{t^{(5)}(n)} \in \cdot \,\middle|\, \mathcal{F}_{t^{(1)}(n)}, S_{t^{(1)}(n)} \in \mathcal{S}^{\rho}\right) - \mu_n \circ \mathbf{S}^{-1}\|_{\mathrm{TV}} \\
&\leqslant \mathbb{P}^{OC}_{\sigma_0}\left(\mathbf{S}_{t^{(5)}(n)} \neq \widetilde{\mathbf{S}}_{t^{(5)}(n)} \,\middle|\, \mathcal{F}_{t^{(1)}(n)}, S_{t^{(1)}(n)} \in \mathcal{S}^{\rho}\right) \leqslant 5\epsilon\,.
\end{aligned}$$

Then from Jensen's inequality we obtain

$$\begin{aligned}
&\|\mathbb{P}_{\sigma_0}\left(\sigma_{t^{(5)}(n)} \in \cdot\right) - \mu_n\|_{\mathrm{TV}} \qquad (4.14)\\
&\leqslant \mathbb{E}^{OC}_{\sigma_0}\left[\|\mathbb{P}^{OC}_{\sigma_0}\left(\sigma_{t^{(5)}(n)} \in \cdot \,\middle|\, \mathcal{F}_{t^{(1)}(n)}\right) - \mu_n\|_{\mathrm{TV}} \,\middle|\, S_{t^{(1)}(n)} \in \mathcal{S}^{\rho}\right] + \mathbb{P}^{OC}_{\sigma_0}\left(S_{t^{(1)}(n)} \notin \mathcal{S}^{\rho}\right) \\
&\leqslant 5\epsilon + \epsilon = 6\epsilon.
\end{aligned}$$

Now $t^{(5)}(n) = t^{\alpha_1}_{\gamma}(n)$ (as defined in (4.1)) with $\gamma = \gamma_1 + \gamma_3 + \gamma_4 + \gamma_5$ and since σ_0 is arbitrary and ϵ can be made arbitrarily small, by choosing γ large enough, this establishes the upper bound for the cutoff. □

5. Mixing in the Supercritical Regime

5.1. **Proof of Theorem 3.** We first give the proof for the case $\beta_s < \beta < \beta_c$. Recall (Subsection 2.2) that $\beta_c(q) = \frac{(q-1)}{q-2}\log(q-1)$ for $q \geqslant 3$ and $\beta_c(2) = 1$. We claim that for $q \geqslant 3$

$$\beta_c(q)\big(1 - 1/q\big) < \beta_c(q-1)\,. \tag{5.1}$$

It can be checked for $q = 3$ and for $q \geqslant 4$, it suffices to prove that $f(x) := \frac{x-1}{x(x-2)}\log(x-1)$ is decreasing in x on $[3, \infty)$. We compute the derivative and obtain that

$$f'(x) = -\frac{1}{x(x-2)}\Big(\frac{x^2 - 2x + 2}{x(x-2)}\log(x-1) - 1\Big)\,,$$

which is negative for $x \geqslant 3$.

Now fix $\delta > 0$ and notice that if $s^1 \in [1/q, 1-\delta]$, conditional on $\{S^1 = s^1\}$, $(S^i/(1-s^1) : 2 \leqslant i \leqslant q)$ is distributed as the proportions vector for the $(q-1)$-states Curie-Weiss Potts model on $(1-s^1)n$ vertices with $\beta' = \beta(1-s^1) \leqslant (1-1/q)\beta \leqslant (1-1/q)\beta_c(q) < \beta_c(q-1)$. Therefore, for all $\delta_1 > 0$

$$\mu_n\Big(\exists 2 \leqslant i \leqslant q \text{ such that } \big|S^i - \frac{1-s^1}{q-1}\big| \geqslant \delta_1 \mid S^1 = s^1\Big) \to 0\,. \tag{5.2}$$

as $n \to \infty$ uniformly in $s^1 \in [1/q, 1-\delta]$. Also uniformly in $s \in \mathcal{S}_n$, recall that:

$$\mathbb{E}\left[\,S^1_{t+1} - S^1_t \big|\, S_t = s\right] = \frac{1}{n}d_\beta(s) + O\left(n^{-2}\right)\,,$$

where $d_\beta(s) = -s^1 + g^1_\beta(s)$. It now follows from the uniform continuity of $d_\beta(s)$ and (5.2) that uniformly in $s^1 \in [1/q, 1-\delta]$,

$$\mathbb{E}_{\mu_n}[S^1_{t+1} - S^1_t \mid S^1_t = s^1] = \tfrac{1}{n}(D_\beta(s^1) + o(1))\,,$$

where $D_\beta(s^1) = d_\beta\left(s^1, \frac{1-s^1}{q-1}, \ldots, \frac{1-s^1}{q-1}\right)$.

Now if $\beta > \beta_s(q)$, from Proposition 3.1, there exists δ_2, such that $D_\beta(s^1)$ is uniformly positive in a δ_2-neighborhood of $s^*(\beta)$. All together we infer that there exists $\epsilon > 0$ such that uniformly in $s^1 \in (s^*(\beta) - \delta_2, s^*(\beta) + \delta_2)$ for all n large enough:

$$\mathbb{E}_{\mu_n}[S^1_{t+1} - S^1_t \mid S^1_t = s^1] \geqslant \frac{\epsilon}{n}\,,$$

and also

$$\mathbb{P}_{\mu_n}(S^1_{t+1} = s^1_1 + \frac{j}{n} \mid S^1_t = s^1_1) - \mathbb{P}_{\mu_n}(S^1_{t+1} = s^1_1 + \tfrac{1}{n} + \frac{j}{n} \mid S^1_t = s^1_1 + \tfrac{1}{n}) = o(1)\,,$$

for $j \in -1, 0, 1$ where the last inequality follows from the concentration of the conditioned measure as well as the continuity of the probability to stay put. These two formulas together imply that

$$\mathbb{P}_{\mu_n}(S^1_{t+1} = s^1_1 + \frac{1}{n} \mid S^1_t = s^1_1) \geqslant \lambda\mathbb{P}_{\mu_n}(S^1_{t+1} = s^1_1 \mid S^1_t = s^1_1 + \frac{1}{n})\,,$$

for some fixed constant $\lambda > 1$ for all $s^1 \in (s^*(\beta) - \delta_2, s^*(\beta) + \delta_2)$ when n is sufficiently large. Since $(S_t)_{t \geqslant 0}$ is a reversible Markov chain, with μ_n its stationary measure,

$$\mathbb{P}_{\mu_n}(S^1_{t+1} = s^1_1 + \frac{1}{n}, S^1_t = s^1_1) = \mathbb{P}_{\mu_n}(S^1_{t+1} = s^1_1, S^1_t = s^1_1 + \frac{1}{n})\,,$$

and therefore, for all $s^1 \in (s^*(\beta) - \delta_2, s^*(\beta) + \delta_2)$

$$\mu_n(S^1 = s^1 + \frac{1}{n}) \geqslant \lambda \mu_n(S^1 = s^1)\,,$$

and hence

$$\mu_n(S^1 = s^*(\beta) + \delta_2) \geqslant \lambda^{2\delta_2 n} \mu_n(S^1 = s^*(\beta) - \delta_2)\,. \tag{5.3}$$

Now select the set $A = \{S_1 \geqslant s^*(\beta) - \delta_2\}$. By (5.3), $\frac{\mu_n(\partial_{P_n} A)}{\mu_n(A)} \leqslant \lambda^{-\delta_2 n}$, where

$$\partial_{P_n} A = \{x \in A \ : \ P_n(x, y) > 0 \text{ for some } y \notin A\}$$

and P_n is the transition kernel of the Glauber dynamics. Since $\beta < \beta_c(q)$ and $s^*(\beta) - \delta_2 > 1/q$ we also have $\mu_n(A) = o(1)$ as $n \to \infty$. Therefore Cheeger's inequality (Theorem 2.5) immediately implies an exponential lower bound on the mixing time.

The case $\beta \geqslant \beta_c(q)$ is simpler. As the large deviations analysis in Subsection 2.2 shows, we may find $A = \{\|S - \check{s}_{\beta,q}\|_2 < \delta\}$, where $\check{s}_{\beta,q}$ is defined in (2.4) and $\delta > 0$ is small enough such that $\limsup_{n\to\infty} n^{-1} \log \mu_n(\partial_{P_n} A) < 0$ and $\liminf_{n\to\infty} n^{-1} \log \mu_n(A) = 0$. Since symmetry implies $\mu_n(A) \leqslant 1/q$ (if δ is sufficiently small), exponential mixing time follows immediately from another application of Cheeger's inequality (Theorem 2.5). □

6. Mixing Near Criticality

We now assume $\beta(n) = \beta_s(q) - \xi(n)$, with $\xi(n) \to 0$ as $n \to \infty$. Once $\beta(n)$ approaches β_s with n, we no longer have a uniform negative upper bound on the drift to the right of $1/q$ for each coordinate. Instead, near $s^*(\beta)$, the drift will be of order $\xi(n)$, possibly even positive and hence it will take longer than linear time to get close to $\widehat{\mathsf{e}}$ and this may have an effect on the order of the mixing time and cutoff window. Accordingly, in addition to the coalescence time analysis near $\widehat{\mathsf{e}}$, one has to obtain sharp asymptotics for the passage time near $s^*(\beta)$. This is achieved using several propositions which we state in Subsection 6.1. Their proofs will be deferred until the end of the section in favor of first showing how they are used along with the previous coalescence analysis to find the mixing time near criticality which gives the proof of Theorem 2.

Both the analysis and the results in Theorem 2 are qualitatively different, depending on whether $\xi(n)$ decays faster or slower than some threshold rate. Accordingly, we shall distinguish between two regimes and write:

$$\begin{array}{ll} \xi \in [\mathrm{CR}] & \text{if } \lim_{n\to\infty} n^{2/3}\xi(n) = \infty, \quad \xi(n) = o(1) \\ \xi \in [\mathrm{NCR}] & \text{if } 0 \leqslant \liminf_{n\to\infty} n^{2/3}\xi(n) \leqslant \limsup_{n\to\infty} n^{2/3}\xi(n) < \infty \end{array} \tag{6.1}$$

([CR] stands for Cutoff Regime and [NCR] stands for No-Cutoff Regime). For $a > 0$, define also

$$t_\gamma^{\xi,a}(n) = \begin{cases} \frac{\pi}{\sqrt{a}} \frac{n}{\sqrt{|\xi(n)|}} + \gamma \left(\frac{n^{1/2}}{|\xi(n)|^{5/4}} \vee n \right) & \text{if } \xi \in [\mathrm{CR}] \\ e^\gamma n^{4/3} & \text{if } \xi \in [\mathrm{NCR}]. \end{cases} \tag{6.2}$$

Both (6.1) and (6.2) will be used for sequences other than ξ as well. We shall also employ the following notation for hitting times. Given a real-valued process $(X_t)_{t \geqslant 0}$ and a number $x \in \mathbb{R}$ we shall write

$$\tau_x^+ = \inf\{t : \ X_t \geqslant x\} \quad \text{and} \quad \tau_x^- = \inf\{t : \ X_t \leqslant x\}$$

for the right and left hitting time of X at x. Notice that this notation does not carry an indication for the process for which τ_x^+ is a hitting time and in case this is not clear from the context, it will be mentioned explicitly.

6.1. **Drift Analysis Near $s^*(\beta)$.** The following proposition states several properties of the function D_β near $s^*(\beta)$.

Proposition 6.1. *For all $q \geqslant 3$ the following holds:*

(1) The point $s^*(\beta_s)$ is the unique $s \in (\frac{1}{q}, 1]$ such that $D_{\beta_s}(s) = 0$.

(2) For $k = 0, 1, \ldots,$ the functions $D_k^*(\beta) \triangleq \frac{d^k}{ds^k} D_\beta(s^*(\beta))$ are C^∞ in a neighborhood of β_s. Furthermore:
- $\frac{d}{d\beta} D_0^*(\beta_s) > 0$.
- $D_2^*(\beta_s) < 0$.

(3) For all $\rho > 0$, there exists $\delta > 0$ such that:

$$\sup\left\{D_\beta(s) : \ s \in \left(\tfrac{1}{q} + \rho, 1\right], |s - s^*(\beta)| > \rho, \ |\beta - \beta_s| < \delta\right\} < 0\,. \tag{6.3}$$

The next lemma gives sharp asymptotics for the passage time near 0 for a process with certain drift assumptions near 0 (given by (6.4) below). The one coordinate process will fall into this category if we analyze it near $s^*(\beta)$.

Formally, let $\left((Z_t)_{t \geqslant 0}^n \ ; \ n \geqslant 0\right)$ be a sequence of discrete time processes. For all n, suppose that $(Z_t)_{t \geqslant 0} = (Z_t)_{t \geqslant 0}^n$ is adapted to $(\mathcal{F}_t)_{t \geqslant 0}^n$, satisfies $n|Z_{t+1} - Z_t| \in \{-1, 0, 1\}$ with probability 1, and

$$\mathbb{E}[Z_{t+1} - Z_t | \mathcal{F}_t] = \frac{1}{n}\left(\zeta(n) + aZ_t^2 + bZ_t^3 + O\left(\zeta(n)Z_t^2 + Z_t^4\right)\right) \tag{6.4}$$

where $a > 0$, $b \in \mathbb{R}$ and $\zeta(n)$ is a sequence satisfying $\zeta(n) \to 0$ as $n \to \infty$. We allow both $\zeta \in$ [CR] and $\zeta \in$ [NCR], but in the latter case, we assume in addition the existence of $d > 0$ such that for all n

$$\mathbb{V}\mathrm{ar}[n(Z_{t+1} - Z_t)|\mathcal{F}_t] \geqslant d. \tag{6.5}$$

Write $\mathbb{P}_{z_0}$ for the probability measure under which this process is defined and starts from z_0.

Lemma 6.2. *Fix $\rho > 0$ sufficiently small. Then for $z_0 = -\rho$ there exist functions $L^*, U^* : (-\infty, \infty) \to [0, 1]$ satisfying $\lim_{\gamma \to -\infty} L^*(\gamma) = \lim_{\gamma \to \infty} U^*(\gamma) = 0$ such that for all γ,*

$$\limsup_{n \to \infty} \mathbb{P}_{z_0}\left(\tau_\rho^+ > t_\gamma^{\zeta,a}(n)\right) \quad \leqslant \quad U^*(\gamma), \tag{6.6}$$

$$\limsup_{n \to \infty} \mathbb{P}_{z_0}\left(\tau_\rho^+ < t_\gamma^{\zeta,a}(n)\right) \quad \leqslant \quad L^*(\gamma), \tag{6.7}$$

where τ_ρ^+ is a hitting time for Z. Moreover, if $\zeta \in$ [NCR] we can chose U^* such that for all γ we have

$$U^*(\gamma) < 1. \tag{6.8}$$

Remark 6.3. *The upper (lower) bound in the lemma still holds if $(Z_t)_{t \geqslant 0}$ satisfies (6.4) with $\geqslant$ ($\leqslant$) in place of the equality sign or if in place of $z_0 = -\rho$ we have $z_0 \geqslant -\rho$ ($z_0 \leqslant -\rho$). Since $(Z_t)_{t \geqslant 0}$ has $0, \pm\frac{1}{n}$ steps this can be shown by a simple coupling argument.*

The next proposition shows that the drift of one coordinate stays close to its upper bound $D_\beta(\cdot)$ for sufficiently long time. More precisely, for $\sigma_0 \in \Sigma_n$, $t > 0$, $\delta > 0$, $y \in [0, 1]$ let

$$K_n(\sigma_0, t, y, \delta) = \mathbb{P}_{\sigma_0}\left(\max_{0 \leqslant \theta \leqslant \min\{t, \tau_y^-\}} D_\beta(S_\theta^1) - n\mathbb{E}_{\sigma_0}\left[S_{\theta+1}^1 - S_\theta^1 \mid \mathcal{F}_\theta\right] > \delta\right), \tag{6.9}$$

where τ_y^- is a hitting time for S_t^1. Then,

Proposition 6.4. *Suppose that $\beta \leqslant q/2$ and set $\sigma_0 \equiv 1$. Then for any $y > \frac{1}{q}$:*

(1) If $t(n) = o(n^2)$ and $\delta(n)n^2 t(n)^{-1} \to \infty$ then $\lim_{n\to\infty} K_n(\sigma_0, t(n), y, \delta(n)) = 0$.
(2) If $t(n) = \gamma n^{4/3}$ then for all $\delta > 0$ we have

$$\lim_{\gamma\to 0} \limsup_{n\to\infty} K_n(\sigma_0, t(n), y, \delta n^{-2/3}) = 0.$$

6.2. **Proof of Theorem 2.**

6.2.1. *Upper Bound on Mixing Time.* Fix $\rho > 0$ small enough and let $\sigma_0 \in \Sigma_n$ be given. By Proposition 6.1 Part (1), we can find $\delta > 0$ so that

$$\sup\{D_\beta(s) : \ |\beta - \beta_s| < \delta, \ 1/q + \rho/2 < s < 1, \ |s - s^*| > \rho/2\} < -\delta. \tag{6.10}$$

where we use s^* in place of $s^*(\beta)$. Then by Lemma 2.1 Part (1), we have that,

$$\mathbb{P}_{\sigma_0}\left(\tau^-_{(s^*+\rho)} > (2/\delta)n\right) = o(1)$$

where this and all hitting times below are of S_t^1. Define now $Z_t = s^* - S^1_{t+\tau^-_{(s^*+\rho)}}$. Using (3.1), (3.3) Proposition 6.1 and applying Taylor's expansion for $D_\beta(s)$ around s^* and then again for $D_0^*(\beta)$ around β_s, we infer that there exist $a > 0$, $\alpha \neq 0$, $b \in \mathbb{R}$ such that

$$\mathbb{E}[Z_{t+1} - Z_t|\mathcal{F}_t] \geqslant \frac{1}{n}\left(\zeta(n) + aZ_t^2 + bZ_t^3 + O\left(\zeta(n)Z_t^2 + Z_t^4\right)\right),$$

where $\zeta(n) = \alpha\xi(n) + O(\xi(n)^2 + n^{-1})$ and also (6.5) holds (if needed), since the probability of choosing any new color at time $t+1$ is bounded above and below, uniformly in n and S_t. Hence by Lemma 6.2 and Remark 6.3, for all γ

$$\mathbb{P}_{\sigma_0}\left(\tau^-_{(s^*-\rho)} - \tau^-_{(s^*+\rho)} > t_\gamma^{\zeta,a}(n)\right) \leqslant U^*(\gamma) + o(1)$$

Now, using the relation between $\zeta(n)$ and $\xi(n)$, it is not difficult to verify that $t_\gamma^{\zeta,a}(n) \leqslant t_{\gamma'}^{\xi,a'}(n)$ for all γ', where $a' = \alpha a$ and $\gamma = F(\gamma')$ for some F such that $\gamma \to \infty$ if $\gamma' \to \infty$.

From Lemma 2.1 Part (3), applied to the process $(S^1_{t+\tau^-_{(s^*-\rho)}} - (s^* - \rho) : \ t \geqslant 0)$, it follows that with $1 - o(1)$ probability $S^1_{t+\tau^-_{(s^*-\rho)}}$ stays to the left of $s^* - \rho/2$ for all $t < n^2$. Then we may apply Lemma 2.1 Part (1) to the process $(S^1_{t+\tau^-_{(s^*-\rho)}} - (1/q + \rho/2) : \ t \geqslant 0)$ to conclude

$$\mathbb{P}_{\sigma_0}\left(\tau^-_{(q^{-1}+\rho/2)} - \tau^-_{(s^*-\rho)} > (2/\delta)n\right) = o(1)$$

Finally another application of Lemma 2.1 Part (3) gives $S^1_{t+\tau^-_{(q^{-1}+\rho/2)}} \leqslant 1/q + \rho$ for all $t < n^2$ with $1 - o(1)$ probability. For the [CR] case, we use union bound (over all coordinates):

$$\mathbb{P}_{\sigma_0}\left(S_{t_{\gamma'}^{\xi,a'}(n)} \notin \mathcal{S}^{\rho+}\right) \leqslant qU^*(\gamma) + o(1). \tag{6.11}$$

For the [NCR] case, define $\tau^{(1)} = \tau^-_{(q^{-1}+\rho/2)}$ and $\tau^{(k)} = \inf\{t \geqslant \tau^{(k-1)} : S_t^k \leqslant q^{-1} + \rho/2\}$ for $k > 1$. Then, by inductive conditioning we obtain

$$\mathbb{P}_{\sigma_0}\left(\tau^{(k)} \leqslant t_{\gamma'}^{\xi,a}(n) : \ k = 1, \ldots, q\right) \geqslant (1 - U^*(\gamma))^q + o(1).$$

Since also $S^k_{t+\tau^{(k)}} \leqslant 1/q+\rho$ for all $k \in [1,q]$, $t < n^2$ with $1-o(1)$ probability, we arrive to

$$\mathbb{P}_{\sigma_0}\left(S_{t^{\xi,a'}_{\gamma'}(n)} \notin \mathcal{S}^{\rho+}\right) \leqslant 1-(1-U^*(\gamma))^q + o(1). \tag{6.12}$$

We now re-employ the overall coupling in Sub-section 4.6, but in view of (6.11) and (6.12) we change step (1) and instead of running the two chains for $\gamma_1 n$ time, we run them for $t^{(1)}(n) = t^{\xi,a'}_{\gamma'}(n)$. As (6.11), (6.12) show, we can choose γ' large enough such that $\mathbb{P}^{OC}_{\sigma_0}(S_{t^{\xi,a'}_{\gamma'}(n)} \notin \mathcal{S}^{\rho+}) \leqslant \epsilon$ for n sufficiently large. The remaining steps in the coupling are left unchanged and we choose the same parameter values, as in the proof of Theorem 1.

Using the analysis of the modified step (1) given by (6.11) and (6.12), together with the analysis in Sub-section 4.7 of the remaining steps - which carries over (uniformly in β near $\beta_s(q)$), since it only required $\beta < \beta_c(q)$, we recover (4.14), namely

$$\|\mathbb{P}_{\sigma_0}\left(\sigma_{t^{(5)}(n)} \in \cdot\right) - \mu_n\|_{\mathrm{TV}} \leqslant 6\epsilon.$$

The time is now given by

$$t^{(5)}(n) = t^{\alpha_1}_{\gamma}(n) + t^{\xi,a'}_{\gamma'}(n),$$

for some $\gamma > 0$. Since σ_0 is arbitrary and ϵ can be made arbitrarily small, by having γ, γ' large enough, this completes the proof for the upper bound in (1.4) and (1.5) with $\alpha_2 = \pi/\sqrt{\alpha a}$.

6.2.2. *No Cutoff in* NCR *Case.* Using the modified overall coupling as introduced above, we obtain from (4.14), (6.12) and (6.8) for any γ' and sufficiently large γ

$$\|\mathbb{P}_{\sigma_0}\left(\sigma_{t^{(5)}(n)} \in \cdot\right) - \mu_n\|_{\mathrm{TV}} \leqslant 1-\epsilon,$$

for all σ_0, large enough n and some $\epsilon > 0$. Then, since in the [NCR] case

$$t^{(5)}(n) = t^{\alpha_1}_{\gamma}(n) + t^{\xi,a'}_{\gamma'}(n) \leqslant t^{\xi,a'}_{\gamma'+C}(n),$$

this shows that there is no cut-off.

6.2.3. *Lower Bound on Mixing Time.* Fix $\rho > 0$ small enough and start with $\sigma_0 \equiv 1$ - the all '1' configuration. Define:

$$\delta(n) = \begin{cases} n^{-1}\xi(n)^{-1/2}A(n) & \text{if } \xi \in [\mathrm{CR}] \\ \delta_1 n^{-2/3} & \text{if } \xi \in [\mathrm{NCR}] \end{cases}$$

where $A(n)$ is a sequence tending to ∞ sufficiently slowly and $\delta_1 > 0$. Set

$$N = \inf\left\{t:\ D_\beta(S^1_t) - n\mathbb{E}\left[S^1_{t+1} - S^1_t \mid \mathcal{F}_t\right] > \delta(n)\right\}$$

and define the process Y_t which is equal to S^1_t up to time N, but after this time evolves like a birth-and-death processes with $\pm 1/n$ increments and drift $-n^{-1}D_\beta(S^1_t)$. Then $(Z_t \triangleq s^* - Y_t:\ t \geqslant 0)$ satisfies

$$\mathbb{E}[Z_{t+1} - Z_t|\mathcal{F}_t] \leqslant \frac{1}{n}\left(\zeta(n) + aZ_t^2 + bZ_t^3 + O\left(\zeta(n)Z_t^2 + Z_t^4\right)\right),$$

with a, b, α as in the upper bound case, but with $\zeta(n) = \alpha\xi(n) + \delta(n) + O(\xi(n)^2 + n^{-1})$ and condition (6.5) holds (if needed) as before. Then, using Lemma 6.2 and Remark 6.3, we have for n large enough $\mathbb{P}_{\sigma_0}\left(\tau^+_\rho < t^{\zeta,a}_{\gamma}(n)\right) \leqslant 2L^*(\gamma)$, where τ^+_ρ is a hitting time for Z and $\gamma \in \mathbb{R}$. As before, it is not difficult to verify that if $A(n)$ is increasing slowly enough, $t^{\zeta,a}_{\gamma}(n) \geqslant t^{\xi,a'}_{\gamma'}(n)$, where $\gamma = F(\gamma')$ satisfies $\gamma \to -\infty$ if $\gamma' \to -\infty$.

Now define $T = \inf\{t : S_t \in \mathcal{S}^{\rho+}\}$ and τ' as τ_ρ^+, only with S_t^1 in place of Y_t. Then

$$\begin{aligned}\mathbb{P}_{\sigma_0}\left(T < t_{\gamma'}^{\xi,a'}(n)\right) &\leqslant \mathbb{P}_{\sigma_0}\left(\tau_\rho^+ < t_{\gamma'}^{\xi,a'}(n)\right) + \mathbb{P}_{\sigma_0}\left(N < \tau' \wedge t_{\gamma'}^{\xi,a'}(n)\right)\\ &\leqslant 2L^*(\gamma) + K_n(\sigma_0, t_{\gamma'}^{\xi,a'}(n), s^* - \rho, \delta(n))\end{aligned}$$

where K_n is defined in (6.9). Then, if ρ is sufficiently small, we can use (4.2) for $\widehat{S}_t$ starting from time T to obtain for all $r > 0$ and γ'':

$$\begin{aligned}&\mathbb{P}_{\sigma_0}\left(\|\widehat{S}_{t_{\gamma''}^{\alpha_1}(n)+t_{\gamma'}^{\xi,a'}(n)}\|_2 < \frac{r}{\sqrt{n}}\right)\\ &\leqslant \mathbb{P}_{\sigma_0}\left(T < t_{\gamma'}^{\xi,a'}(n)\right) + \mathbb{P}_{\sigma_0}\left(\|\widehat{S}_{t_{\gamma''}^{\alpha_1}(n)+t_{\gamma'}^{\xi,a'}(n)}\|_2 < \frac{r}{\sqrt{n}}\middle| T \geqslant t_{\gamma'}^{\xi,a'}(n)\right)\\ &\leqslant 2L^*(\gamma) + K_n(\sigma_0, t_{\gamma'}^{\xi,a'}(n), s^* - \rho, \delta(n)) + O((e^{-C_2\gamma''} - r)^{-2})\end{aligned}$$

Using Proposition 6.4 for the middle term, the last inequality gives (4.3) with $t_{\gamma''}^{\alpha_1}(n) + t_{\gamma'}^{\xi,a'}(n)$ in place of $t_\gamma^{\alpha_1}(n)$. The remaining of the proof is identical to the subcritical case and this shows the lower bound for both parts of Theorem 2 with $\alpha_2 = \pi/\sqrt{\alpha a}$. □

6.3. **Proofs for Subsection 6.1.**

Proof of Proposition 6.1. First observe that for all β, $D_\beta(\frac{1}{q}) = 0$ and for all $s > \frac{1}{q}$,

$$\frac{d}{d\beta}D_\beta(s) = \frac{d}{d\beta}\left(-s + \frac{1}{1 + (q-1)e^{-\frac{2\beta q}{q-1}(s-\frac{1}{q})}}\right) > 0\,.$$

Now since $D_\beta(s)$ is smooth as a function of s and β and $d_0(s) = -s + \frac{1}{q}$ it follows that $\beta_s > 0$. We have that

$$\begin{aligned}\frac{d}{ds}D_\beta(s) &= -1 + \frac{2\beta q e^{-\frac{2\beta q}{q-1}(s-\frac{1}{q})}}{\left(1 + (q-1)e^{-\frac{2\beta q}{q-1}(s-\frac{1}{q})}\right)^2}\,,\\ \frac{d^2}{ds^2}D_\beta(s) &= \frac{4\beta^2 q^2 e^{-\frac{2\beta q}{q-1}(s-\frac{1}{q})}(1 - (q-1)e^{-\frac{2\beta q}{q-1}(s-\frac{1}{q})})}{(q-1)\left(1 + (q-1)e^{-\frac{2\beta q}{q-1}(s-\frac{1}{q})}\right)^3}\,,\end{aligned} \tag{6.13}$$

and so

$$\frac{d}{ds}D_\beta(s)\bigg|_{s=\frac{1}{q}} = -1 + \frac{2\beta}{q}, \qquad \frac{d^2}{ds^2}D_\beta(s)\bigg|_{s=\frac{1}{q}} = \frac{4\beta^2(q-2)}{q(q-1)} > 0\,. \tag{6.14}$$

which implies that $d_{q/2}(s) > 0$ when $s \in (\frac{1}{q}, \frac{1}{q} + \epsilon)$ for some small ϵ. This implies that $\beta_s < q/2$. It follows that

$$\frac{d}{ds}D_{\beta_s}(s)\bigg|_{s=\frac{1}{q}} < 0\,,$$

and so $D_\beta(s) < 0$ when $\beta \in [\beta_s, \beta_s + \epsilon]$ and $s \in (\frac{1}{q}, \frac{1}{q} + \epsilon)$ for some small ϵ. It follows by compactness then that for some $\frac{1}{q} + \epsilon \leqslant s^*(\beta_s) \leqslant 1$ that $D_{\beta_s}(s^*(\beta_s)) = 0$. By the definition of β_s and since $D_\beta(s)$ is smooth we have that

$$\frac{d}{ds}D_{\beta_s}(s)\bigg|_{s=s^*(\beta_s)} = 0, \qquad \frac{d^2}{ds^2}D_{\beta_s}(s)\bigg|_{s=s^*(\beta_s)} \leqslant 0\,. \tag{6.15}$$

The equation $\frac{d}{ds}D_{\beta_s}(s)=0$ is equivalent to

$$2\beta q e^{-\frac{2\beta_s q}{q-1}(s-\frac{1}{q})} = \left(1+(q-1)e^{-\frac{2\beta_s q}{q-1}(s-\frac{1}{q})}\right)^2 ,$$

which is a quadratic equation in $e^{-\frac{2\beta_s q}{q-1}(s-\frac{1}{q})}$ and hence has at most 2 solutions which we denote s_1, s_2 with $s_1 < s_2$. Since $\frac{d}{ds}D_{\beta_s}(0) < 0$ then $D_{\beta_s}(s_1) < 0$ and so $s^*(\beta_s) = s_2$. In particular this implies that $s^*(\beta_s)$ is the unique $s \in (\frac{1}{q}, 1]$ such that $D_{\beta_s}(s) = 0$. Also it follows that $\frac{d}{ds}D_{\beta_s}(s) > 0$ for $s \in (s_1, s^*(\beta_s))$ and that there exists $s' \in (s_1, s^*(\beta_s))$ such that $\frac{d^2}{ds^2}D_{\beta_s}(s') = 0$. Since by equation (6.13) there is at most one s such that $\frac{d^2}{ds^2}D_{\beta_s}(s) = 0$ it follows that

$$\frac{d^2}{ds^2}D_{\beta_s}(s)\Big|_{s=s^*(\beta_s)} < 0.$$

Hence by the Inverse Function Theorem $s^*(\beta)$ is a smooth function of β when β is in a small neighborhood of β_s. Then we have that

$$\begin{aligned}\frac{d}{d\beta}D_0^*(\beta_s) &= \frac{d}{d\beta}D_\beta(s^*(\beta_s))\Big|_{\beta=\beta_s} + \left(\frac{d}{d\beta}s^*(\beta)\Big|_{\beta=\beta_s}\right)\frac{d}{ds}D_{\beta_s}(s)\Big|_{s=s^*(\beta_s)} \\ &= \frac{d}{d\beta}D_\beta(s^*(\beta_s))\Big|_{\beta=\beta_s} \\ &> 0\,,\end{aligned}$$

since $\frac{d}{ds}D_{\beta_s}(s)|_{s=s^*(\beta_s)} = 0$ which completes the proof of the second part.

We now turn to prove the third part. As we have observed $D_{\beta_s}(s)$ is a smooth function satisfying

$$D_{\beta_s}(s^*(\beta_s)) = 0\,,\ \frac{d}{ds}D_{\beta_s}(s)|_{s=s^*(\beta_s)} = 0\,,\ \text{and}\ \frac{d^2}{ds^2}D_{\beta_s}(s)|_{s=s^*(\beta_s)} < 0\,.$$

Therefore, we deduce that for any $\rho > 0$, there exists $\delta_1 > 0$ such that

$$D_{\beta_s}(s) < -2\delta_1 \text{ for all } s \in \left\{s : |s - s^*(\beta_s)| \geqslant \rho/2, s \in \left(\frac{1}{q}+\rho, 1\right]\right\}\,. \tag{6.16}$$

Since $\frac{d^2}{ds^2}D_{\beta_s}(s)\Big|_{s=s^*(\beta)} < 0$ for all $\rho > 0$, there exists $\delta_2 > 0$ such that

$$|s^*(\beta) - s^*(\beta_s)| < \rho/2 \text{ for all } |\beta - \beta_s| < \delta_2\,.$$

Combined with (6.16), it follows that

$$D_{\beta_s}(s) < -2\delta_1,\ \text{for all } |s - s^*(\beta)| \geqslant \rho \text{ and } |\beta - \beta_s| < \delta_2\,. \tag{6.17}$$

Now that $D_\beta(s)$ can be viewed as a continuous function of (β, s) and by compactness, there exists $\delta_3 > 0$ such that $|D_\beta(s) - D_{\beta_s}(s)| \leqslant \delta_1$ for all $s \in [1/q, 1]$ and $|\beta - \beta_s| \leqslant \delta_3$. Combined with (6.17), it completes the proof by taking $\delta = \delta_1 \wedge \delta_2 \wedge \delta_3$. □

Proof of Lemma 6.2. We do not lose anything by assuming that

$$\mathbb{E}[Z_{t+1} - Z_t|\mathcal{F}_t] = f(Z_t) \quad ; \qquad f(z) = \tfrac{1}{n}(\zeta(n) + az^2 + bz^3 + cz^4)\mathbb{1}_{[-\rho_0,+\rho_0]}(z) \tag{6.18}$$

for some $a > 0$, b, c with $\rho_0 = 2\rho$ and that once Z_t exits $[-\rho_0, +\rho_0]$ it is stopped. Indeed, having f vanish outside of $[-\rho_0, +\rho_0]$, does not change the asymptotics of the passing time. Clearly, this is the case for $z > +\rho_0$. For $z < -\rho_0$, it follows from

$$\mathbb{P}_{-\rho}(\tau^-_{-\rho_0} < t^{\zeta,a}_\gamma(n)) = o(1), \tag{6.19}$$

for all γ, which is a consequence of Lemma 2.1 part (3) since the drift of Z_t is at least $\frac{c}{n}$ on $[-\rho_0, -\rho]$ for some positive c uniformly in n (if ρ is small enough).

As for replacing the error term in (6.4) by cZ_t^4, as the proof below shows, the functions U^*, L^* in the lemma restricted to condition (6.18) can be chosen to be continuous in a in a small interval $[a_0 - \epsilon, a_0 + \epsilon]$ and the limits (6.6), (6.7) hold uniformly in a in this interval. This together with Remark 6.3 implies the existence of U^*, L^* under which (6.6), (6.7) hold in the general case (6.4) with $t^{\zeta,a+O(\zeta(n))}_\gamma(n)$. Now, it is not difficult to see that the latter is bounded above and below by $t^{\zeta,a}_{\gamma\pm C}(n)$ for some $C > 0$ and hence (6.6), (6.7) hold with $t^{\zeta,a}_\gamma(n)$. Similar considerations apply for (6.8).

Set

$$\Psi(z) = \int_0^z \frac{1}{f(x)}dx \quad \text{and} \qquad Y_t = \Psi(Z_t) - \Psi(Z_0) - t.$$

The motivation behind the above definitions comes from a continuous time deterministic analog of (6.18) in the form of an ODE

$$\dot{z}(t) = f(z(t)) \tag{6.20}$$

for which $z(t) = \Psi^{-1}(t - t_0)$ is a solution (roughly speaking Y_t measures how far behind or ahead "in schedule" Z_t is, judging from its position).

Start with the [CR] case and set

$$t^{\zeta,a}(n) = t^{\zeta,a}_0(n) = \frac{\pi}{\sqrt{a}}\frac{n}{\sqrt{\zeta(n)}} \quad ; \qquad w^\zeta(n) = \frac{n^{1/2}}{\zeta^{5/4}(n)} \vee n.$$

In the deterministic setting the time it takes for $z(t)$ to pass from $z(0) = -\rho$ to ρ is

$$\Psi(\rho) - \Psi(-\rho) = \int_{-\rho}^{\rho} \frac{n\,dx}{\zeta(n) + ax^2 + bx^3 + cx^4} = t^{\zeta,a}(n) + O(n)$$

if ρ is small enough. This will be shown in Proposition 6.5 below. Thus, bounding the passage time τ^+_ρ around $t^{\zeta,a}(n)$ can be achieved by bounding $|Y_{t^{\zeta,a}(n)}|$.

Accordingly, let $Y_t = M_t + A_t$ be the Doob-decomposition of Y_t, with M_t a zero-mean martingale and A_t the predictable process. The next proposition will allow us to bound Y_t. The proof of this proposition will be deferred to the end of this section.

Proposition 6.5. *If ρ is small enough and $\zeta \in$ [CR] then*

(1) $\Psi(\rho) - \Psi(-\rho) = t^{\zeta,a}(n) + O(n)$.
(2) $\mathbb{E}M^2_{t^{\zeta,a}_\gamma(n)} = O(w^\zeta(n)^2)$.
(3) $A_{t^{\zeta,a}_\gamma(n)} = o(w^\zeta(n))$ *with probability* 1.

Now using the monotonicity of Ψ in $[-\rho_0, +\rho_0]$ we have

$$\begin{aligned}
\mathbb{P}_{-\rho}(\tau_\rho^+ > t_\gamma^{\zeta,a}(n)) &\leqslant \mathbb{P}_{-\rho}\left(Z_{t_\gamma^{\zeta,a}(n)} < \rho\right) \\
&\leqslant \mathbb{P}_{-\rho}\left(Y_{t_\gamma^{\zeta,a}(n)} < t^{\zeta,a}(n) - t_\gamma^{\zeta,a}(n) + O(n)\right) \\
&\leqslant \mathbb{P}_{-\rho}\left(M_{t_\gamma^{\zeta,a}(n)} < -\gamma w^\zeta(n) + O(n) + o\left(w^\zeta(n)\right)\right) \\
&\leqslant \left(\frac{O(w^\zeta(n))}{(\gamma + O(1))w^\zeta(n)}\right)^2,
\end{aligned}$$

where the last inequality is a second moment bound. This shows (6.6). For the lower bound, if $-\gamma$ is large enough, we may write

$$\begin{aligned}
\mathbb{P}_{-\rho}\left(\tau_\rho^+ < t_\gamma^{\zeta,a}(n)\right) &= \mathbb{P}_{-\rho}\left(\exists t < t_\gamma^{\zeta,a}(n) \ : \ Z_t \geqslant \rho\right) \\
&\leqslant \mathbb{P}_{-\rho}\left(\exists t < t_\gamma^{\zeta,a}(n) \ : \ Y_t \geqslant t^{\zeta,a}(n) - t_\gamma^{\zeta,a}(n) + O(n)\right) \\
&\leqslant \mathbb{P}_{-\rho}\left(\exists t < t_\gamma^{\zeta,a}(n) \ : \ M_t \geqslant -\gamma w^\zeta(n) + O(n) + o\left(w^\zeta(n)\right)\right) \\
&\leqslant \frac{O(w^\zeta(n))}{(-\gamma + O(1))w^\zeta(n)}
\end{aligned}$$

where the last inequality follows from Doob's inequality. This shows (6.7)

Next, we address the [NCR] case. We can no longer use the means analysis (6.20) throughout the entire passage interval $[-\rho, \rho]$ as Z_t is not concentrated around its mean near 0. Accordingly, we analyze the passage time in each of the following segments separately:

$$[-\rho, -rn^{-1/3}] \quad ; \qquad [-rn^{-1/3}, +rn^{-1/3}] \quad ; \qquad [+rn^{-1/3}, +\rho],$$

for some $r > 0$ to be chosen later.

We start with the upper bound. For the sequel, let $w = rn^{-1/3}$. The upper bound will follow if we show the following:

(1) *Segment* $[-\rho, -w]$. For any γ,

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}_{-\rho}(\tau_{-w}^+ > t_\gamma^{\zeta,a}(n)) = 0 \tag{6.21}$$

(2) *Segment* $[+w, +\rho]$. For any γ,

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}_{w}(\tau_{\rho}^+ > t_\gamma^{\zeta,a}(n)) = 0 \tag{6.22}$$

(3) *Segment* $[-w, +w]$. For any $r > 0$, there exists $u : \mathbb{R} \to [0, 1)$ such that

$$\limsup_{n\to\infty} \mathbb{P}_{-w}\left(\tau_w^+ > t_\gamma^{\zeta,a}(n)\right) \leqslant u(\gamma) < 1, \tag{6.23}$$

for all γ. Furthermore, $u(\gamma) \to 0$ as $\gamma \to \infty$.

All are hitting times for Z. Indeed, by first choosing large enough r and then choosing large enough γ both (6.6) and (6.8) will follow by multiplication. We proceed to prove each of the above statements.

Segments $[-\rho, -w]$, $[+w, +\rho]$. Here we can use the means analysis as in the [CR] case. As before, we do not change the asymptotics of the passage time through these intervals, if we assume that $f(z)$ in (6.18) satisfies:

$$f(z) = \tfrac{1}{n}(\zeta(n) + az^2 + bz^3 + cz^4)\mathbb{1}_{[-\rho_0,-w_0]\cup[+w_0,+\rho_0]}(z)$$

where a, b, c, ρ_0 are as before, $w_0 = w/2$ and once Z_t exits $[-\rho_0, -w_0] \cup [+w_0, +\rho_0]$ it is stopped. Indeed this follows from the same reasoning and in addition since for any γ

$$\lim_{r\to\infty} \mathbb{P}_w(\tau^-_{w_0} < t^{\zeta,a}_\gamma(n)) = 0\,,$$

uniformly in n (large enough) as it follows from Lemma 2.1 part (2) since the drift of Z_t is non-negative on $[w_0, w]$.

We use the same definitions for Y_t, M_t and A_t as above. In place of Proposition 6.5 we have

Proposition 6.6. *Assume $\zeta \in$ [NCR]. There exists $k(r)$ satisfying $k(r) \to 0$ as $r \to \infty$ such that for any ρ small enough, r large enough, n large enough and all t:*

(1) $\Psi(-w) - \Psi(-\rho)$, $\Psi(\rho) - \Psi(w) \leqslant k(r)n^{4/3}$.
(2) $\mathbb{E}M_t^2 \leqslant k(r)tn^{4/3}$.
(3) $|A_t| \leqslant k(r)t$ with probability 1.

The proof is again deferred. Now, as before

$$\begin{aligned}
\mathbb{P}_{-\rho}\left(\tau^+_{-w} > t^{\zeta,a}_\gamma(n)\right) &\leqslant \mathbb{P}_{-\rho}\left(Z_{t^{\zeta,a}_\gamma(n)} < -w\right)\\
&\leqslant \mathbb{P}_{-\rho}\left(Y_{t^{\zeta,a}_\gamma(n)} < \Psi(-w) - \Psi(-\rho) - t^{\zeta,a}_\gamma(n)\right)\\
&\leqslant \mathbb{P}_{-\rho}\left(M_{t^{\zeta,a}_\gamma(n)} < k(r)n^{4/3} + k(r)e^\gamma n^{4/3} - e^\gamma n^{4/3}\right)\\
&\leqslant \frac{k(r)e^\gamma}{((1-k(r))e^\gamma - k(r))^2} \qquad (6.24)
\end{aligned}$$

where the last inequality is Chebyshev. This goes to zero as $r \to \infty$ for any γ. This shows (6.21). Similarly,

$$\begin{aligned}
\mathbb{P}_w\left(\tau^+_\rho > t^{\zeta,a}_\gamma(n)\right) &\leqslant \mathbb{P}_w\left(Z_{t^{\zeta,a}_\gamma(n)} < \rho\right)\\
&\leqslant \frac{k(r)e^\gamma}{((1-k(r))e^\gamma - k(r))^2}
\end{aligned}$$

and this shows (6.22).

Segment $[-w, w]$. Here we still assume (6.18), but instead of absorbing Z_t at the boundaries, we shall now suppose that Z_t evolves like a symmetric random walk with $\pm n^{-1}$ steps, once it exits $[-\rho_0, +\rho_0]$.

We first show that u can be chosen to vanish at infinity. Consider the process $U_t = (U_0 - (Z_t - Z_0 + \delta n^{-5/3}t)$, for $\delta > 0$ with U_0 to be chosen later and set $N = \inf\{t : U_t \leqslant 0\}$. Then, by the definition of the [NCR] regime for n large enough $U_{t\wedge N}$ is a non-negative supermartingale satisfying the requirements of Lemma 2.3 and hence

$$\begin{aligned}
\mathbb{P}_{-w}(\tau^+_w > t^{\zeta,a}_\gamma(n)) &= \mathbb{P}_{-w}(Z_t < w \;\; \forall t \leqslant t^{\zeta,a}_\gamma(n))\\
&\leqslant \mathbb{P}_{-w}(U_t > U_0 + Z_0 - (w + \delta e^\gamma n^{-1/3}) \;\; \forall t \leqslant t^{\zeta,a}_\gamma(n))\\
&= \mathbb{P}_{-w}(N > t^{\zeta,a}_\gamma(n))\\
&\leqslant \frac{4(w + \delta e^\gamma n^{-1/3} + w)}{\sqrt{d}n^{-1}e^{\gamma/2}n^{2/3}} \leqslant C(r+1)e^{-\gamma/2}
\end{aligned}$$

where we choose $U_0 = w + \delta e^\gamma n^{-1/3} - Z_0$ and $\delta = e^{-\gamma}$. The last expression can be made arbitrarily small by taking γ large enough, uniformly in n if it is sufficiently large.

To show that u can satisfy $u(\gamma) < 1$ for all γ, we have to show that Z can cross from $-w$ to w in $t_\gamma^{\zeta,a}(n)$-time for arbitrarily small γ. If ρ is small and n is large, then $X_t = n(Z_t - (-w))$ satisfies the conditions in Lemma 2.2 with $\delta = \zeta(n)^-$ and $a = d$. Therefore

$$\begin{aligned}\mathbb{P}_{-w}(\tau_w^+ > t_\gamma^{\zeta,a}(n)) &\geqslant \mathbb{P}_{-w}(\exists t \leqslant t_\gamma^{\zeta,a}(n) : X_t \geqslant 2rn^{2/3}) \\ &\geqslant C_1 \exp\{-C_2(2re^{-\gamma/2} + e^{\gamma/2}\zeta(n)^- n^{2/3})^2\} + O(n^{-2/3})\end{aligned}$$

which is positive for all γ, once n is large enough. This proves (6.23) and concludes the proof of the upper bound.

To show the lower bound in the [NCR] case, set $V_t = Z_t - \delta n^{-5/3} t + w$ and choose $\delta, r > 0$ such that $V_{t\wedge\tau_w^+}$ has non-positive drift whenever $V_{t\wedge\tau_w^+} \geqslant 0$. Then,

$$\begin{aligned}\mathbb{P}_{-\rho}(\tau_\rho^+ < t_\gamma^{\zeta,a}(n)) &\leqslant \mathbb{P}_{-\rho}(\exists t < t_\gamma^{\zeta,a}(n) \ : Z_t \geqslant w) \\ &\leqslant \mathbb{P}_{-\rho}(\exists t < t_\gamma^{\zeta,a}(n) \ : V_{t\wedge\tau_w^+} \geqslant 2w - \delta e^\gamma n^{-1/3}) \\ &= \mathbb{P}_{-\rho}(\exists t < t_\gamma^{\zeta,a}(n) \ : V_{t\wedge\tau_w^+} \geqslant (2r - \delta e^\gamma) n^{-1/3})\end{aligned}$$

and part (2) of Lemma 2.1 shows that the last expression goes to 0 as $\gamma \to -\infty$ uniformly in n (large enough). This proves (6.7) and completes the [NCR] case. □

It remains to prove Propositions 6.4–6.6.

Proof of Proposition 6.4. Let $\tau^* = \min\left\{\tau_y^-, t(n), \min_{i\geqslant 2}\min\{t \geqslant 0 : S_t^i \geqslant \frac{1}{q}\}\right\}$. Fix some $2 \leqslant i < j \leqslant q$ and set

$$Y_t = S_t^i - S_t^j.$$

Let $U_{t-1} = Y_t - Y_{t-1} - \mathbb{E}_{\sigma_0}[Y_t - Y_{t-1} \mid \mathcal{F}_{t-1}]$ and then since $|Y_t - Y_{t-1}| \leqslant \frac{2}{n}$ we have that $|U_i| \leqslant \frac{4}{n}$. Define the process Z_t by $Z_0 = 0$ and

$$Z_t - Z_{t-1} := \mathrm{sign}(Z_{t-1})\mathrm{sign}(Y_{t-1})U_{t-1}$$

where

$$\mathrm{sign}(x) = \begin{cases} 1 & x \geqslant 0, \\ -1 & x < 0. \end{cases}$$

With this definition Z_t is clearly a martingale and since $|Z_t - Z_{t-1}| \leqslant \frac{4}{n}$, then $\mathbb{E}_{\sigma_0} Z_t^2 \leqslant \frac{16t}{n^2}$ and so by Doob's maximal inequality,

$$\mathbb{E}_{\sigma_0}\left[\max_{0\leqslant t\leqslant t(n)} |Z_t|\right]^2 \leqslant 2\mathbb{E}_{\sigma_0} Z_{t(n)}^2 \leqslant \frac{32t(n)}{n^2}. \tag{6.25}$$

Now when $t < \tau^*$ we have that $S_t^i, S_t^j < \frac{1}{q}$ and so

$$\left|e^{2\beta(S_t^i - \frac{1}{q})} - e^{2\beta(S_t^j - \frac{1}{q})}\right| \leqslant 2\beta\left|S_t^i - S_t^j\right|.$$

By Jensen's inequality $\sum_{k=1}^q e^{2\beta(S_t^i - \frac{1}{q})} \geqslant q$ so

$$\begin{aligned}&\mathbb{E}_{\sigma_0}[\mathrm{sign}(Y_{t-1})(Y_t - Y_{t-1}) \mid \mathcal{F}_{t-1}] \\ &= \mathbb{E}_{\sigma_0}\left[\mathrm{sign}(S_{t-1}^i - S_{t-1}^j)\frac{1}{n}\left(\frac{e^{2\beta(S_t^i - \frac{1}{q})} - e^{2\beta(S_t^j - \frac{1}{q})}}{\sum_{k=1}^q e^{2\beta(S_t^i - \frac{1}{q})}} - (S_{t-1}^i - S_{t-1}^j)\right)\right] \\ &\leqslant -\frac{q - 2\beta}{qn}\left|S_t^i - S_t^j\right| \leqslant 0.\end{aligned} \tag{6.26}$$

Now when $|Y_{t-1}| \geqslant \frac{2}{n}$ we have that $|Y_t| - |Y_{t-1}| = \text{sign}(Y_{t-1})(Y_t - Y_{t-1})$ and we always have

$$|Z_t| - |Z_{t-1}| \geqslant \text{sign}(Z_{t-1})(Z_t - Z_{t-1}) = \text{sign}(Y_{t-1})U_{t-1}$$

with equality when $\text{sign}(Z_t) = \text{sign}(Z_{t-1})$. Hence it follows that when $|Y_{t-1}| \geqslant \frac{2}{n}$ and $t \leqslant \tau^*$,

$$\begin{aligned} |Y_t| - |Y_{t-1}| &= \text{sign}(Y_{t-1})(Y_t - Y_{t-1}) \\ &\leqslant \text{sign}(Y_{t-1})U_{t-1} \\ &\leqslant |Z_t| - |Z_{t-1}| \end{aligned}$$

where the first inequality follows from equation (6.26). It follows by induction that $|Z_t| \geqslant |Y_t| - \frac{3}{n}$ for all $t \leqslant \tau^*$. In particular we have by equation (6.25) that

$$\mathbb{E}_{\sigma_0}\left[\max_{2 \leqslant i<j \leqslant q} \max_{0 \leqslant t \leqslant \tau^*} \left|S_t^i - S_t^j\right|\right]^2 = O\left(\frac{t(n)}{n^2}\right) = o(1). \tag{6.27}$$

By Markov's inequality with probability tending to 1 we have that $\left|S_{\tau^*}^i - S_{\tau^*}^j\right| = o(1)$ for every pair $2 \leqslant i, j \leqslant q$. Now by construction $S_{\tau^*}^1 \geqslant y - \frac{1}{n}$ so with high probability we have that $S_{\tau^*}^i \leqslant \frac{1}{q} - \frac{y-\frac{1}{q}}{q-1} + o(1) < \frac{1}{q}$ which implies that with high probability $\tau^* = \min\{t(n), \tau_y^-\}$.

Now, by Taylor series expansions,

$$\begin{aligned} 0 \leqslant \left(\sum_{i=2}^q e^{2\beta S_t^i}\right) - (q-1)e^{2\beta\frac{1-S_t^1}{q-1}} &\leqslant \sum_{i=2}^q (2\beta S_t^i - 2\beta\frac{1-S_t^1}{q-1}) + \sum_{i=2}^q O\left((2\beta S_t^i - 2\beta\frac{1-S_t^1}{q-1})^2\right) \\ &\leqslant O\left(\left(\max_{2 \leqslant i<j \leqslant q}\left|S_t^i - S_t^j\right|\right)^2\right), \end{aligned}$$

where the first inequality is by Jensen, and we have used the fact that $\sum_{i=2}^q S_t^i = 1 - S_t^1$. It therefore follows that with high probability for all $0 \leqslant t \leqslant \max\{t(n), \tau_y^-\}$ that

$$\begin{aligned} n\mathbb{E}_{\sigma_0}\left[S_t^1 - S_{t-1}^1 \mid \mathcal{F}_{t-1}\right] &= \frac{e^{2\beta S_{t-1}^1}}{\sum_{i=1^q} e^{2\beta S_{t-1}^i}} - S_{t-1}^1 \\ &= \frac{e^{2\beta S_{t-1}^1}}{e^{2\beta S_{t-1}^1} + (q-1)e^{2\beta\frac{1-S_{t-1}^1}{q-1}}} - S_{t-1}^1 - O\left(\left(\max_{2 \leqslant i<j \leqslant q}\left|S_t^i - S_t^j\right|\right)^2\right) \\ &= D_\beta(S_{t-1}^1) - O\left(\left(\max_{2 \leqslant i<j \leqslant q}\left|S_t^i - S_t^j\right|\right)^2\right) \end{aligned}$$

and hence that

$$\begin{aligned} &\max_{0 \leqslant t \leqslant \tau^*} D_\beta(S_{t-1}^1) - n\mathbb{E}_{\sigma_0}\left[S_t^1 - S_{t-1}^1 \mid \mathcal{F}_{t-1}\right] \\ &\qquad \leqslant O\left(\left(\max_{2 \leqslant i<j \leqslant q} \max_{0 \leqslant t \leqslant \tau^*}\left|S_t^i - S_t^j\right|\right)^2\right) \end{aligned}$$

which combined with equation (6.27) and Markov's inequality completes the result. □

Proof of Proposition 6.5. Starting with part (1),

$$\begin{aligned}
&\Psi(\rho)-\Psi(-\rho)\\
&= \int_{-\rho}^{\rho}\frac{n\,dx}{\zeta(n)+ax^2}+\int_{-\rho}^{\rho}\left(\frac{n}{\zeta(n)+ax^2}-\frac{n}{\zeta(n)+ax^2+bx^3+cx^4}\right)dx\\
&= \frac{2n}{\sqrt{a\zeta(n)}}\tan^{-1}\left(\frac{\rho\sqrt{a}}{\sqrt{\zeta(n)}}\right)+n\int_{-\rho}^{\rho}\left(-\frac{bx^3+cx^4}{(\zeta(n)+ax^2)^2}+O\left(\frac{\left(bx^3+cx^4\right)^2}{(\zeta(n)+ax^2)^3}\right)\right)dx\\
&= \frac{\pi}{\sqrt{a}}\frac{n}{\sqrt{\zeta(n)}}+O(n)+O\left(n\int_{-\rho}^{\rho}\left(-\frac{cx^4}{a^2x^4}+\frac{\left(bx^3+cx^4\right)^2}{a^3x^6}\right)dx\right)\\
&= t^{\zeta,a}(n)+O(n)+O\left(n\int_{-\rho}^{\rho}dx\right)=t^{\zeta,a}(n)+O(n).
\end{aligned}$$

To prove part (2), we use the law of total variance:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}M_t &= \mathbb{V}\mathrm{ar}\,\mathbb{E}[M_t|\mathcal{F}_{t-1}]\;+\;\mathbb{E}\,\mathbb{V}\mathrm{ar}[M_t|\mathcal{F}_{t-1}]\\
&= \mathbb{V}\mathrm{ar}M_{t-1}\;+\;\mathbb{E}\,\mathbb{V}\mathrm{ar}[Y_t|\mathcal{F}_{t-1}]\\
&\leqslant \mathbb{V}\mathrm{ar}M_{t-1}\;+\;\max_{|z|\leqslant 2\rho_0}\left|\Psi'(z)\right|^2\mathbb{E}\,\mathbb{V}\mathrm{ar}[Z_t-Z_{t-1}|\mathcal{F}_{t-1}]\\
&\leqslant \mathbb{V}\mathrm{ar}M_{t-1}\;+\;\frac{n^2}{\zeta^2(n)}\frac{1}{n^2}.
\end{aligned}$$

Hence by induction

$$\mathbb{E}M^2_{t_\gamma^{\zeta,a}(n)}=\mathbb{V}\mathrm{ar}M_{t_\gamma^{\zeta,a}(n)}\leqslant\frac{t_\gamma^{\zeta,a}(n)}{\zeta^2(n)}=O(w^\zeta(n)^2).$$

As for part (3),

$$\begin{aligned}
A_{t+1}-A_t &= \mathbb{E}[Y_{t+1}-Y_t|\mathcal{F}_t]\\
&= \mathbb{E}\left[\Psi(Z_{t+1})-\Psi(Z_t)|\,\mathcal{F}_t\right]-1\\
&= \mathbb{E}\left[\Psi'(Z_t)(Z_{t+1}-Z_t)\;+\;O(\max_{|z|\leqslant 2\rho}\left|\Psi''(z)\right|(Z_{t+1}-Z_t)^2)\;\middle|\;\mathcal{F}_t\right]-1\\
&= \Psi'(Z_t)\mathbb{E}[Z_{t+1}-Z_t|\mathcal{F}_t]\;+\;O(\max_{|z|\leqslant 2\rho_0}\left|\Psi''(z)\right|n^{-2})-1\\
&= O(\max_{|z|\leqslant 2\rho}\left|\Psi''(z)\right|n^{-2})=O\left(\frac{1}{n\zeta^{3/2}(n)}\right),
\end{aligned}$$

where the last inequality follows from

$$\begin{aligned}
\left|\frac{d^2}{dz^2}\Psi(z)\right| &= \frac{n|2az+3bz^2+4cz^3|}{(\zeta(n)+az^2+bz^3+cz^4)^2}\\
&\leqslant \frac{nC_1|z|}{(\zeta(n)+C_2z^2)^2}=O\left(\frac{n}{\zeta^{3/2}(n)}\right).
\end{aligned}$$

if ρ is small enough. Then again by induction, we conclude that

$$A_{t_\gamma^{\zeta,a}(n)}=O\left(\frac{t_\gamma^{\zeta,a}(n)}{n\zeta^{3/2}(n)}\right)=o\left(w^\zeta(n)\right).$$

□

Proof of Proposition 6.6. If r is large enough and ρ is small, we have $f(z) \geqslant n^{-1}(a/2)z^2$ for all $w_0 < |z| < \rho_0$, where as before $w_0 = w/2$ and $\rho_0 = 2\rho$. This immediately gives part (1) with $k(r) = Cr^{-1}$.

The proof for parts (2),(3) are similar to the ones in Proposition 6.5. This time the bounds on the derivatives become

$$\max_{w_0/2<|z|<2\rho_0} \left|\Psi'(z)\right|^2 \leqslant Cr^{-4}n^{10/3} \quad ; \quad \max_{w_0/2<|z|<2\rho_0} \left|\Psi''(z)\right| \leqslant Cr^{-3}n^2.$$

Proceeding by induction as before, we obtain (3), (2) with $k(r) = Cr^{-4}$ and $k(r) = Cr^{-3}$ respectively. □

7. Essential Mixing

Proof of Theorem 4. As the reader can verify, most statements in Sections 3 and 4 hold when $\beta < \beta_c(q)$ and even $\beta < q/2$ (the restrictions on β are indicated before each statement there). The only time $\beta < \beta_s(q) < \beta_c(q)$ is required is in step (1) of the overall coupling, where the condition ensures that the drift of each single coordinate S_t^i is negative in all $(1/q, 1]$, which, in turn, implies that for any initial configuration, after $t = O(n)$ time, $\sigma_t \in \Sigma_n^\rho$, which is a necessary starting point for the couplings that follow.

Now, if $\beta \geqslant \beta_s(q)$, but still $\beta < \beta_c(q)$, we may replace this step, with the requirement that σ_0 is initially chosen from $\widetilde{\Sigma}_n = \Sigma_n^\rho$. The analysis of the overall coupling will remain the same, with the coalescence time being even smaller (but just by a linear term, which can be absorbed in the cutoff-window term). Thus, the restricted mixing time $t_{\mathrm{MIX}(\epsilon)}^{\widetilde{\Sigma_n}}(n)$ will be upper bounded as before. In addition, the lower bound in Subsection 4.1 will also hold for $t_{\mathrm{MIX}(\epsilon)}^{\widetilde{\Sigma_n}}(n)$, since as initial configuration, we may take any $\sigma_0 \in \partial_{P_n}\Sigma_n(\rho)$ for $\rho > 0$.

It remains to show that $\Sigma_n \setminus \widetilde{\Sigma}_n$ has an exponentially decreasing probability under μ_n. This follows immediately from the large deviations analysis in Subsection 2.2. If $\beta < \beta_c(q)$, the rate function $I_{\beta,q}$ is strictly positive away from $\widehat{\mathsf{e}}$ and in particular there exist $C_1 > 0$, $C_2 > 0$, such that

$$\mu_n\left(\Sigma_n \setminus \widetilde{\Sigma}_n\right) = \pi_n\left(\mathcal{S}_n \setminus \mathcal{S}_n^{\rho_1}\right) \leqslant C_1 e^{-C_2 n}.$$

This concludes the proof of the theorem. □

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