QUASI-POLYNOMIAL MIXING OF CRITICAL 2D RANDOM CLUSTER MODELS

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Abstract. We study the Glauber dynamics for the random cluster (FK) model on the torus $(\mathbb{Z}/n\mathbb{Z})^2$ with parameters (p, q), for $q \in (1, 4]$ and p the critical point p_c. The dynamics is believed to undergo a *critical slowdown*, with its continuous-time mixing time transitioning from $O(\log n)$ for $p \neq p_c$ to a power-law in n at $p = p_c$. This was verified at $p \neq p_c$ by Blanca and Sinclair, whereas at the critical $p = p_c$, with the exception of the special integer points $q = 2, 3, 4$ (where the model corresponds to Ising/Potts models) the best-known upper bound on mixing was exponential in n. Here we prove an upper bound of $n^{O(\log n)}$ at $p = p_c$ for all $q \in (1, 4]$, where the key ingredient is controlling the number of disjoint long-range crossings at criticality.

1. Introduction

The random cluster (FK) model is an extensively studied model in statistical physics, generalizing electrical networks, percolation, and the Ising and Potts models, to name a few, under a single unifying framework. It is defined on a graph $G = (V, E)$ with parameters $0 < p < 1$ and $q > 0$ as the probability measure over subsets $\omega \subset E$ (or equivalently, configurations $\omega \in \{0, 1\}^E$), given by

$$\pi_{G,p,q}(\omega) \propto p^{|\omega|}(1 - p)^{|E|-|\omega|}q^{k(\omega)} ,$$

where $k(\omega)$ is the number of connected components (clusters) in the graph (V, ω).

At $q = 1$, the FK model reduces to bond percolation on G, and for integer $q \geq 2$ it corresponds via the Edwards–Sokal coupling [9] to the Ising $(q = 2)$ and Potts $(q \geq 3)$ models on $V(G)$. Since its introduction around 1970, this model has been well-studied both in its own right and as a means of analyzing the Ising and Potts models, with an emphasis on $\mathbb{Z}^d$ as the underlying graph. There, for every $q \in [1, \infty)$, the model enjoys monotonicity, and exhibits a phase transition at a critical $p_c(q)$ w.r.t. the existence (almost surely) of an infinite cluster (see, e.g., [11] and the references therein).

Significant progress has been made on the model at $d = 2$, in particular for $1 \leq q \leq 4$ where the model is expected to be conformally invariant (see [21, Problem 2.6]). It is known [1] that $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ on $\mathbb{Z}^2$ for all $q \geq 1$. Moreover, while the phase transition at this p_c is conjectured to be discontinuous if $q > 4$ (as confirmed [13] for all $q > 25$), it is continuous for $1 \leq q \leq 4$ (as established in [8]). There, the probability that x belongs to the cluster of the origin decays as $\exp(-c|x|)$ at $p < p_c$, as a power-law $|x|^{-\eta}$ at the critical p_c, and is bounded away from 0 at $p > p_c$.

Here we study heat-bath Glauber dynamics for the two-dimensional FK model, where the following *critical slowdown* phenomenon is expected: on an $n \times n$ torus, for all $p \neq p_c$ the mixing time of the dynamics should have order $\log n$ (recently shown by [2]), yet at $p = p_c$ it should behave as $\exp(cn)$ in the presence of a discontinuous phase transition, and as n^z for some universal $z > 0$ in the presence of a continuous phase transition. The critical behavior in the former case was established in a companion paper [10], as was a critical power-law in the cases $q = 2$ ([16]) and $q = 3$ ([10]). In this work we obtain a quasi-polynomial upper bound for non-integer $1 < q \leq 4$ at criticality.

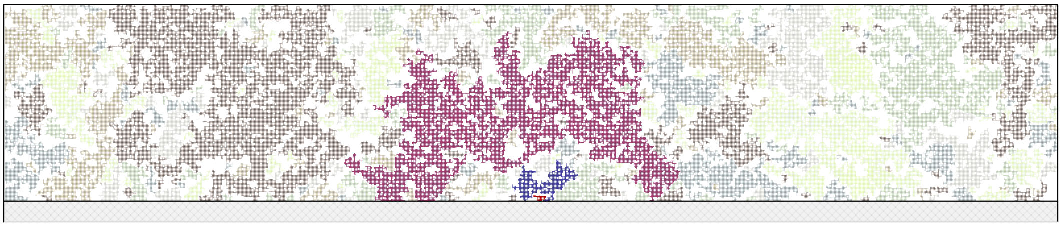

FIGURE 1. A critical FK configuration that induces three distinct boundary components bridging over an edge on the bottom segment.

More precisely, Glauber dynamics for the FK measure $\pi_{G,p,q}$ is the continuous-time Markov chain $(X_t)_{t\geq 0}$ that assigns each edge $e \in E$ a rate-1 independent Poisson clock, where upon ringing, $X_t(e)$ is resampled via $\pi_{G,p,q}$ conditioned on the values of $X_t\restriction_{E-\{e\}}$. This Markov chain is reversible by construction w.r.t. $\pi_{G,p,q}$, and may hence be viewed both as a natural model for the dynamical evolution of this interacting particle system, and as a simple protocol for sampling from its equilibrium measure. A central question is then to estimate the time it takes this chain to convergence to stationarity, measured in terms of the total variation mixing time t_{MIX} (see §2.2 for the related definitions).

For $p \neq p_c$, the fact that $t_{\text{MIX}} \asymp \log n$ was established in [2] using the aforementioned exponential decay of cluster diameters in the high-temperature regime $p < p_c$: on finite boxes with certain boundary conditions, this translates to a property known as *strong spatial mixing*, implying that the number of disagreements between the states of two chains started at different initial states decreases exponentially fast, thus $t_{\text{MIX}} \asymp \log n$; this result readily extends to $p > p_c$ by the duality of the two-dimensional FK model.

At $p = p_c$, where there no longer is an exponential decay of correlations, polynomial upper bounds on t_{MIX} were obtained for the Ising model in [16] and the 3-state Potts model—along with a quasi-polynomial bound for the 4-state model—in [10], using a multiscale approach that reduced the side length of the box by a constant factor in each step at a cost of a constant factor in the upper bound via a coupling argument; these carry over to the FK model for $q = 2, 3, 4$ via the comparison estimates of [22]. However, for non-integer q, FK configurations may form macroscopic clusters along the boundary of smaller-scale boxes, destroying the coupling—a situation that is prevented for integer q thanks to the special relation between FK/Potts models (see Fig. 1).

It was recently shown [12] that for $q = 2$ the FK Glauber dynamics on *any* graph $G = (V, E)$ has $t_{\text{MIX}} \leq |E|^{O(1)}$; the technique there, however, is highly specific to the case of $q = 2$. Indeed, this bound does not hold on $\mathbb{Z}^d$, for any $d \geq 2$, at $p = p_c$ and q large enough, as follows from the exponential lower bounds of [3, 4] on the mixing time of Swendsen–Wang dynamics for the Potts model (see, e.g., [6, 10] for further details). The best prior bound on non-integer $1 < q < 4$ was $t_{\text{MIX}} \leq \exp(O(n))$.

In the present paper, we prove that for periodic boundary conditions (as well as a wide class of others, including wired and free; see Remark 1.1), the following holds:

Theorem 1. *Let $q \in (1, 4]$ and consider the Glauber dynamics for the critical FK model on $(\mathbb{Z}/n\mathbb{Z})^2$. There exists $c = c(q) > 0$ such that $t_{\text{MIX}} \lesssim n^{c \log n}$.*

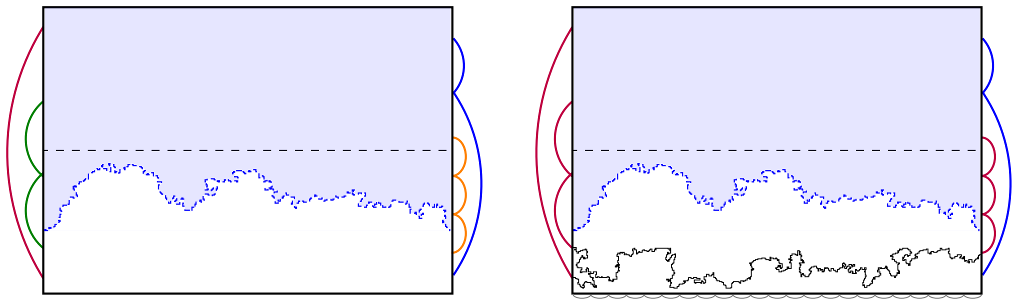

FIGURE 2. Macroscopic disjoint boundary bridges prevent the coupling of FK configurations sampled under two different boundary conditions on $\partial_{\mathrm{S}}\Lambda$ from being coupled past a common horizontal dual-crossing.

Remark 1.1. Theorem 1 holds for rectangles with uniformly bounded aspect ratio, under any set of boundary conditions with the following property: for every edge e on the boundary of the box, there are $O(\log n)$ distinct boundary components connecting vertices on either side of e (see Definitions 5.1–5.2 and Theorem 5.4). This includes, in particular, the wired and free boundary conditions, as well as, with high probability, "typical" boundary conditions: those that are sampled under $\pi_{\mathbb{Z}^2,p_c,q}$ (see Lemma 5.7).

Remark 1.2. For $q \in \{2, 3, 4\}$, the comparison estimates of [22] carry the upper bounds on the mixing time of the Potts model to the FK model, yet only for a limited class of boundary conditions (e.g., the partition of boundary vertices can have at most one cluster of size larger than n^{ε}, in contrast to "typical" ones as above). The above theorem thus extends the class of FK boundary conditions for which t_{MIX} is quasi-polynomial.

Remark 1.3. Theorem 1 implies analogous upper bounds for other flavors of Glauber dynamics (e.g., Metropolis) by standard comparison estimates [14, Lemma 13.22], as well as global cluster dynamics (e.g., Chayes–Machta [5]) via the estimates of [22].

1.1. **Main techniques.** As mentioned above, in [16] and then [10], the polynomial upper bounds on the mixing time at the critical temperature of the models at $q \in \{2, 3\}$ relied on Russo–Seymour–Welsh (RSW) bounds [7, 8] to expose a dual-interface in the FK representation of the model, beyond which block dynamics chains could be coupled. However, the fact that chains, started from any two initial configurations, could be coupled past that dual-interface, relied on a certain (exponentially rare) conditional event, implicit in the relation between the FK and Potts models at integer q (namely that no two distinct boundary components were connected in the interior). Outside of this conditional space, connections between two distinct components on one side of a rectangle could dramatically alter the boundary conditions elsewhere (see Fig. 2). We refer to such distinct boundary components as *bridges*, as they cross over dual-interfaces in the interior of the box and allow long-range interactions (see Definition 3.4 in §3.2).

Such a difficulty was pointed out in [2] in the high-temperature regime and later in [10] at criticality when the phase transition is discontinuous. In both cases, however, the exponential decay of correlations under $\pi^0_{\mathbb{Z}^2}$ ensured such bridges would be, with high probability, microscopic: in [2] these bridges were disregarded by only considering side-homogenous (wired or free on sides) boundary conditions, while in [10], the relevant boundary segments could be completely disconnected by brute-force modifications.

In contrast, in the present setting, due to the power-law decay of correlations, for FK boundary conditions induced by $\pi_{\mathbb{Z}^2}$ there will typically be many boundary edges with some $c \log n$ distinct macroscopically-long bridges over each (see Corollary 3.13). Therefore, to obtain a quasi-polynomial upper bound at criticality, it is necessary to control the effect of such bridges on mixing; we do so by restricting our attention to boundary conditions that have $O(\log n)$ distinct boundary bridges over any one edge.

Using the RSW bounds of [8], we obtain upper and lower exponential tails on the number of disjoint bridges over a fixed edge under the FK measure (see Proposition 3.9, which may be of independent interest). Consequently, we prove that if we sample from the FK measure on a $2n \times 2n$ box Λ with *arbitrary* boundary conditions, the boundary conditions this induces on the concentric inner $n \times n$ box will belong to our permissible class of "typical" boundary conditions with probability $1 - O(n^{-c})$ (see Lemma 5.7).

To maintain "typical" (as opposed to worst-case) boundary conditions throughout the multi-scale analysis, we turn to the Peres–Winkler censoring inequalities [19] for a monotone spin system, that were used in [18] (then later in [15]) for the Ising model under "plus" boundary, a class of boundary conditions that *dominate the plus phase*.

A major issue when attempting to carry out this approach—adapting the analysis of the low temperature Ising model to the critical FK model—is the stark difference between the nature of the corresponding equilibrium estimates needed to drive the multi-scale analysis. In the former, crucial to maintaining "plus" boundary condition throughout the induction of [18] was that in the presence of favorable boundary conditions, the multiscale analysis could be controlled except with super-polynomially small probability. This yielded a bound on coupling the dynamics started at the extremal (plus and minus) initial configurations, which a standard union bound over the $O(n^2)$ sites of the box (see Fact 2.4) then transformed to a bound on t_{MIX}.

In our case, we again wish to couple the dynamics from the extremal (wired and free) initial configurations, since starting from an arbitrary state may induce boundary conditions on the smaller scales that are not "typical" ones. However, even in the ideal scenario where boundary conditions on a given block have *no* bridges, the probability that we can couple the dynamics from wired and free initial states on that block is at best $\varepsilon > 0$ (as per the RSW estimates), rather than $1 - o(n^{-2})$. In particular, we cannot afford the $O(n^2)$ factor of translating this to a bound on t_{MIX} even in this ideal setting—and the actual setting is far worse, replacing ε by n^{-c}.

Instead, we construct a censored dynamics that mimics a systematic block dynamics chain (one where the order of updated blocks is deterministically chosen), and bound the total variation distance between their distributions in terms of the probability that we encounter unfavorable boundary conditions on the sub-blocks along the way (see Proposition 4.8). By doing so, we compare the censored block dynamics to the systematic block dynamics with boundary conditions modified to eliminate all $O(\log n)$ bridges over a particular edge, paying a cost of n^c. We may then let the systematic block dynamics run some n^c rounds, and use the absence of bridges at a particular edge, to boost the probability of coupling from $o(1)$ to $1 - o(1)$. This extends to our censored dynamics via induction on the smaller scale blocks (enjoying favorable boundary conditions except with negligible probability), yielding a bound on t_{MIX}.

2. Preliminaries

In this section we define the random cluster (FK) model and the FK dynamics that will be the object of study in this paper. We also recall various important results from the equilibrium theory of the FK model in §2.1, and the general theory of Markov chain mixing times (§2.2), including, in particular, that of monotone chains. Throughout the paper, for sequences $f(N), g(N)$ we will write $f \lesssim g$ if there exists a constant $c > 0$ such that $f(N) \leq cg(N)$ for all N and $f \asymp g$ if $f(N) \lesssim g(N) \lesssim f(N)$.

For a more detailed exposition of much of §2.1 see [11], and for a more detailed exposition of the main ideas in §2.2 see [14].

2.1. The FK model.

Throughout the paper, we identify an FK configuration $\omega \subset E$ with an assignment $E \to \{0,1\}$, referring to an edge e with $\omega(e) = 1$ as open and to an edge e with $\omega(e) = 0$ as closed. We will drop the subscripts p, q from $\pi_{G,p,q}$ whenever their value is clear from the context.

Boundary conditions. For a graph G, one can fix an arbitrary subset of the vertices to be the boundary ∂G so that a *boundary condition* ξ on ∂G is a partition of $V(\partial G)$ into distinct components. The *wired* boundary condition consists of just one component consisting of all $v \in V(\partial G)$, and the *free* boundary condition consists of only singletons, each corresponding to one vertex in $V(\partial G)$. For ease of notation, in the former case we say $\xi = 1$ and in the latter case we say $\xi = 0$. Denote the interior of G by $G^o = G - \partial G$.

For two domains $R_1 \subset R_2$, we say that a configuration ω on R_2 with boundary condition ξ induces a boundary condition ζ on R_1 if ζ is the boundary condition induced by $\omega\restriction_{R_2 - R_1^o} \cup \xi$: here the union of two boundary conditions denotes the partition arising from all connections through $\omega\restriction_{R_2 - R_1^o}$ and ξ together. In such situations, when we write $\omega\restriction_{\partial R_1}$ we mean the boundary condition induced on R_1 by ω on $R_2 - R_1^o$ and ξ. If two sites x, y are in the same component of a boundary condition ξ, we write $x \xleftrightarrow{\xi} y$.

Domain Markov property. For any q, the FK model satisfies the Domain Markov property: that is to say, for any graph G and any boundary conditions ξ on ∂G, for every subgraph $F \subset G$ and configuration η on $G - F$,

$$\pi_F^{\xi \cup \eta}(\omega \in \cdot) = \pi_G^{\xi}(\omega\restriction_F \in \cdot \mid \omega\restriction_{G-F} = \eta)\,.$$

Monotonicity and FKG inequalities. There is a natural partial ordering to configurations and boundary conditions in the FK model: for two configurations $\omega, \omega' \in \Omega$ we say $\omega \geq \omega'$ if $\omega(e) \geq \omega'(e)$ for every edge $e \in E$, and for any two boundary conditions ξ, ξ' we say that $\xi \geq \xi'$ if $x \xleftrightarrow{\xi'} y$ implies $x \xleftrightarrow{\xi} y$ for every pair of sites $x, y \in V(\partial G)$, which is to say that ξ' corresponds to a finer partition than ξ of the vertices $V(\partial G)$.

An event A is *increasing* if it is closed under addition of edges so that if $\omega \leq \omega'$, then $\omega \in A$ implies $\omega' \in A$; analogously, it is decreasing if it is closed under removal of edges. The FK model satisfies *FKG inequalities* for all $q \geq 1$ (i.e., it is positively correlated) so that for any two increasing events A, B,

$$\pi_G^{\xi}(A \cap B) \geq \pi_G^{\xi}(A)\pi_G^{\xi}(B)\,.$$

This leads to *monotonicity in boundary conditions* for all $q \geq 1$. For any pair of boundary conditions ξ, ξ' with $\xi' \leq \xi$, and any increasing event A,

$$\pi_G^{\xi'}(A) \leq \pi_G^{\xi}(A)\,,$$

whence we say that π_G^{ξ} stochastically dominates ($\succeq$) $\pi_G^{\xi'}$.

Planar duality. For the purposes of this paper, we now restrict our attention to graphs that are subsets of $\mathbb{Z}^2$, the graph with vertices at $\mathbb{Z}^2$ and edges between nearest-neighbors in Euclidean distance. For a connected graph $G \subset \mathbb{Z}^2$, we let ∂G consist of all $v \in V$ having a $\mathbb{Z}^2$-neighbor in $\mathbb{Z}^2 - G$ along with all edges between such vertices.

For a graph $G \subset \mathbb{Z}^2$ (in fact for any planar graph), there is a powerful duality between the FK model on G and the FK model on G^* where G^* is the dual graph of G. Given a planar graph G, we can identify to any configuration ω a dual configuration ω^* on G^* where (identifying to each $e \in E(G)$, the unique dual edge e^* passing through e), $\omega^*(e^*) = 1$ if and only if $\omega(e) = 0$. We sometimes identify edges with their midpoints.

For any boundary condition ξ on a planar graph G, for all $q \geq 1$, the map $p \mapsto p^*$ where $pp^* = q(1-p)(1-p^*)$ can be seen to satisfy

$$\pi_{G,p,q}^{\xi} \stackrel{d}{=} \pi_{G^*,p^*,q}^{\xi^*}$$

where the boundary condition ξ^* is determined on a case by case basis so that $(\xi^*)^* = \xi$ (in particular, the wired and free boundary conditions are dual to each other).

Planar notation. The graphs we consider will be rectangular subsets of $\mathbb{Z}^2$, denoted,

$$\Lambda_{n,m} = [\![0,n]\!] \times [\![0,m]\!]\,,$$

where throughout the paper, $[\![0,n]\!] := \{k \in \mathbb{Z} : 0 \leq k \leq n\}$. When n, m are fixed and understood from context, we drop them from the notation. Then we denote the sides of $\partial\Lambda$ by $\partial_{\mathrm{W}}\Lambda = \{0\} \times [\![0,m]\!]$ and the analogously defined $\partial_{\mathrm{N}}\Lambda, \partial_{\mathrm{S}}\Lambda, \partial_{\mathrm{E}}\Lambda$. We collect multiple sides into their union by including both subscripts, e.g., $\partial_{\mathrm{N,S}}\Lambda = \partial_{\mathrm{N}}\Lambda \cup \partial_{\mathrm{S}}\Lambda$.

Consider the FK model on a rectangular graph Λ. For any $x, y \in \Lambda$, we write $x \longleftrightarrow y$ if x and y are part of the same component of ω on $\Lambda - \partial\Lambda$ (there exists a connected set of open edges with one edge adjacent x and one adjacent y). For a subset $R \subset \Lambda$, we write $x \stackrel{R}{\longleftrightarrow} y$ to denote the existence of such a crossing within $R - \partial R$, and for two sets $A, B \subset \Lambda$ we write $A \longleftrightarrow B$ if there exists $a \in A, b \in B$ such that $a \longleftrightarrow b$.

We now define the vertical crossing event for a rectangle Λ as

$$\mathcal{C}_v(\Lambda) = \partial_{\mathrm{S}}\Lambda \stackrel{\Lambda}{\longleftrightarrow} \partial_{\mathrm{N}}\Lambda\,,$$

and analogously define the horizontal crossing event $\mathcal{C}_h(\Lambda)$. One can similarly define the dual-crossing events $\mathcal{C}_v^*(\Lambda), \mathcal{C}_h^*(\Lambda)$ (where abusing notation, the fact that the crossings occur on Λ^* is understood) and more generally, writing $x^* \stackrel{*}{\longleftrightarrow} y^*$ denotes the existence of a connection in the dual graph. Then, crucially, planarity and self-duality of $\mathbb{Z}^2$ imply that for a rectangle Λ, we have $\mathcal{C}_v(\Lambda) = (\mathcal{C}_h^*(\Lambda))^c$.

Finally for two rectangles $\Lambda' \subset \Lambda$, an annulus $A = \Lambda - \Lambda'$, denote the existence of an *open circuit* (connected set of open edges with nontrivial homology w.r.t. A) by $\mathcal{C}_o(A)$.

Gibbs measures and the FK phase transition. Infinite-volume Gibbs measures can be derived by taking limits of $\pi^{\xi_n}_{\Lambda_{n,n}}$ as $n \to \infty$ for a prescribed sequence of boundary conditions ξ_n: natural choices of such boundary conditions are $\xi_n = 1, 0$ or periodic so that the graph is $(\mathbb{Z}/n\mathbb{Z})^2$. If such limits exist weakly, we denote them by $\pi^{\xi}_{\mathbb{Z}^2}$, and they satisfy the DLR conditions (see, e.g., [11]).

By the self-duality of $\mathbb{Z}^2$ (up to translation), one sees that at the fixed point of $p \mapsto p^*$, ($p_{\mathrm{sd}} = \frac{\sqrt{q}}{1+\sqrt{q}}$), one has $\pi^1_{\mathbb{Z}^2} \overset{d}{=} \pi^0_{(\mathbb{Z}^2)^*}$, and we say the model is self-dual. The FK model for $q \geq 1$ exhibits a sharp phase transition between a high temperature phase (p small) where there is no infinite component, and a low temperature phase (p large) where there is almost surely an infinite component, through a critical point $p_c(q) = \inf\{p \in [0,1] : \pi_{\mathbb{Z}^2,p,q}(0 \longleftrightarrow \infty) > 0\}$. It was proved in [1] that for all $q \geq 1$, $p_c(q) = p_{\mathrm{sd}}(q)$, and later in [8] that for all $q \in [1,4]$, we have that $\pi^1_{\mathbb{Z}^2,p_c,q}(0 \longleftrightarrow \infty) = 0$, implying $\pi^1_{\mathbb{Z}^2,p_c,q} \overset{d}{=} \pi^0_{\mathbb{Z}^2,p_c,q}$ and continuity of the phase transition (these were established much earlier for the cases of bond percolation $q = 1$ and the Ising model $q = 2$).

Russo–Seymour–Welsh estimates. A key ingredient in the proof of the continuity of the phase transition for all $q \in [1,4]$ was the following set of Russo–Seymour–Welsh (RSW) type estimates on crossing probabilities of rectangles uniform in the boundary conditions (such results were obtained for $q = 1$ in [20] and for $q = 2$ in [7]), which were central to all available mixing time upper bounds at p_c on $\mathbb{Z}^2$ (see [10, 16]):

Theorem 2.1 ([8, Theorem 3])**.** *Let $q \in (1,4]$ and consider the critical FK model on $\Lambda_{n,n'}$ where $n' = \lfloor \alpha n \rfloor$ for some $\alpha > 0$. For every $\varepsilon > 0$, if $R_\varepsilon = [\![\varepsilon n, (1-\varepsilon)n]\!] \times [\![\varepsilon n', (1-\varepsilon)n']\!]$, there exists a $p(\alpha,\varepsilon,q) > 0$ such that,*

$$\pi^0_\Lambda(\mathcal{C}_v(R_\varepsilon)) \geq p\,.$$

Corollary 2.2. *Let $q \in (1,4]$ and consider the critical FK model on $\Lambda_{n,n'}$. Let R_ε be as in Theorem 2.1; then there exists a $p(\alpha,\varepsilon,q) > 0$ such that*

$$\pi^0_\Lambda(\mathcal{C}_o(\Lambda - R_\varepsilon)) \geq p\,.$$

When $1 < q < 4$, we have the a stronger bound uniform in boundary conditions:

Proposition 2.3 ([8, Theorem 7])**.** *Let $q \in (1,4)$ and consider the critical FK model on $\Lambda_{n,n'}$ where $n' = \lfloor \alpha n \rfloor$ for $\alpha > 0$. There exists $p(\alpha,q) > 0$ such that,*

$$\pi^0_\Lambda(\mathcal{C}_v(\Lambda)) \geq p\,.$$

Such a bound is in fact not expected to hold for $q = 4$, where, for instance, it is believed (see [8]) that under free boundary conditions the crossing probability goes to 0 as $N \to \infty$.

2.2. **Markov chain mixing times.** In this section we introduce the dynamical notation we will be using along with several important results in the theory of Markov chain mixing times, and in particular the theory of Markov chains on monotone spin systems, that we will use in the proof of Theorem 1.

Mixing times. Consider a Markov chain $(X_t)_{t\geq 0}$ with finite state space Ω, and (in discrete time) transition kernel P with invariant measure π. In the continuous-time setup, instead of P^t we consider, for $\omega_0, \omega \in \Omega$, the heat kernel

$$H_t(\omega_0, \omega) = \mathbb{P}_{\omega_0}(X_t = \omega) = e^{t\mathcal{L}}(\omega_0, \omega)\,,$$

where $\mathbb{P}_{\omega_0}$ is the probability w.r.t. the law of the chain $(X_t)_{t\geq 0}$ given $X_0 = \omega_0$, and $\mathcal{L}$ is the infinitesimal generator for the Markov process.

For two measures μ, ν on Ω, define the total variation distance

$$\|\mu - \nu\|_{\mathrm{TV}} = \sup_{A\subset\Omega} |\mu(A) - \nu(A)| = \inf\{\mathbb{P}(X \neq Y) \mid X \sim \mu, Y \sim \nu\}\,,$$

where the infimum is over all couplings (μ, ν). The worst-case total variation distance of X_t from π is denoted

$$d_{\mathrm{TV}}(t) = \max_{\omega_0\in\Omega} \|\mathbb{P}_{\omega_0}(X_t \in \cdot) - \pi\|_{\mathrm{TV}}\,,$$

and the *total variation mixing time* of the Markov chain is given by (for $\varepsilon \in (0, 1)$),

$$t_{\mathrm{MIX}}(\varepsilon) = \inf\{t \geq 0 : d_{\mathrm{TV}}(t) \leq \varepsilon\}\,.$$

For any $\varepsilon \leq \frac{1}{4}$, $t_{\mathrm{MIX}}(\varepsilon)$ is submultiplicative and the convergence to π in total variation distance is thenceforth exponentially fast. As such, we write t_{MIX}, omitting the parameter ε to refer to the standard choice $\varepsilon = 1/(2e)$.

The FK dynamics. The present paper is almost exclusively concerned with continuous-time *heat-bath Glauber dynamics* $(X_t)_{t\geq 0}$ for the random cluster model on Λ with boundary conditions ξ: this is a reversible Markov chain w.r.t. π^ξ_Λ defined as follows: assign i.i.d. rate-1 Poisson clocks to every edge in $\Lambda - \partial\Lambda$; whenever the clock at an edge rings, resample its edge value according to $\pi^\xi_\Lambda(\omega\restriction_e \in \cdot \mid \omega\restriction_{\Lambda-\{e\}} = X_t\restriction_{\Lambda-\{e\}})$. In particular, for $e = (v, w) \in \Lambda - \partial\Lambda$, the transition rate from ω to $\omega \cup \{e\}$ is

$$\begin{cases} p & \text{if } v \longleftrightarrow w \text{ in } \Lambda - \{e\} \cup \xi\,, \\ p/[p + q(1 - p)] & \text{otherwise}\,. \end{cases}$$

An alternative view of the heat-bath dynamics is the *random mapping representation* of this dynamics: the edge updates correspond to a sequence $(J_i, U_i, T_i)_{i\geq 1}$, in which $T_1 < T_2 < \ldots$ are the clock ring times, the J_i's are i.i.d. uniformly selected edges in $\Lambda - \partial\Lambda$, and the U_i's are i.i.d. uniform random variables on $[0, 1]$: at time T_i, for $J_i = (v, w)$, the dynamics replaces the value of $\omega(J_i)$ by $\mathbf{1}\{U_i \leq p\}$ if $v \longleftrightarrow w$ in $\Lambda - \{J_i\} \cup \xi$ and by $\mathbf{1}\{U_i \leq p/[p + q(1 - p)]\}$ otherwise.

Monotonicity. As a result of the monotonicity of the FK model for $q \geq 1$, the heat-bath Glauber dynamics for the FK model is *monotone*: for two initial configurations $\omega' \geq \omega$, we have that for all times $t \geq 0$,

$$H_t(\omega', \cdot) \succeq H_t(\omega, \cdot)\,.$$

Using the random mapping representation, we define the *grand coupling* of the set of Markov chains with all possible initial configurations, which corresponds to the identity coupling of all three random variables $(J_i, U_i, T_i)_{i\geq 1}$ amongst all the chains; for $q \geq 1$, this coupling preserves the partial ordering on initial states for all subsequent times.

The following standard fact is obtained via the grand coupling (see, e.g., [18, Eq. 2.10] in the context of the Ising model, as well as [10] in the context of the FK model).

Fact 2.4. *Consider a set E and a monotone Markov chain $(X_t)_{t\geq 0}$ on $\Omega = \{0,1\}^E$ with extremal configurations $\{0,1\}$. For every $t \geq 0$,*

$$d_{\mathrm{TV}}(t) \leq |E| \|\mathbb{P}_1(X_t \in \cdot) - \mathbb{P}_0(X_t \in \cdot)\|_{\mathrm{TV}}\,.$$

Combined with the triangle inequality one obtains for the sub-multiplicative quantity

$$\bar{d}_{\mathrm{TV}}(t) = \max_{\omega_1,\omega_2\in\Omega} \|P_{\omega_1}(X_t \in \cdot) - \mathbb{P}_{\omega_2}(X_t \in \cdot)\|_{\mathrm{TV}}\,,$$

that, in the FK setting,

$$d_{\mathrm{TV}}(t) \leq \bar{d}_{\mathrm{TV}}(t) \leq 2|E(G)| \|\mathbb{P}_1(X_t \in \cdot) - \mathbb{P}_0(X_t \in \cdot)\|_{\mathrm{TV}}\,. \tag{2.1}$$

Censoring. Key to our proof will be the Peres–Winkler [19] censoring inequality for monotone spin systems (its formulation in continuous-time follows from the same proof of [19, Theorem 1.1]; see [18, Theorem 2.5]).

Theorem 2.5 ([19])**.** *Let μ_T be the law of continuous-time Glauber dynamics at time T of a monotone spin system on Λ with invariant measure π, whose initial distribution μ_0 is such that μ_0/π is increasing. Set $0 = t_0 < t_1 < \ldots < t_k = T$ for some k, let $(B_i)_{i=1}^k$ be subsets of Λ, and let $\bar{\mu}_T$ be the law at time T of the censored dynamics, started at μ_0, where only updates within B_i are kept in the time interval $[t_{i-1}, t_i)$. Then $\|\mu_T - \pi\|_{\mathrm{TV}} \leq \|\bar{\mu}_T - \pi\|_{\mathrm{TV}}$ and $\mu_T \preceq \bar{\mu}_T$; moreover, μ_T/π and $\bar{\mu}_T/\pi$ are both increasing.*

Boundary modifications. Let ξ, ξ' be a pair of boundary conditions on Λ with corresponding mixing times $t_{\mathrm{MIX}}, t'_{\mathrm{MIX}}$; define

$$M_{\xi,\xi'} = \|\pi^\xi_\Lambda / \pi^{\xi'}_\Lambda\|_\infty \;\vee\; \|\pi^{\xi'}_\Lambda / \pi^{\xi}_\Lambda\|_\infty\,.$$

It is well-known (see, e.g., [18, Lemma 2.8]) that for some c independent of n, ξ, ξ',

$$t_{\mathrm{MIX}} \leq c M^3_{\xi,\xi'} |E(\Lambda)| t'_{\mathrm{MIX}} \tag{2.2}$$

(this follows from first bounding t_{MIX} via its spectral gap, then using the variational characterization of the spectral gap: the Dirichlet form, expressed in terms of local variances, gives a factor of $M^2_{\xi,\xi'}$, and the variance produces another factor of $M_{\xi,\xi'}$).

3. Equilibrium estimates

In what follows, fix $q \in (1, 4]$, let $p = p_c(q)$ and drop p, q from the notation henceforth.

3.1. Crossing probabilities.

In this subsection we present estimates on crossing probabilities that will be used to prove the desired mixing time bounds. The following is a slight extension of [10, Theorem 3.4].

Proposition 3.1. *Let $q \in (1, 4]$ and fix $\alpha \in (0, 1]$. Consider the critical FK model on $\Lambda = \Lambda_{n,n'}$ with $\lfloor \alpha n \rfloor \leq n' \leq \lceil \alpha^{-1} n \rceil$. For every $\varepsilon > 0$, there exists $c_\star(\alpha, \varepsilon, q) > 0$ such that for every $x \in [\![\varepsilon n, 1 - \varepsilon n]\!]$, and every boundary condition ξ on $\partial\Lambda$, one has*

$$\pi^\xi_\Lambda\big((x, 0) \longleftrightarrow (x, \lfloor n' \rfloor)\big) \gtrsim n^{-c_\star}\,.$$

Proof. The proposition was proved in the case $n' = \lfloor \alpha n \rfloor$ in [10, Theorem 3.4] by stitching together crossings of rectangles and using the RSW estimates of Theorem 2.1. Since the crossing probabilities of Theorem 2.1 are monotone in the aspect ratio, each is bounded away from zero for aspect ratios in $[\alpha, \alpha^{-1}]$, yielding the desired extension. ∎

The next two results are for $q = 4$ (Proposition 2.3 implies both for $1 < q < 4$).

Lemma 3.2. *Let $q = 4$ and fix $\alpha \in (0,1]$. Consider the critical FK model on $\Lambda = \Lambda_{n,n'}$ with $\lfloor \alpha n \rfloor \leq n' \leq \lceil \alpha^{-1} n \rceil$ and $(1,0)$ boundary conditions denoting wired on $\partial_{\mathrm{S}}\Lambda$ and free elsewhere. For every $\varepsilon > 0$, there exists $p(\alpha, \varepsilon) > 0$ such that*

$$\pi_\Lambda^{1,0}\left(\mathcal{C}_v(\llbracket 0, n\rrbracket \times \llbracket 0, (1-\varepsilon)n'\rrbracket)\right) \geq p(\varepsilon).$$

Proof. Note that for an $n \times n$ square with wired boundary conditions on the N, S sides, and free boundary conditions elsewhere, the probability of a vertical crossing is, by self-duality, $1/2$. By bounding the Radon–Nikodym derivative, it is easy to see that under the same boundary conditions but with the north and south sides disconnected from each other, the same probability is bounded below by some $p_0(q) > 0$.

Moreover, by Theorem 2.1 and monotonicity in boundary conditions, there exists $p_1(\varepsilon) > 0$ such that

$$\pi_\Lambda^{1,0}\left(C_h\left(\llbracket \tfrac{\varepsilon}{4} n, (1-\tfrac{\varepsilon}{4})n\rrbracket \times \llbracket (1-\varepsilon)\alpha n, (1-\tfrac{\varepsilon}{2})\alpha n\rrbracket\right)\right) \geq p_1\,.$$

The measure on $\llbracket (\varepsilon/4)n, (1-\varepsilon/4)n\rrbracket \times \llbracket 0, (1-\varepsilon)n\rrbracket$ conditioned on the above crossing event stochastically dominates the measure induced on it by wired on the N, S sides and free on the E, W sides of $\llbracket (1-\alpha+\frac{\varepsilon\alpha}{2})\frac{n}{2}, (1+\alpha-\frac{\varepsilon\alpha}{2})\frac{n}{2}\rrbracket \times \llbracket 0, (1-\frac{\varepsilon}{2})\alpha n\rrbracket$. By monotonicity in boundary conditions inequality the probability of a vertical crossing in $\llbracket 0, n\rrbracket \times \llbracket 0, (1-\varepsilon)\alpha n\rrbracket$ is thus bigger than $p_0 p_1$. Finally, by Corollary 2.2 and monotonicity of crossing probabilities in aspect ratio, there exists $p_2(\alpha, q) > 0$ such that

$$\pi_\Lambda^{1,0}\left(\mathcal{C}_0(\Lambda - \llbracket (1-\alpha)\tfrac{n}{2}, (1+\alpha)\tfrac{n}{2}\rrbracket \times \llbracket (1-\varepsilon)\alpha n, (1-\varepsilon)n'\rrbracket)\right) \geq p_2$$

holds for every $\lfloor \alpha n \rfloor \leq n' \leq \lceil \alpha^{-1} n \rceil$. By the FKG inequality, stitching the three crossings together implies the desired lower bound for $p = p_0 p_1 p_2$. ∎

Corollary 3.3. *Let $q = 4$ and fix $\alpha \in (0,1]$. Consider the critical FK model on $\Lambda = \Lambda_{n,n'}$ with $\lfloor \alpha n \rfloor \leq n' \leq \lceil \alpha^{-1} n \rceil$ and boundary conditions, denoted by $(1,0,1,0)$, that are wired on $\partial_{\mathrm{N,S}}\Lambda$ and free on $\partial_{\mathrm{E,W}}\Lambda$. Then there exists $p(\alpha) > 0$ such that*

$$\pi_\Lambda^{1,0,1,0}(\mathcal{C}_v(\Lambda)) \geq p\,.$$

Proof. For all $n' \leq n$ this follows immediately from self-duality and monotonicity in boundary conditions. For $n \leq n' \leq \lceil \alpha^{-1} n \rceil$, by monotonicity in boundary conditions and Lemma 3.2, for any $\varepsilon \in (0,1)$, there is a $p(1,\varepsilon) > 0$ such that,

$$\pi_\Lambda^{1,0,1,0}\left(\mathcal{C}_v(\llbracket 0, n\rrbracket \times \llbracket 0, \varepsilon n\rrbracket)\right) \geq p\,,$$

and by reflection symmetry, $\pi_\Lambda^{1,0,1,0}\left(\mathcal{C}_v(\llbracket 0, n\rrbracket \times \llbracket n' - \varepsilon n, n'\rrbracket)\right) \geq p$. Let

$$A_\varepsilon = \Lambda - \llbracket \varepsilon n, (1-\varepsilon)n\rrbracket \times \llbracket \varepsilon n, n' - \varepsilon n\rrbracket\,.$$

Since $\mathcal{C}_o(A_\varepsilon)$ can be lower bounded by four crossings of rectangles, each of whose probabilities is monotone in the aspect ratio and thus bounded away from 0 uniformly

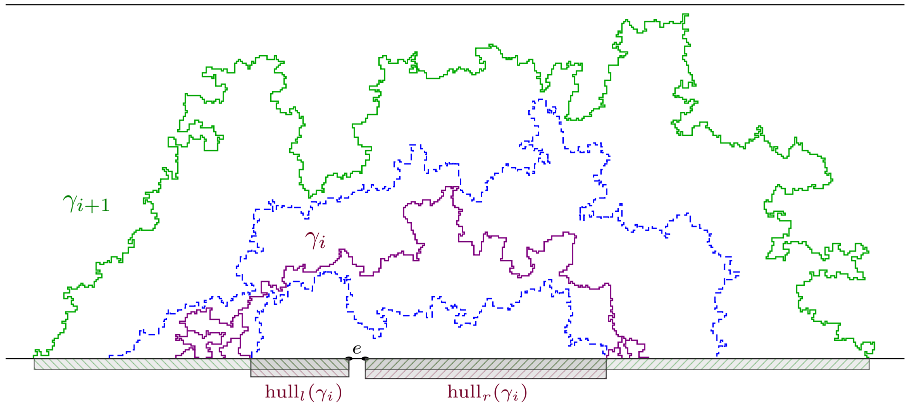

FIGURE 3. A pair of boundary bridges, γ_i, γ_{i+1}, over $e \in \partial_{\mathrm{N}} R$ induced by a configuration on $\Lambda - R$, and separated by a dual-bridge over e.

over $n \le n' \le \lceil \alpha^{-1} n \rceil$, we have that $\pi_\Lambda^{1,0,1,0}(\mathcal{C}_v(A_\varepsilon)) \ge p'$ uniformly over $n \le n' \le \lceil \alpha^{-1} n \rceil$ for some $p'(\alpha, \varepsilon)$. Now observe that

$$\Big(\mathcal{C}_v([\![0,n]\!] \times [\![0,\varepsilon n]\!]) \cap \mathcal{C}_v([\![0,n]\!] \times [\![n'-\varepsilon n, n']\!]) \cap \mathcal{C}_o(A_\varepsilon)\Big) \subset \mathcal{C}_v(\Lambda)\,.$$

After fixing any small $\varepsilon > 0$, by the FKG inequality, there exists some $p(\alpha) > 0$ such that for every $n \le n' \le \lceil \alpha^{-1} n \rceil$, one has $\pi_\Lambda^{1,0,1,0}(\mathcal{C}_v(\Lambda)) \ge p$, as required. ∎

3.2. **Boundary bridges.** In this subsection we define boundary bridges of the FK model and related notation. As explained in detail in §1.1, the presence of boundary bridges will be the key obstacle to coupling and, in turn, to mixing time bounds.

Definition 3.4. Consider a rectangle $\Lambda = \Lambda_{n,n'}$ with boundary conditions ξ, and a connected segment $L = [\![a,b]\!] \times \{n'\} \subset \partial_{\mathrm{N}}\Lambda$. A component $\gamma \subset \partial_{\mathrm{N}}\Lambda$ of ξ is a *bridge over* L if there exist $v = (v_1, v_2), w = (w_1, w_2) \in \gamma$ such that $v \overset{\xi}{\longleftrightarrow} w$ and

$$v_1 < a \qquad \text{and} \qquad w_1 > b\,.$$

Note that every two distinct bridges $\gamma_1 \neq \gamma_2$ over L are disjoint in ξ. Denote by $\Gamma^L = \Gamma^L(\xi)$ the set of all bridges over the segment L. Define bridges on subsets of $\partial\Lambda_{\mathrm{S}}, \partial\Lambda_{\mathrm{E}}, \partial\Lambda_{\mathrm{W}}$ analogously.

Definition 3.5 (hull and length of a bridge)**.** The *left and right hulls* of a bridge γ over $L = [\![a,b]\!] \times \{n'\}$ are defined as

$$\mathrm{hull}_l(\gamma) = [\![\max\{x \le a : (x,y) \in \gamma\}, a]\!] \times \{n'\}\,,$$
$$\mathrm{hull}_r(\gamma) = [\![b, \min\{x \ge b : (x,y) \in \gamma\}]\!] \times \{n'\}\,,$$

so that the hulls of a bridge γ are connected subsets of $\partial_{\mathrm{N}}\Lambda$ (see Fig. 3. The *left and right lengths* of γ are defined to be

$$\ell_l(\gamma) = |\,\mathrm{hull}_l(\gamma)|\,, \qquad \ell_r(\gamma) = |\,\mathrm{hull}_r(\gamma)|\,.$$

Given the above convention, for any L and ξ we can define a *right-ordering* of $\Gamma^L(\xi)$ as $(\gamma_1, \gamma_2, ..., \gamma_{|\Gamma^L|})$ where, for all $i < j$,

$$\ell_r(\gamma_i) < \ell_r(\gamma_j)\,.$$

Note that, in this ordering of the bridges, $\mathrm{hull}_r(\gamma_i) \subsetneq \mathrm{hull}_r(\gamma_j)$ for all $i < j$. Define a *left-ordering* of Γ^L analogously.

Definition 3.6. For a subset $R \subset \Lambda$, an *induced boundary condition on ∂R* is one that can be identified with the component structure of an edge configuration $\omega\restriction_{\Lambda - R^o}$ along with the boundary condition on Λ.

Using the above definitions, and planarity, one can check the following useful facts (depicted in Fig. 3). For concreteness we use the right-ordering of $\Gamma^L = \{\gamma_1, ..., \gamma_{|\Gamma^L|}\}$.

Fact 3.7. *Let $\Lambda \supset R$ with boundary conditions ξ, and let $L \subset \partial_{\mathrm{N}} R$. If γ_i, for $i < |\Gamma^L|$, is the i-th bridge in the right-ordering of Γ^L, then either the two connected components of $\partial_{\mathrm{N}} R - (\mathrm{hull}_l(\gamma_i) \cup L \cup \mathrm{hull}_r(\gamma_i))$ are connected in $\Lambda - R$, or each of these components is connected to $\partial\Lambda$ in $\Lambda - R$.*

Fact 3.8. *Let $\Lambda \supset R$ with boundary conditions ξ, and let $L \subset \partial_{\mathrm{N}} R$. For every two induced bridges $\gamma_1 \neq \gamma_2$ over a segment L such that $\mathrm{hull}_r(\gamma_1) \subset \mathrm{hull}_r(\gamma_2)$, either the two sets $(\mathrm{hull}_l(\gamma_2) \triangle \mathrm{hull}_l(\gamma_1))$ and $(\mathrm{hull}_r(\gamma_2) \triangle \mathrm{hull}_r(\gamma_1))$ are dual-connected in $\Lambda - R$, or each of these sets is dual-connected to $\partial\Lambda$ in $\Lambda - R$.*

3.3. **Estimating the number of boundary bridges.** In this subsection, we bound the number of distinct induced boundary bridges over a segment of ∂R.

When sampling boundary conditions on $R \subset \Lambda$ under π_Λ^ξ, the induced bridges over e and all properties of them, are measurable w.r.t. $\omega\restriction_{\Lambda - R^o}$. For any configuration ω, we denote by $\Gamma^e = \Gamma^e(\omega\restriction_{\Lambda - R^o}, \xi)$ the set of all bridges over e corresponding to that configuration on Λ, with the above defined left and right orderings.

The main estimate on $|\Gamma^e|$, that will be key to the proof of Theorem 1, is the following.

Proposition 3.9. *Let $q \in (1, 4]$ and fix $\alpha \in (0, 1]$. Consider the critical FK model on $\Lambda = \Lambda_{n,n'}$ with $n' \geq \lfloor \alpha n \rfloor$, along with the subset $R = \Lambda_{n,n'/2}$. There exists $c(\alpha, q) > 0$ such that for every $e \in \partial_{\mathrm{N}} R$, every boundary condition ξ, and every $K > 0$,*

$$\pi_\Lambda^\xi(\omega : |\Gamma^e(\omega\restriction_{\Lambda - R^o}, \xi)| \geq K \log n) \lesssim n^{-cK}\,. \tag{3.1}$$

Moreover, there exists $c'(\alpha, q) > 0$, and for every $\varepsilon > 0$ there is some $K_0(\varepsilon)$, such that for every $e \in [\![n^\varepsilon, n - n^\varepsilon]\!] \times \{\lfloor \frac{n'}{2} \rfloor\}$, every boundary condition ξ, and every $K < K_0$,

$$\pi_\Lambda^\xi(\omega : |\Gamma^e(\omega\restriction_{\Lambda - R^o}, \xi)| \geq K \log n) \gtrsim n^{-c'K}\,.$$

Before proving Proposition 3.9, we present some notation and two lemmas central to the proposition, whose proofs are deferred until after the proof of the proposition.

For the rest of this subsection, since e is fixed, if e is in the left half of $\partial_{\mathrm{N}} R$ then we will use the right-ordering of Γ^e and otherwise we will use the left-ordering of Γ^e. If e is in the left half of $\partial_{\mathrm{N}} R$ define the following subsets of Γ^e:

$$\Gamma_1^e = \Gamma_1^e(\omega\restriction_{\Lambda - R^o}, \xi) = \left\{\gamma_i \in \Gamma^e \,:\, \ell_r(\gamma_{i-1}) \leq \tfrac{n}{6}\,,\ \ell_r(\gamma_i) \leq 2\ell_r(\gamma_{i-1})\right\}\,,$$
$$\Gamma_2^e = \Gamma_2^e(\omega\restriction_{\Lambda - R^o}, \xi) = \left\{\gamma_i \in \Gamma^e \,:\, \ell_r(\gamma_{i-1}) \geq \tfrac{n}{6}\,,\ \ell_r(\gamma_i) \leq \tfrac{1}{2}\left(\ell_r(\gamma_{i-1}) + n - x\right)\right\}\,.$$

For e in the right half of $\partial_{\mathrm{N}} R$, define Γ_1^e and Γ_2^e analogously, by replacing ℓ_r with ℓ_l and $n-x$ with x. For convenience, let γ_0 be the possibly nonexistent bridge given by the two vertices incident to the edge e, which will allow us to treat γ_1 as we would treat the other γ_i's.

Lemma 3.10. *There exists $c_1(\alpha,q)>0$ such that for every $e\in\partial_{\mathrm{N}} R$ and ξ,*

$$\pi_\Lambda^\xi\left(\omega : |\Gamma_1^e(\omega\restriction_{\Lambda-R^o},\xi)|\geq m\right)\leq e^{-c_1 m}\,.$$

Lemma 3.11. *There exists $c_2(\alpha,q)>0$ such that for every $e\in\partial_{\mathrm{N}} R$ and ξ,*

$$\pi_\Lambda^\xi\left(\omega : |\Gamma_2^e(\omega\restriction_{\Lambda-R^o},\xi)|\geq m\right)\leq e^{-c_2 m}\,.$$

With these two lemmas in hand the proof of Proposition 3.9 is greatly simplified.

Proof of Proposition 3.9. We begin with the upper bound. Fix an edge $e\in\partial_{\mathrm{N}} R$ and a boundary condition ξ on $\partial\Lambda$. Without loss of generality suppose that e is in the left half of $\partial_{\mathrm{N}} R$ and use the right-ordering of $\Gamma^e=\{\gamma_1,\gamma_2,...,\gamma_{|\Gamma^e|}\}$. Define

$$\Gamma_0^e=\{\gamma_i\in\Gamma^e\ :\ \ell_r(\gamma_i)\leq 2\ell_r(\gamma_{i-1})\,,\, d_r(\gamma_{i-1},\partial)\leq 2d_r(\gamma_i,\partial)\}\subset\Gamma_1^e\cup\Gamma_2^e\,.$$

Observe that violating the second condition in Γ_1^e means that $\ell_r(\gamma_i)$ has doubled the length of its predecessor, whereas violating the second condition in Γ_2^e means that $n-x-\ell_r(\gamma_i)$ is half the corresponding quantity of its predecessor. Since $\ell_r(\gamma_i)\leq n$ and $n-x-\ell_r(\gamma_i)\leq n$ for all i, deterministically

$$|\Gamma-\Gamma_0^e|\leq 2\log_2 n\leq 3\log n\,.$$

Using a union bound,

$$\pi_\Lambda^\xi\left(|\Gamma_0^e|\geq (K-3)\log n\right)\leq \pi_\Lambda^\xi\left(|\Gamma_1^e|\geq \tfrac{K-3}{2}\log n\right)+\pi_\Lambda^\xi\left(|\Gamma_2^e|\geq \tfrac{K-3}{2}\log n\right)\,.$$

The bounds on the two terms on the right-hand side are given by Lemmas 3.10–3.11, respectively. Taking the minimum of c_1,c_2 in those lemmas and taking $m=\frac{K-3}{2}\log n$ then implies that there exists $c(\alpha,q)>0$ such that

$$\pi_\Lambda^\xi(|\Gamma^e|\geq K\log n)\leq 2e^{-c\frac{K-3}{2}\log n}\,.$$

In order to prove the lower bound, for any $\varepsilon>0$, fix any edge $e=(x,\lfloor\frac{n'}{2}\rfloor)$ with $x\in[\![n^\varepsilon,n-n^\varepsilon]\!]$. For $i\geq 1$, suppressing the dependence on e, define

$$\begin{aligned}\tilde R_i^{\mathrm{N}}&=[\![x-2^{i+1},x+2^{i+1}]\!]\times[\![\lfloor\tfrac{n'}{2}\rfloor+2^i,\lfloor\tfrac{n'}{2}\rfloor+2^{i+1}]\!]\,,\\ \tilde R_i^{\mathrm{E}}&=[\![x+2^{i},x+2^{i+1}]\!]\times[\![\lfloor\tfrac{n'}{2}\rfloor-2^i,\lfloor\tfrac{n'}{2}\rfloor+2^{i+1}]\!]\,,\\ \tilde R_i^{\mathrm{W}}&=[\![x-2^{i+1},x-2^{i}]\!]\times[\![\lfloor\tfrac{n'}{2}\rfloor-2^i,\lfloor\tfrac{n'}{2}\rfloor+2^{i+1}]\!]\,,\end{aligned}\tag{3.2}$$

and their respective subsets,

$$\begin{aligned}R_i^{\mathrm{N}}&=[\![x-2^{i+1}+2^{i-1},x+2^{i+1}-2^{i-1}]\!]\times[\![\lfloor\tfrac{n'}{2}\rfloor+2^i,\lfloor\tfrac{n'}{2}\rfloor+2^{i}+2^{i-1}]\!]\,,\\ R_i^{\mathrm{E}}&=[\![x+2^{i},x+2^{i}+2^{i-1}]\!]\times[\![\lfloor\tfrac{n'}{2}\rfloor,\lfloor\tfrac{n'}{2}\rfloor+2^i+2^{i-1}]\!]\,.\\ R_i^{\mathrm{W}}&=[\![x-2^{i}-2^{i-1},x-2^{i}]\!]\times[\![\lfloor\tfrac{n'}{2}\rfloor,\lfloor\tfrac{n'}{2}\rfloor+2^i+2^{i-1}]\!]\,,\end{aligned}$$

When $K < K_0 := \frac{\varepsilon \log 4}{4}$, for every $i \leq 2K \log n$, we have $\tilde{R}_i^{\mathrm{W}}, \tilde{R}_i^{\mathrm{N}}, \tilde{R}_i^{\mathrm{E}} \subset \Lambda$. Also define the following crossing events.

$$\mathcal{A}_i = \mathcal{C}_v(R_i^{\mathrm{W}}) \cap \mathcal{C}_h(R_i^{\mathrm{N}}) \cap \mathcal{C}_v(R_i^{\mathrm{E}})\,,$$
$$\mathcal{A}_i^* = \mathcal{C}_v^*(R_i^{\mathrm{W}}) \cap \mathcal{C}_h^*(R_i^{\mathrm{N}}) \cap \mathcal{C}_v^*(R_i^{\mathrm{E}})\,.$$

Then by definition of distinct bridges in $\Lambda - R^o$, we observe that for each k,

$$\{|\Gamma^e| \geq K \log n\} \supset \bigcap_{i=1}^{K\log n} \mathcal{A}_{2i-1} \cap \mathcal{A}_{2i}^*\,. \tag{3.3}$$

By monotonicity in boundary conditions, the FKG inequality, and Theorem 2.1, there exists $p(\alpha, q) > 0$ such that for every $i \leq 2K \log n$,

$$\pi_\Lambda^\xi(\mathcal{A}_i) \geq \pi^0_{\tilde{R}_i^{\mathrm{W}}}(\mathcal{C}_v(R_i^{\mathrm{W}}))\pi^0_{\tilde{R}_i^{\mathrm{N}}}(\mathcal{C}_h(R_i^{\mathrm{N}}))\pi^0_{\tilde{R}_i^{\mathrm{E}}}(\mathcal{C}_v(R_i^{\mathrm{E}})) \geq p\,,$$
$$\pi_\Lambda^\xi(\mathcal{A}_i^*) \geq \pi^1_{\tilde{R}_i^{\mathrm{W}}}(\mathcal{C}_v^*(R_i^{\mathrm{W}}))\pi^1_{\tilde{R}_i^{\mathrm{N}}}(\mathcal{C}_h^*(R_i^{\mathrm{N}}))\pi^1_{\tilde{R}_i^{\mathrm{E}}}(\mathcal{C}_v^*(R_i^{\mathrm{E}})) \geq p\,.$$

Thus, if $K < K_0$, we have $\pi_\Lambda^\xi(|\Gamma^e| \geq K \log n) \geq p^{2K\log n}$, as required. ■

We now prove Lemmas 3.10–3.11, whose proofs constitute the majority of the work in obtaining Proposition 3.9.

Proof of Lemma 3.10. Assume without loss of generality that $e = (x, \lfloor \frac{n'}{2} \rfloor)$ is such that $x \leq \frac{n}{2}$ and fix the right-ordering of Γ_1^e so that $\Gamma_1^e = \{\gamma_1, \gamma_2, ..., \gamma_{|\Gamma_1^e|}\}$. We prove by induction on m that $\pi_\Lambda^\xi(|\Gamma_1^e| \geq m) \leq p^m$, for the choice of

$$p = 1 - p_1 p_2 p_3\,, \tag{3.4}$$

where

- p_1 is given by Proposition 2.3 with aspect ratio 1 for $1 < q < 4$ and by Lemma 3.2 with the choice $\varepsilon = 1/2$ and aspect ratio $1/2$ for $q = 4$,
- p_2 is the probability given by Theorem 2.1 for $\varepsilon = 1/4$ and aspect ratio $6 \vee \alpha^{-1}$,
- p_3 is the probability given by Theorem 2.1 for $\varepsilon = 1/3$ and aspect ratio 1,

so that $p \in (0,1)$ and depends only on α and q. Let $m \geq 1$, and suppose by induction (starting with the trivial case $m = 0$) that $\pi_\Lambda^\xi(|\Gamma_1^e| \geq m-1) \leq p^{m-1}$. We see that

$$\begin{aligned}\pi_\Lambda^\xi(|\Gamma_1^e| \geq m) &\leq p^{m-1}\pi_\Lambda^\xi(|\Gamma_1^e| \geq m \mid |\Gamma_1^e| \geq m-1) \\ &\leq p^{m-1}\sum_{k=1}^{n/6}\pi_\Lambda^\xi\big(\ell_r(\gamma_{m-1}) = k \;\big|\; |\Gamma_1^e| \geq m-1\big)\pi_\Lambda^\xi\big(|\Gamma_1^e| \geq m \;\big|\; \ell_r(\gamma_{m-1}) = k\big)\end{aligned}$$

where we used the fact that $\ell_r(\gamma_{m-1}) > 0$ implies that $|\Gamma_1^e| \geq m-1$. The proof will therefore follow if we show that, uniformly over $1 \leq k \leq n/6$,

$$\pi_\Lambda^\xi\left(|\Gamma_1^e| \geq m \mid \ell_r(\gamma_{m-1}) = k\right) \leq p\,. \tag{3.5}$$

Let $F = [\![0, n]\!] \times [\![\lfloor \frac{n'}{2} \rfloor, n']\!]$ (so that $\omega\restriction_F, \xi$ is the set of connections with respect to which the existence/properties of bridges are measurable) and fix any $1 \leq k \leq n/6$.

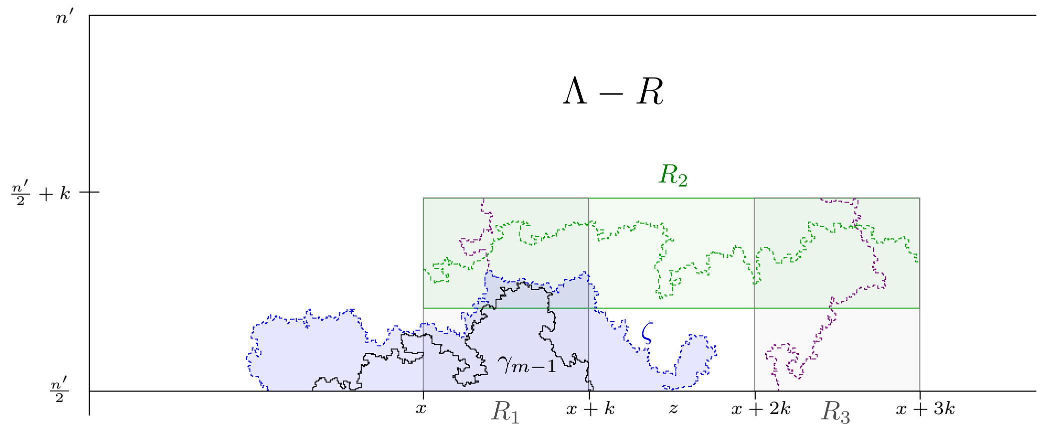

FIGURE 4. After conditioning on ζ (via the configuration in the blue shaded region), the probability of the purple and green dual-crossings is greater than $p_1p_2p_3$, bounding the probability of $\{|\Gamma_1^e| \geq m\}$.

First observe that by Fact 3.8 and the definition of $\mathrm{hull}_r(\gamma_{m-1})$, if L_l, L_r are the two connected subsets of $\partial_{\mathrm{N}} R - \mathrm{hull}_r(\gamma_{m-1})$, the event

$$\mathcal{E}_m = \left\{ L_l \stackrel{F^*}{\longleftrightarrow} L_r \text{ or } L_r \stackrel{F^*}{\longleftrightarrow} \partial\Lambda \right\}$$

satisfies $\mathcal{E}_m \supset \{|\Gamma_1^e| \geq m\}$. Therefore,

$$\pi_\Lambda^\xi\left(|\Gamma_1^e| \geq m \mid \ell_r(\gamma_{m-1}) = k\right) \leq \pi_\Lambda^\xi\left(|\Gamma_1^e| \geq m \mid \ell_r(\gamma_{m-1}) = k,\, \mathcal{E}_m\right)\,, \tag{3.6}$$

so that it suffices to upper bound the right-hand side. In particular, we can condition on the first dual-bridge over $\mathrm{hull}_r(\gamma_{m-1})$, connecting the segments L_l, L_r in F, or if there is no such dual-bridge in F then the west-most dual-crossing from L_r to $\partial\Lambda$, and then average over such choice of inner-most dual-bridge/crossing. Namely, conditioned on $\mathcal{E}_m$ and $\ell_r(\gamma_{m-1}) = k$, reveal the component of $e + (k - \frac{1}{2}, 0)$ in F; if it forms a bridge in F then beyond it is the desired dual-bridge; otherwise, this vertex is connected to $\partial\Lambda$, thus we have revealed the west-most dual-crossing east of $\mathrm{hull}_r(\gamma_{m-1})$. Denote the dual-bridge/crossing revealed in this manner by ζ (see Fig. 4) and let $(z, \lfloor \frac{n'}{2} \rfloor)$ be the west-most point of $\zeta \cap \partial_{\mathrm{N}} R$. Since $\ell_r(\gamma_i) \leq 2\ell_r(\gamma_{i-1})$ for all $i = 1, \ldots, |\Gamma_1^e|$, we see that if $|\Gamma_1^e| \geq m$ then necessarily $\ell_r(\gamma_m) \leq 2k$ and so

$$(z, \lfloor \tfrac{n'}{2} \rfloor) \in [\![x+k, x+2k]\!] \times \{\lfloor \tfrac{n'}{2} \rfloor\} =: I\,.$$

We will establish the desired upper bound of p uniformly over all such ζ.

Note that conditional on $\ell_r(\gamma_{m-1}) = k$, $\mathcal{E}_m$, and ζ, by Fact 3.7, the event $\{|\Gamma_1^e| \geq m\}$ implies the event S, stating that either ζ is a dual-bridge and I is primal-connected in $F \cup \xi$ to the left component of $\partial_{\mathrm{N}} R - \mathrm{hull}_r(\zeta)$, or alternatively ζ is a dual-crossing to $\partial\Lambda$ and I is primal-connected to $\partial\Lambda$ in F. Thus, in this conditional space,

$$\{|\Gamma_1^e| < m\} \supset \left\{ \zeta \stackrel{F^*}{\longleftrightarrow} [\![x+2k, x+3k]\!] \times \{\lfloor \tfrac{n'}{2} \rfloor\} \right\}\,, \tag{3.7}$$

since the right-hand side of Eq. (3.7) implies S^c which implies the left-hand side.

In order to lower bound the probability of the last event in Eq. (3.7), let D^* be the outer (if ζ is a dual-crossing in F, then eastern) connected component of $F^* - \zeta$, and let D be its dual. Define also the following subsets of Λ:

$$\begin{aligned} R_1 =& [\![x, x+k]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, \lfloor \tfrac{n'}{2} \rfloor + \min\{k, \tfrac{\alpha n}{4}\}]\!]\,, \\ R_2 =& [\![x, x+3k]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor + \min\{\tfrac{k}{2}, \tfrac{\alpha n}{8}\}, \lfloor \tfrac{n'}{2} \rfloor + \min\{k, \tfrac{\alpha n}{4}\}]\!]\,, \\ R_3 =& [\![x+2k, x+3k]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, \lfloor \tfrac{n'}{2} \rfloor + \min\{k, \tfrac{\alpha n}{4}\}]\!]\,, \end{aligned}$$

whereby, the event in the right-hand side of Eq. (3.7) can be written as $\{\zeta \xleftrightarrow{F^*} \partial_{\mathrm{S}} R_3\}$. For any $i = 1, 2, 3$, define the following crossing events (see Fig. 4):

$$\begin{aligned} \mathcal{C}_v^*(R_i \cap D) &= \left\{ \partial_{\mathrm{E}} R_i \overset{R_i \cap D}{\not\longleftrightarrow} \partial_{\mathrm{W}} R_i \right\}\,, \\ \mathcal{C}_h^*(R_i \cap D) &= \left\{ \partial_{\mathrm{N}} R_i \overset{R_i \cap D}{\not\longleftrightarrow} \partial_{\mathrm{S}} R_i \right\}\,. \end{aligned} \tag{3.8}$$

(observe that implicit in $(\mathcal{C}_v^*(R_i \cap D))^c$ is the event $\{\partial_{\mathrm{E}} R_i \cap D \neq \emptyset\} \cap \{\partial_{\mathrm{W}} R_i \cap D \neq \emptyset\}$, and similarly, implicit in $(\mathcal{C}_h^*(R_i \cap D))^c$ is the event $\{\partial_{\mathrm{N}} R_i \cap D \neq \emptyset\} \cap \{\partial_{\mathrm{S}} R_i \cap D \neq \emptyset\}$).

Claim 3.12. *Conditional on $\ell_r(\gamma_{m-1}) = k$, $\mathcal{E}_m$, and ζ,*

$$\{|\Gamma_1^e| < m\} \supset \Big(\mathcal{C}_v^*(R_1 \cap D) \cap \mathcal{C}_h^*(R_2 \cap D) \cap \mathcal{C}_v^*(R_3 \cap D) \Big)\,.$$

Proof. Suppose that ω satisfies the events on the right-hand. Recall that ζ is such that $\partial_{\mathrm{E}} R_3 \cap D \neq \emptyset$ and $\partial_{\mathrm{W}} R_3 \cap D \neq \emptyset$, and $\partial_{\mathrm{E}} R_3 \not\longleftrightarrow \partial_{\mathrm{W}} R_3$ in $R_3 \cap D$ since $\omega \in \mathcal{C}_v^*(R_3 \cap D)$. Consider $R_3 \cap D$ with boundary conditions wired on $\partial_{\mathrm{E,W}} R_3 \cap D$ and free on ζ and $\partial_{\mathrm{N,S}} R_3 \cap D$; then the boundary conditions on $R_3 \cap D$ alternate between free and wired on boundary curves ordered clockwise as $L_1^w, L_1^f, L_2^w, L_2^f ...$; by planarity and the choice of generalized Dobrushin boundary conditions, for any two wired boundary curves L_i^w, L_{i+1}^w, either $L_i^w \longleftrightarrow L_{i+1}^w$, or $L_i^f \overset{*}{\longleftrightarrow} L_j^f$ for some $j \neq i$. Picking the two wired segments of $\partial_{\mathrm{E,W}} R_3 \cap D$ closest to $\partial_{\mathrm{S}} R_3$, the aforementioned fact that $\partial_{\mathrm{E}} R_3 \not\longleftrightarrow \partial_{\mathrm{W}} R_3$ in $R_3 \cap D$ implies that either $\partial_{\mathrm{S}} R_3 \overset{*}{\longleftrightarrow} \zeta$ or $\partial_{\mathrm{S}} R_3 \overset{*}{\longleftrightarrow} \partial_{\mathrm{N}} R_3$. In the former, $\{|\Gamma_1^e| < m\}$ holds by Eq. (3.7), so suppose only the latter holds and call the dual-crossing ζ_3.

Since $\partial_{\mathrm{S}} R_3 \overset{*}{\longleftrightarrow} \partial_{\mathrm{N}} R_3$, both $\partial_{\mathrm{S}} R_2 \cap D$ and $\partial_{\mathrm{N}} R_2 \cap D$ are nonempty. Clearly, ζ_3 splits $R_2 \cap D$ into the subset to its east, U_{E}, and that to its west, U_{W}. Consider the set to its east, U_{E}, with boundary conditions that are wired on $\partial_{\mathrm{S,N}} R_2 \cap D$ and free on ζ and on $\partial_{\mathrm{E,W}} R_2 \cap D$. Since ζ_3 and ζ are vertex-disjoint (by our assumption that $\partial_{\mathrm{S}} R_3 \overset{*}{\not\longleftrightarrow} \zeta$ in $R_3 \cap D$), and the wired boundary segments adjacent to ζ_3 are disconnected in U_{E}, it must be that either $\zeta_3 \overset{*}{\longleftrightarrow} \zeta$ or $\zeta_3 \overset{*}{\longleftrightarrow} \partial_{\mathrm{E}} R_2$ in U_{E}. Using the same reasoning on U_{W}, either $\zeta_3 \overset{*}{\longleftrightarrow} \zeta$ or $\zeta_3 \overset{*}{\longleftrightarrow} \partial_{\mathrm{W}} R_2$ in U_{W}. Combining these, either $\zeta \overset{*}{\longleftrightarrow} \zeta_3$, in which case $\zeta \overset{*}{\longleftrightarrow} \partial_{\mathrm{S}} R_3$, or alternatively $\partial_{\mathrm{E}} R_2 \overset{*}{\longleftrightarrow} \zeta_3 \overset{*}{\longleftrightarrow} \partial_{\mathrm{W}} R_2$ in $R_2 \cap D$. In the former case, by Eq. (3.7), $\{|\Gamma_1^e| < m\}$; assume therefore that only the latter case holds, and let ζ_2 be a dual-crossing between $\partial_{\mathrm{E}} R_2$ to $\partial_{\mathrm{W}} R_2$ that intersects ζ_3.

Finally, we can deduce that $\partial_{\mathrm{E}} R_1 \cap D$ and $\partial_{\mathrm{W}} R_1 \cap D$ are nonempty as ζ_2 and ζ are vertex-disjoint (by our assumptions $\zeta \overset{*}{\not\longleftrightarrow} \zeta_3$ and $\zeta_2 \overset{*}{\longleftrightarrow} \zeta_3$). Considering now U_{S}, the subset of $R_1 \cap D$ south of ζ_2 with wired boundary conditions on $\partial_{\mathrm{E,W}} R_1 \cap D$ and free elsewhere, as before we deduce that either $\zeta_2 \xleftrightarrow{*} \zeta$ or $\zeta_2 \xleftrightarrow{*} \partial_{\mathrm{S}} R_1$ in U_{S}. Since, by definition of ζ, deterministically $\partial_{\mathrm{S}} R_1 \cap D = \emptyset$, the former must hold, and $\zeta \xleftrightarrow{*} \partial_{\mathrm{S}} R_3$ through ζ_2 and ζ_3, and Eq. (3.7) concludes the proof. ∎

We will next bound the probability of each of the events $\mathcal{C}_v^*(R_1 \cap D)$, $\mathcal{C}_h^*(R_2 \cap D)$ and $\mathcal{C}_v^*(R_3 \cap D)$, which, using the above claim, will translate to a bound on $\{|\Gamma_1^e| < m\}$.

To see this, first note that by planarity, for all $i = 1, 2, 3$ and every subset D,

$$\mathcal{C}_v^*(R_i \cap D) \supset (\mathcal{C}_h(R_i))^c = \mathcal{C}_v^*(R_i)\,, \tag{3.9}$$

and likewise for horizontal crossing events. Define the rectangle $\tilde{R}_1 \supset R_1$ by

$$\tilde{R}_1 = [\![x, x+k]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, n']\!] \subset \Lambda\,.$$

Let the boundary conditions $(1,0)$ on $\tilde{R}_1$ be free on $\partial_{\mathrm{S}} \tilde{R}_1$ and wired on $\partial_{\mathrm{N,E,W}} \tilde{R}_1$. Combining Eq. (3.9), monotonicity in boundary conditions, and the domain Markov property, we get for $p_1(\alpha, q) > 0$ given by Eq. (3.4),

$$\begin{aligned}\pi_\Lambda^\xi\left(\mathcal{C}_v^*(R_1 \cap D) \mid \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta\right) &\geq \pi_{\tilde{R}_1}^1(\mathcal{C}_v^*(R_1 \cap D) \mid \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta) \\ &\geq \pi_{\tilde{R}_1}^{1,0}(\mathcal{C}_v^*(R_1)) \geq p_1\,,\end{aligned}$$

where the last inequality follows from Proposition 2.3, Lemma 3.2 and self-duality. We stress that wiring of $\partial_{\mathrm{N,E,W}} \tilde{R}_1$ allowed us to ignore the information revealed on $R_1 - D$ as far as the configuration in $R_1 \cap D$ is concerned, and the fact that $\omega\restriction_\zeta$ is closed allowed us to place a free boundary on $\partial_{\mathrm{S}} \tilde{R}_1$, supporting Lemma 3.2.

Next, consider the rectangle $\tilde{R}_2 \supset R_2$ defined by

$$\tilde{R}_2 = [\![x-k, x+4k]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, n']\!]\,,$$

so that $\tilde{R}_2 \subset \Lambda$ since $k = \ell_r(\gamma_{m-1}) \leq n/6$. By monotonicity in boundary conditions and Eq. (3.9), we get that for the choice of $p_2(\alpha, q) > 0$ given by Eq. (3.4),

$$\begin{aligned}\pi_\Lambda^\xi(\mathcal{C}_h^*(R_2 \cap D) \mid \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta) &\geq \pi_{\tilde{R}_2}^1(\mathcal{C}_h^*(R_2 \cap D) \mid \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta) \\ &\geq \pi_{\tilde{R}_2}^1(\mathcal{C}_h^*(R_2)) \geq p_2\,.\end{aligned}$$

Similarly, applying the exact same treatment of $\tilde{R}_2$ to

$$\tilde{R}_3 = [\![x+k, x+4k]\!] \times [\![\tfrac{n'}{4}, n']\!] \subset \Lambda\,,$$

(it is possible to encapsulate R_3 by a rectangle with wired boundary conditions since ζ does not intersect $\partial_{\mathrm{S}} R_3$ in our conditional space) shows that

$$\begin{aligned}\pi_\Lambda^\xi(\mathcal{C}_v^*(R_3 \cap D) \mid \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta) &\geq \pi_{\tilde{R}_3}^1(\mathcal{C}_v^*(R_3 \cap D) \mid \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta) \\ &\geq \pi_{\tilde{R}_3}^1(\mathcal{C}_v^*(R_3)) \geq p_3\,,\end{aligned}$$

for $p_3(\alpha, q) > 0$ as defined in Eq. (3.4).

Therefore, by the FKG inequality and Claim 3.12, for any $1 \leq k \leq n/6$,

$$\pi_\Lambda^\xi\left(|\Gamma_1^e| < m \;\middle|\; \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta\right) \geq p_1 p_2 p_3\,,$$

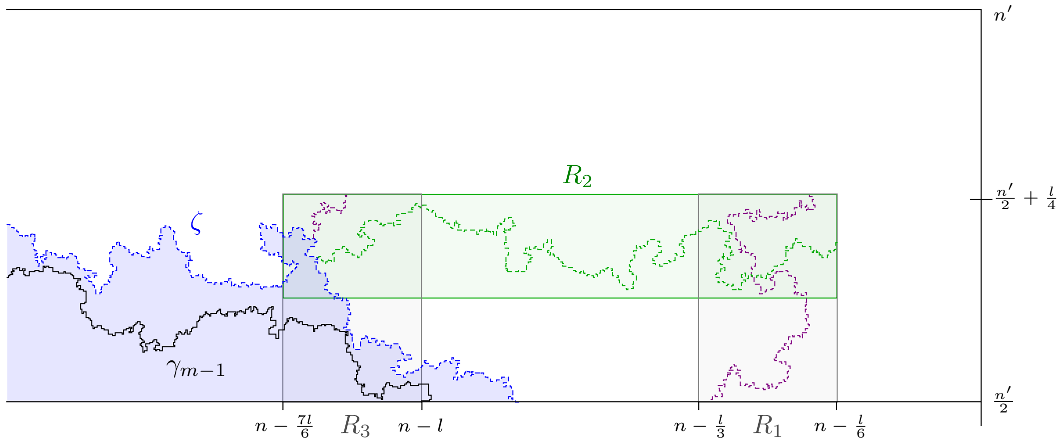

FIGURE 5. After revealing ζ (via the blue shaded region), the existence of the three dual-crossings depicted precludes $\{|\Gamma_2^e| \geq m\}$.

Since this bound is uniform over all k, ζ, by Eq. (3.6), for p given by Eq. (3.4), we have

$$\pi_\Lambda^\xi \left(|\Gamma_1^e| \geq m \mid \ell_r(\gamma_{m-1}) = k\right) \leq p\,,$$

thereby establishing Eq. (3.5), as desired. ∎

Proof of Lemma 3.11. Without loss of generality suppose e is in the left half of $\partial_{\mathrm{N}} R$ and use the right-ordering of bridges so that $\Gamma_2^e = \{\gamma_1, ..., \gamma_{|\Gamma_2^e|}\}$. For a fixed m let

$$l = n - (x + \ell_r(\gamma_{m-1}))\,,$$

measurable w.r.t. $\mathrm{hull}_r(\gamma_{m-1})$.

The proof follows the same argument used to prove Lemma 3.10, applying induction on m. In what follows we describe the necessary modifications that are needed here. Our goal is to prove that $\pi_\Lambda^\xi(|\Gamma_2^e| \geq m) \leq p^m$ for the choice of $p = 1 - p_1 p_2 p_3$ where,

- p_1 is given by Proposition 2.3 with aspect ratio α for $1 < q < 4$ and by Lemma 3.2 with the choice $\varepsilon = 1/2$ and aspect ratio $\alpha/2$ for $q = 4$,
- p_2 is the probability given by Theorem 2.1 for $\varepsilon = 1/8$ and aspect ratio $6/\alpha$,
- p_3 is the probability given by Theorem 2.1 for $\varepsilon = 1/3$ and aspect ratio α,

so that $p = p(\alpha, q) \in (0,1)$ as desired.

Let L_l, L_r be the left and right connected components of $\partial_{\mathrm{N}} R - \mathrm{hull}_r(\gamma_{m-1})$. As in the proof of Lemma 3.10, condition on $\ell_r(\gamma_{m-1}) = k$, where now $\frac{n}{6} \leq k \leq n$, then on

$$\mathcal{E}_m = \left\{ L_r \xleftrightarrow{F^*} L_l \text{ or } L_r \xleftrightarrow{F^*} \partial\Lambda \right\}\,,$$

and finally on ζ (the dual-bridge/crossing for which $\mathcal{E}_m$ is satisfied), whose west-most vertex of intersection with $\partial_{\mathrm{N}} R$ is marked by $(z, \lfloor \frac{n'}{2} \rfloor)$. (Recall that this is achieved by revealing the component of $e + (k - \frac{1}{2}, 0)$ in F.) If $n - z < l/2$, deterministically $|\Gamma_2^e| < m$

(as argued in the proof of Proposition 3.9), hence we may assume that $n - z \geq l/2$; moreover, since $k \geq \frac{n}{6}$, it must be that $\frac{l}{6} \leq k$. Define the following subsets of Λ:

$$R_1 = [\![n - l - \tfrac{l}{6}, n - l]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, \lfloor \tfrac{n'}{2} \rfloor + \tfrac{\alpha l}{6}]\!] \,,$$
$$R_2 = [\![n - l - \tfrac{l}{6}, n - \tfrac{l}{6}]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, \lfloor \tfrac{n'}{2} \rfloor + \tfrac{\alpha l}{6}]\!] \,,$$
$$R_3 = [\![n - \tfrac{l}{3}, n - \tfrac{l}{6}]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, \lfloor \tfrac{n'}{2} \rfloor + \tfrac{\alpha l}{6}]\!] \,.$$

Define $\mathcal{C}_v^*(R_1 \cap D), \mathcal{C}_h^*(R_2 \cap D), \mathcal{C}_v^*(R_3 \cap D)$ as in Eq. (3.8). As in Claim 3.12,

$$\{|\Gamma_2^e| < m\} \supset \Big(\mathcal{C}_v^*(R_1 \cap D) \cap \mathcal{C}_h^*(R_2 \cap D) \cap \mathcal{C}_v^*(R_3 \cap D) \Big) \,.$$

Finally, for $\tilde{R}_i$, $i = 1, 2, 3$ given by

$$\tilde{R}_1 = [\![n - l - \tfrac{l}{6}, n - l]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor, \lfloor \tfrac{n'}{2} \rfloor + \tfrac{\alpha l}{3}]\!] \,,$$
$$\tilde{R}_2 = [\![n - l - \tfrac{l}{3}, n]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor - \tfrac{\alpha l}{6}, \lfloor \tfrac{n'}{2} \rfloor + \tfrac{\alpha l}{3}]\!] \,,$$
$$\tilde{R}_3 = [\![n - \tfrac{l}{2}, n, \tfrac{l}{2}]\!] \times [\![\lfloor \tfrac{n'}{2} \rfloor - \tfrac{\alpha l}{6}, \lfloor \tfrac{n'}{2} \rfloor + \tfrac{\alpha l}{3}]\!] \,,$$

(note that all three are subsets of Λ, by the fact that $l \leq n$ and $n' \geq \lfloor \alpha n \rfloor$), the same monotonicity argument used in the proof of Lemma 3.10 now implies (see Fig. 5) that

$$\pi_\Lambda^\xi \left(|\Gamma_2^e| < m \;\middle|\; \ell_r(\gamma_{m-1}) = k, \mathcal{E}_m, \zeta \right) \geq p_1 p_2 p_3 \,.$$

Since this bound is uniform in ζ and $\mathcal{E}_m \supset \{|\Gamma_2^e| \geq m\}$, we see that for every $\frac{n}{6} \leq k \leq n$,

$$\pi_\Lambda^\xi \left(|\Gamma_2^e| \geq m \;\middle|\; \ell_r(\gamma_{m-1}) = k \right) \leq p \,. \qquad \blacksquare$$

The following estimate proves that boundary conditions induced by the FK measure will, with high probability, have edges with order $\log n$ bridges over them; when $p \neq p_c$, the typical maximum number of bridges over an edge is actually $o(\log n)$.

Corollary 3.13. *Let $q \in (1, 4]$ and $\alpha \in (0, 1]$. Consider a rectangle $\Lambda = \Lambda_{n,n'}$ with $n' \geq \lfloor \alpha n \rfloor$, along with the subset $R = \Lambda_{n,n'/2}$. There exist $K' > K > 0$ and $c(\alpha, q) > 0$ such that for every boundary condition ξ,*

$$\pi_\Lambda^\xi \left(\max_{e \in \partial_{\mathrm{N}} R} |\Gamma^e| \notin [\![K \log n, K' \log n]\!] \right) \lesssim n^{-cK'} \,.$$

Proof. By a union bound, the upper bound on $\max_{e \in \partial_{\mathrm{N}} R} |\Gamma^e|$ follows from the choice of $K' > 1/c$ for c given by Eq. (3.1) of Proposition 3.9.

Fix any $\varepsilon > 0$ and consider the edges $e_k = (\lfloor k n^\varepsilon \rfloor + \frac{1}{2}, \lfloor \frac{n'}{2} \rfloor)$ for $k = 1, \ldots, n^{1-\varepsilon} - 1$. It suffices to prove that for some $K(\varepsilon) > 0$,

$$\pi_\Lambda^\xi \left(\max_{k=1}^{n^{1-\varepsilon}-1} |\Gamma^{e_k}| \leq K \log n \right) \lesssim n^{-cK'} \,.$$

In order to prove the above, for fixed K, let $\chi_e = \{|\Gamma^e| \geq K \log n\}$ so that

$$\pi_\Lambda^\xi \left(\max_{k=1}^{n^{1-\varepsilon}-1} |\Gamma^{e_k}| \leq K \log n \right) \geq \pi_\Lambda^\xi \left(\sum_{k=1}^{n^{1-\varepsilon}-1} \mathbf{1}\{\chi_{e_k}\} = 0 \right) \,.$$

Recall for each e_k the definitions of the rectangles $\tilde R_i^{\mathrm{N}}, \tilde R_i^{\mathrm{E}}, \tilde R_i^{\mathrm{W}}$ given by (3.2). Also, denote by χ'_e the event on the right-hand side of (3.3), so that $\mathbf{1}\{\chi_{e_k}\} \geq \mathbf{1}\{\chi'_{e_k}\}$. Observe that for each $k = 1, ..., n^{1-\varepsilon}-1$, the event χ'_{e_k} is measurable w.r.t. the rectangle $R_{e_k} = \bigcup_{i=1}^{K \log n} \tilde R_i^{\mathrm{N}} \cup \tilde R_i^{\mathrm{E}} \cup \tilde R_i^{\mathrm{W}}$. By similar considerations as before, there exists some $p = p(\alpha, q) > 0$ such that, for every $i = 1, ..., 2K \log n$, every e_k, and every η,

$$\pi_\Lambda^\xi(\mathcal{A}_i \mid \omega\restriction_{\Lambda - \tilde R_i^{\mathrm{N,E,W}}} = \eta) \geq \pi^0_{\tilde R_i^{\mathrm{W}}}(\mathcal{C}_v(R_i^{\mathrm{W}}))\pi^0_{\tilde R_i^{\mathrm{E}}}(\mathcal{C}_v(R_i^{\mathrm{E}}))\pi^0_{\tilde R_i^{\mathrm{N}}}(\mathcal{C}_h(R_i^{\mathrm{N}})) \geq p\,,$$

$$\pi_\Lambda^\xi(\mathcal{A}_i^* \mid \omega\restriction_{\Lambda - \tilde R_i^{\mathrm{N,E,W}}} = \eta) \geq \pi^1_{\tilde R_i^{\mathrm{W}}}(\mathcal{C}^*_v(R_i^{\mathrm{W}}))\pi^1_{\tilde R_i^{\mathrm{E}}}(\mathcal{C}^*_v(R_i^{\mathrm{E}}))\pi^1_{\tilde R_i^{\mathrm{N}}}(\mathcal{C}^*_h(R_i^{\mathrm{N}})) \geq p\,.$$

Observe that for $i \neq j$, the interiors of $\tilde R_i^{\mathrm{N,E,W}}$ and $\tilde R_j^{\mathrm{N,E,W}}$ are disjoint and therefore $\mathbf{1}\{\chi'_{e_k}\} \succeq \mathrm{Bernoulli}(p^{K \log n})$. Combined with $R_{e_j} \cap R_{e_k} = \emptyset$ when $j \neq k$, we see that

$$\sum_{e \in \mathcal{E}} \mathbf{1}\{\chi_e\} \geq \sum_{e \in \mathcal{E}} \mathbf{1}\{\chi'_e\} \succeq \mathrm{Bin}(n^{1-\varepsilon}, p^{K \log n})\,.$$

Choosing $K < K_0(\varepsilon) \wedge \varepsilon/\log(1/p)$, the right-hand side dominates a $\mathrm{Bin}(n^{1-\varepsilon}, n^{-\varepsilon})$ random variable, whose probability of being zero is exponentially small in $n^{1-2\varepsilon}$. ∎

3.4. **Disjoint crossings.** To extend our mixing time bound from favorable boundary conditions (see §5.1) to periodic boundary conditions (which are not in that class) in §5.3, we need an analogous bound on the number of disjoint crossings of a rectangle.

For a rectangle R and a configuration $\omega\restriction_R$, let $\Psi_R = \Psi_R(\omega\restriction_R)$ be the set containing every component $A \subset V(R)$ (connected via the edges of $\omega\restriction_R$) that intersects both $\partial_{\mathrm{S}} R$ and $\partial_{\mathrm{N}} R$. We will need the following equilibrium estimate similar to Proposition 3.9.

Proposition 3.14. *Let $q \in (1,4]$ and $\alpha \in (0,1]$. Consider the critical FK model on $\Lambda = \Lambda_{n,n'}$ with $n' \geq \lfloor \alpha n \rfloor$, and the subset $R = [\![0,n]\!] \times [\![\frac{n'}{3}, \frac{2n'}{3}]\!]$. There exists $c(\alpha,q) > 0$ such that for every boundary condition ξ and every $m \geq 3$,*

$$\pi_\Lambda^\xi\left(|\Psi_R| \geq m\right) \leq e^{-cm}\,.$$

Proof. We will prove by induction that, for all $m \geq 1$,

$$\pi_\Lambda^\xi(|\Psi_R| \geq m) \leq (1-p)^{m-2}\,, \tag{3.10}$$

where $p > 0$ is as given by Proposition 2.3 with aspect ratio $3/\alpha$ when $1 < q < 4$, and is as given by Corollary 3.3 with aspect ratio $\alpha/3$ when $q = 4$.

The cases $m = 1, 2$ are trivially satisfied for any $0 < p < 1$. Now let $m \geq 3$, and suppose that Eq. (3.10) holds for $m-1$; the proof will be concluded once we show that

$$\pi_\Lambda^\xi\left(|\Psi_R| \geq m \;\middle|\; |\Psi_R| \geq m-1\right) \leq 1-p\,.$$

Conditioned on the existence of at least $m-1$ distinct components in Ψ_R, we can condition on the west-most component in Ψ_R (by revealing all dual-components of $\omega\restriction_R$ incident to $\partial_{\mathrm{W}} R$, then revealing the primal-component of the adjacent primal-crossing). We can also condition on the $m-2$ east-most components in Ψ_R (by successively repeating the aforementioned procedure from east to west, i.e., replacing $\partial_{\mathrm{W}} R$ above by $\partial_{\mathrm{E}} R$ to reveal some component $C \in \Psi_R$, then by its western boundary $\partial_{\mathrm{W}} C$, etc.).

Through this process, we can find two disjoint vertical dual-crossings ζ_1, ζ_2 of R, each one a simple dual-path; the set $(R^* - \zeta_1 - \zeta_2)^*$ consists of three connected subsets of R; let D denote the middle one. There are exactly $m-1$ elements of Ψ_R in $R-D$, thus its m-th element, if one exists, must belong to D. Since every edge in $\zeta_1 \cup \zeta_2$ is dual-open, for any such choice of ζ_1, ζ_2, we then have

$$\pi_\Lambda^\xi\left(|\Psi_R| \geq m \;\middle|\; |\Psi_R| \geq m-1, \zeta_1, \zeta_2\right) = \pi_\Lambda^\xi\left(\mathcal{C}_v(D) \;\middle|\; \zeta_1, \zeta_2\right) ,$$

Using the domain Markov property and monotonicity of boundary conditions,

$$\pi_\Lambda^\xi\left(\mathcal{C}_v(D) \;\middle|\; \zeta_1, \zeta_2\right) \leq \pi_D^{1,0,1,0}(\mathcal{C}_v(D)) ,$$

where $(1,0,1,0)$ boundary conditions on D denote those that are free on ζ_1, ζ_2 and wired on $\partial R \cap D$. Again by monotonicity (in boundary conditions and crossing events),

$$\pi_D^{1,0,1,0}(\mathcal{C}_v(D)) \leq \pi_R^{1,0,1,0}\left(\mathcal{C}_v(D) \;\middle|\; \omega_{\zeta_1} = 0, \omega_{\zeta_2} = 0\right) \leq 1 - \pi_R^{1,0,1,0}(\mathcal{C}_h^*(R)) ,$$

where, following the notation of Corollary 3.3, $(1,0,1,0)$ boundary conditions on a rectangle R are wired on $\partial_{\mathrm{N,S}} R$ and free on $\partial_{\mathrm{E,W}} R$. By monotonicity in boundary conditions and the definition of p, the right-hand side is bounded above by

$$1 - \pi_R^{(1,0,1,0)}\left(\mathcal{C}_h^*([\![0,n]\!] \times [\![\tfrac{n'}{3}, \tfrac{n'}{3} + \tfrac{\alpha n}{3}]\!])\right) \leq 1 - p . \qquad \blacksquare$$

4. Dynamical tools

In this section, we introduce the main techniques we use to control the total variation distance from stationarity for the random cluster heat-bath Glauber dynamics.

4.1. **Modifications of boundary conditions.** Crucial to the proof of Theorem 1 is the modification of boundary bridges so that we can couple beyond FK interfaces as done in [16]; in this subsection we define boundary condition modifications and control the effect such modifications can have on the mixing time.

Definition 4.1 (segment modification)**.** Let ξ be a boundary condition on a rectangle $\partial\Lambda$ which corresponds to a partition $\{\mathcal{P}_1, ..., \mathcal{P}_k\}$ of $\partial\Lambda$, and let $\Delta \subset \partial\Lambda$. The *segment modification on* Δ, denoted by ξ^Δ, is the boundary condition that corresponds to the partition $\{\mathcal{P}_1 - V(\Delta), ..., \mathcal{P}_k - V(\Delta), V(\Delta)\}$ of $\partial\Lambda$.

Definition 4.2 (bridge modification)**.** Let ξ be a boundary condition on $\partial\Lambda$, corresponding to a partition $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ of $\partial\Lambda$. Let Γ^e be the set of disjoint bridges in $\xi\restriction_{\partial_{\mathrm{N}}\Lambda}$ over the edge $e = (x,y) \in \partial_{\mathrm{N}}\Lambda$, corresponding to the components $\{\mathcal{P}_{i_j}\}_{j=1}^{\ell}$, as per Definition 3.4. The *bridge modification* of ξ over e, denoted ξ^e, is the boundary condition associated to the partition where every $\mathcal{P}_{i_j}$ is split into two components,

$$\mathcal{P}_{i_j}^{\mathrm{W}} = \{(v_1, v_2) \in \mathcal{P}_{i_j} : v_1 - x < 0\} \qquad \textit{and} \qquad \mathcal{P}_{i_j}^{\mathrm{E}} = \{(v_1, v_2) \in \mathcal{P}_{i_j} : v_1 - x > 0\} .$$

(Observe that, in particular, ξ^e has no bridges over e.) Define the bridge modification w.r.t. the other sides of $\partial\Lambda$ analogously.

Definition 4.3 (side modification)**.** Let ξ be a boundary condition on $\partial\Lambda$, corresponding to a partition $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ of $\partial\Lambda$. The *side modification* ξ^s is defined as follows. Split every $\mathcal{P}_j$ into its four sides, that is, for $i = \mathrm{N}, \mathrm{S}, \mathrm{E}, \mathrm{W}$, let

$$\mathcal{P}_j^i = \{v \in \mathcal{P}_j : v \in \partial_i \Lambda\} ,$$

where for the corner vertices the choice is arbitrary (for concreteness, associate the corner with the side that follows it clockwise). Then for every ξ, the modified ξ^s has no components that contain vertices in more than one side of $\partial\Lambda$.

It will be useful to have a notion of distance between boundary conditions.

Definition 4.4. For any pair of boundary conditions, ξ, ξ' define the symmetric distance function $d(\xi,\xi')$ as follows: if ξ'' is the unique smallest (in the previously defined partial ordering) boundary condition with $\xi'' \geq \xi$ and $\xi'' \geq \xi'$, define $d(\xi,\xi') = k(\xi'') - k(\xi) + k(\xi'') - k(\xi')$.

If ξ is a boundary condition on Λ and ξ' is a any of the above boundary modifications of ξ, then $\xi' \leq \xi$ and the partition associated to it is a refinement of ξ; this implies that $d(\xi,\xi') = k(\xi) - k(\xi')$. One can easily verify the following.

Fact 4.5. *For a segment Δ, we have $d(\xi,\xi^\Delta) \leq |V(\Delta)|$; for an edge e, we have $d(\xi,\xi^e) = |\Gamma^e|$; for the side modification ξ^s, we have that $d(\xi,\xi^s)$ is bounded above by three times the number of components in ξ with vertices in multiple sides of $\partial\Lambda$.*

We now present a lemma bounding the effect on total variation mixing from modifying the boundary conditions. Recall that for two boundary conditions ξ,ξ' on Λ, we defined in the preliminaries the quantity $M_{\xi,\xi'} = \|\frac{\pi_\Lambda^\xi}{\pi_\Lambda^{\xi'}}\|_\infty \vee \|\frac{\pi_\Lambda^{\xi'}}{\pi_\Lambda^\xi}\|_\infty$, and we have from Eq. (2.2), that $t_{\textsc{mix}} \lesssim M_{\xi,\xi'}^3 |E(\Lambda)| t'_{\textsc{mix}}$. Moreover, using the notation of [18] and [15], for an initial configuration ω_0, and boundary condition ξ, let

$$d_{\textsc{tv}}^{(\omega_0,\xi)}(t) = \|\mathbb{P}_{\omega_0}^\xi(X_t \in \cdot) - \pi_\Lambda^\xi\|_{\textsc{tv}}\,,$$

where here and throughout the paper, for any Markov chain $(X_t)_{t\geq 0}$, $\mathbb{P}_{\omega_0}^\xi(X_t \in \cdot) = \mathbb{P}(X_t \in \cdot \mid X_0 = \omega_0)$ with boundary conditions ξ; when clear from the context we may drop the boundary condition superscript from the notation.

Lemma 4.6. *Let ξ,ξ' be a pair of boundary conditions on $\partial\Lambda$. Then,*

$$M_{\xi,\xi'} \leq q^{d(\xi',\xi)}\,, \tag{4.1}$$

and consequently, there exists an absolute $c > 0$ such that for every $t > 0$,

$$\max_{\omega_0\in\{0,1\}} d_{\textsc{tv}}^{(\omega_0,\xi)}(t) \leq 8 \max_{\omega_0\in\{0,1\}} d_{\textsc{tv}}^{(\omega_0,\xi')}\left(c|E(\Lambda)|^{-2} q^{-4d(\xi',\xi)}\, t\right) + \exp\left(-q^{d(\xi',\xi)}\right)\,. \tag{4.2}$$

Proof. Adapting an argument of [18] to the FK setting, Lemma 5.4 of [10] proves a version of this lemma for two coupled probability measures $\mathbf{P}, \mathbf{P}^\Delta$ over pairs ξ,ξ^Δ. The proof for arbitrary pairs of boundary conditions, ξ,ξ', is identical; letting $\mathbf{P}$ be a point mass at ξ completes the proof. ∎

4.2. **Censored block dynamics.** We next define the censored and systematic block dynamics whose coupling is the core of the dynamical analysis used to prove Theorem 1. This coupling may be of general interest in the study of mixing times of monotone Markov chains, where one only has control on mixing times in the presence of favorable boundary conditions. We therefore present it in more generality than necessary for the proof of Theorem 1: consider the heat-bath dynamics for a monotone spin or edge system on a graph G with boundary ∂G that satisfies the domain Markov property and has extremal configurations $\{0,1\}$ and invariant measure π_G^ξ.

Definition 4.7 (systematic and censored block dynamics)**.** Let $B_0,\dots,B_{s-1}$ denote a finite cover of $E(G)$ (or $V(G)$ for a spin system), and for $k\geq 1$ let $i_k := (k-1) \bmod s$. Further, consider a set Γ_i of *permissible* boundary conditions for B_i, and fix $\varepsilon>0$.

The *systematic block dynamics* $(Y_k)_{k\geq 0}$ is a discrete-time flavor of the block dynamics w.r.t. $\{B_i\}$, with blocks that are updated in a sequential deterministic order: at time k, the chain updates B_{i_k} by resampling $\omega\restriction_{B^o_{i_k}} \sim \pi_G^\xi(\cdot \mid \omega\restriction_{G-(B^o_{i_k})})$. As in the standard block dynamics [17], Y_k is clearly reversible w.r.t. π_G.

The *censored block dynamics* $(\bar X_t)_{t\geq 0}$ is the continuous-time single-bond (single-site) heat-bath dynamics that simulates Y_k as follows. For a given $\varepsilon>0$, define

$$T = T(\varepsilon) = \max_i \max_{\xi\in\Gamma_i} t_{\text{MIX}}^{\xi,B_i}(\varepsilon)\,, \tag{4.3}$$

where t_{MIX}^{ξ,B_i} is the mixing time of standard heat-bath dynamics on the block B_i with boundary conditions ξ. Let the chain $\bar X_t$ be obtained from the standard heat-bath dynamics by censoring, as in Theorem 2.5, for every integer $k\geq 1$, along the interval $((k-1)T,kT]$, all updates except those in B_{i_k}.

Proposition 4.8 (comparison of censored / systematic block dynamics)**.** *Let $(\bar X_t)_{t\geq 0}$ and $(Y_k)_{k\geq 0}$ be the censored and systematic block dynamics, respectively, w.r.t. some blocks $B_0,\dots B_{s-1}$ and permissible boundary conditions Γ_i on G with boundary conditions ξ and initial state ω_0, as per Definition 4.7. Let*

$$\rho := \max_{k\geq 1}\max_{i\in[\![0,s-1]\!]} \mathbb{P}_{\omega_0}\left(Y_k\restriction_{\partial B_i}\notin\Gamma_i\right)\,, \tag{4.4}$$

where $Y_k\restriction_{\partial B_i}$ is the boundary condition induced on ∂B_i by Y_k. Then for every $\varepsilon>0$, every integer $k\geq 0$, and T as in (4.3),

$$\left\|\mathbb{P}_{\omega_0}\left(\bar X_{kT}\in\cdot\right)-\mathbb{P}_{\omega_0}\left(Y_k\in\cdot\right)\right\|_{\text{TV}} \leq k(\rho+\varepsilon)\,. \tag{4.5}$$

Remark 4.9. Although we defined the systematic and censored block dynamics for deterministic block updates, one could easily formulate the same bound for the usual block dynamics with random updates, where the s sub-blocks are each assigned i.i.d. Poisson clocks (cf. [17]), by also randomizing the order in which the censored block dynamics updates sub-blocks, using the identity coupling on the corresponding clocks.

Proof of Proposition 4.8. We now prove Eq. (4.5) by induction on k. Fix any ω_0 and let $\delta_k = \left\|\mathbb{P}_{\omega_0}\left(\bar X_{kT}\in\cdot\right)-\mathbb{P}_{\omega_0}\left(Y_k\in\cdot\right)\right\|_{\text{TV}}$ denote its left-hand side; observe that $\delta_0=0$ by definition, and suppose that $\delta_k\leq k(\rho+\varepsilon)$ for some k. Denote by $i=i_{k+1}$ the block that is updated at time $k+1$ by the systematic block dynamics, and let $\bar X_t^{(i)}$ and $Y_k^{(i)}$ be the censored and systematic chains corresponding to the block sequence $(B_{(i+\ell) \bmod s})_{\ell\geq 0}$ (where the block B_i is the first to be updated). By the Markov property and the triangle inequality,

$$\begin{aligned}\delta_{k+1} &\leq \frac{1}{2}\sum_{\omega,\omega'}\bigg(\Big|\mathbb{P}_{\omega_0}(\bar X_{kT}=\omega')-\mathbb{P}_{\omega_0}(Y_k=\omega')\Big|\,\mathbb{P}_{\omega'}(\bar X^{(i)}_T=\omega)\\ &\qquad\qquad+\Big|\mathbb{P}_{\omega'}(\bar X^{(i)}_T=\omega)-\mathbb{P}_{\omega'}(Y^{(i)}_1=\omega)\Big|\,\mathbb{P}_{\omega_0}(Y_k=\omega')\bigg)\\ &=\delta_k+\sum_{\omega'}\mathbb{P}_{\omega_0}(Y_k=\omega')\,\Big\|\mathbb{P}_{\omega'}(\bar X^{(i)}_T\in\cdot)-\mathbb{P}_{\omega'}(Y^{(i)}_1\in\cdot)\Big\|_{\mathrm{TV}}\,. \end{aligned}\tag{4.6}$$

The last summand in (4.6) satisfies

$$\begin{aligned}&\sum_{\omega}\mathbb{P}_{\omega_0}(Y_k=\omega)\,\Big\|\mathbb{P}_{\omega}(\bar X^{(i)}_T\in\cdot)-\mathbb{P}_{\omega}(Y^{(i)}_1\in\cdot)\Big\|_{\mathrm{TV}}\\ &\quad\leq \mathbb{P}_{\omega_0}(Y_k\restriction_{\partial B_i}\in\Gamma_i)\max_{\omega:\omega\restriction_{\partial B_i}\in\Gamma_i}\Big\|\mathbb{P}_{\omega}(\bar X^{(i)}_T\in\cdot)-\mathbb{P}_{\omega}(Y^{(i)}_1\in\cdot)\Big\|_{\mathrm{TV}}+\mathbb{P}_{\omega_0}(Y_k\restriction_{\partial B_i}\notin\Gamma_i)\\ &\quad\leq (1-\rho)\varepsilon+\rho\,,\end{aligned}$$

by the definition of $T=T_1(\varepsilon)$ and ρ; here we identified the configuration on $G-B_i^o$ with the boundary it induces on ∂B_i. Combined with Eq. (4.6), this completes the proof of Eq. (4.5). ∎

Remark 4.10. In the setting of Proposition 4.8, when the initial state is $\omega_0\in\{0,1\}$ (either minimal or maximal), one can obtain the following improved bound. Set

$$T=\max_i\max_{\xi\in\Gamma_i}t^{\xi,B_i}_{\mathrm{MIX}}(\omega_0\restriction_{B_i},\varepsilon)\,,\tag{4.7}$$

where $t^{\xi,B_i}_{\mathrm{MIX}}(\omega_0,\varepsilon)=\inf\{t:d^{(\omega_0,\xi)}_{\mathrm{TV}}(t)\leq\varepsilon\}$, relaxing the previous definition (4.3) of T to only consider the initial state ω_0. Let $(\bar X_t)$ be the censored block dynamics w.r.t. this new value of T, and denote by $(\bar X'_t)$ the modification of $(\bar X_t)$ where, for every $k\geq 1$, the configuration of the block B_{i_k} (i.e., the block that is to be updated in the interval $((k-1)T,kT])$ is reset at time $(k-1)T$ to the original value of ω_0 on that block. We claim that (4.5) holds[1] for the relaxed value of T in (4.7) if we replace $\bar X_t$ by $\bar X'_t$. Indeed, all the steps in the above proof of Proposition 4.8 remain valid up to the final inequality, at which point the fact that we consider $\bar X'_t$ (as opposed to $\bar X_t$) implies that

$$\max_{\omega:\omega\restriction_{\partial B_i}\in\Gamma_i}\Big\|\mathbb{P}_{\omega}(\bar X^{(i)}_T\in\cdot)-\mathbb{P}_{\omega}(Y^{(i)}_1\in\cdot)\Big\|_{\mathrm{TV}}=\max_{\xi\in\Gamma_i}\Big\|\mathbb{P}_{\omega_0\restriction_{B_i}}(\bar X^{(i)}_T\in\cdot)-\pi^{\xi}_{B_i}\Big\|_{\mathrm{TV}}\,,$$

which is at most ε when T is as defined in (4.7).

5. Proof of main result

In this section, we prove Theorem 1 by combining the equilibrium estimates of §3 with the dynamical tools provided in §4. We first establish an analog of Theorem 1 (Theorem 5.4) for "typical" boundary conditions (defined in §5.1 below), and then, using Proposition 3.14, derive from it the case of periodic boundary conditions in §5.3.

[1]In fact, (4.5) is valid for $\bar X'_t$ with the relaxed T in (4.7) for every ω_0, not just for the maximal and minimal configurations; however, it is when $\omega_0\in\{0,1\}$ that the modified dynamics $\bar X'_t$ can easily be compared to $\bar X_t$, and thereafter to X_t, via the censoring inequality of Theorem 2.5.

The effect of boundary bridges (which may foil the multiscale coupling approach, as described in §1.1) is controlled by restricting the analysis to those boundary conditions that have $O(\log n)$ bridges, and applying Proposition 4.8 to bound the mixing time under such boundary conditions. We now define the favorable boundary conditions for which we prove a mixing time upper bound of $n^{O(\log n)}$.

5.1. **Typical boundary conditions.** We first define the class of "typical" boundary conditions on a segment (e.g., $\partial_{\mathrm{N}}\Lambda$).

Definition 5.1 (typical boundary conditions on a segment)**.** For $K > 0$, $N \geq 1$, and a segment L, let $\Xi_{K,N}$ be the set of boundary conditions ξ on L such that

$$|\Gamma^e(\xi)| \leq K \log N \qquad \text{for every } e \in L\,.$$

We will later see (as a consequence of Lemma 5.7 below) that the boundary conditions on each of the sides of a box Λ induced by the infinite-volume FK measure $\pi_{\mathbb{Z}^2}$ belong to the class of "typical" boundary conditions with high probability.

Next, we define the global property we require of typical boundary conditions.

Definition 5.2 (typical boundary conditions on $\partial\Lambda$)**.** Let $\Upsilon_{K_1,K_2,N} = \Upsilon^{\Lambda}_{K_1,K_2,N}$ be the set of boundary conditions ξ on $\partial\Lambda$ such that $\xi\restriction_{\partial_i\Lambda} \in \Xi_{K_1,N}$ for every $i = \mathrm{N},\mathrm{S},\mathrm{E},\mathrm{W}$, and ξ has at most $K_2 \log N$ distinct components with vertices on different sides of $\partial\Lambda$.

Remark 5.3. The wired and free boundary conditions on a side $\partial_i\Lambda$ are always in $\Upsilon_{K_1,K_2,N}$ whenever $K_1 \log N \geq 1$ and $K_2 \log N \geq 1$ (in the former all vertices are in just one component and in the latter no two vertices are in the same component).

5.2. **Mixing under typical boundary conditions.** Since periodic boundary conditions are not in $\Upsilon_{K_1,K_2,N}$ for any $K_2 > 0$, we first bound the mixing time on rectangles $\Lambda_{N,N'}$ where $N' = \lfloor \bar{\alpha} N \rfloor$ for $\bar{\alpha} \in (0,1]$, with boundary conditions $\xi \in \Upsilon_{K_1,K_2,N}$.

Theorem 5.4. *Let $q \in (1,4]$ and fix $\bar{\alpha} \in (0,1]$ and $K_1, K_2 > 0$. Consider the Glauber dynamics for the critical FK model on $\Lambda_{N,N'}$ with $\bar{\alpha} N \leq N' \leq N$ and boundary conditions $\xi \in \Upsilon_{K_1,K_2,N}$. Then there exists $c = c(\bar{\alpha}, q, K_1, K_2) > 0$ such that*

$$t_{\mathrm{MIX}} \lesssim N^{c \log N}\,.$$

Observe that if we define

$$\Upsilon_{K,N} := \Upsilon_{K,2K,N}\,, \tag{5.1}$$

clearly $\Upsilon_{K_1,K_2,N} \subset \Upsilon_{\max\{K_1,K_2\},N}$, so it suffices to consider $\Upsilon_{K,N}$ for general $K > 0$.

The proof of Theorem 5.4 proceeds by analyzing the censored and systematic block dynamics on Λ, obtaining good control on the systematic block dynamics using the RSW estimates of [8], then comparing it to the censored block dynamics. The choice of parameters for which we will apply Proposition 4.8 is the following.

Definition 5.5 (block choice for censored / systematic block dynamics)**.** Let $q \in (1,4]$ and for any $n' \leq n \leq N$, consider the critical FK Glauber dynamics on $\Lambda_{n,n'}$. Let

$$B_{\mathrm{E}} = [\![\tfrac{n}{4}, n]\!] \times [\![0, n']\!]\,,$$
$$B_{\mathrm{W}} = [\![0, \tfrac{3n}{4}]\!] \times [\![0, n']\!]\,,$$

ordered as $B_0 = B_{\mathrm{E}}, B_1 = B_{\mathrm{W}}$ as in the setup of Proposition 4.8. For $K = \max\{K_1, K_2\}$ given by Theorem 5.4, let $\Gamma_i = \Upsilon_{K,N}$ be the set of permissible boundary conditions for the block B_i in $\Lambda_{n,n'}$.

Before proving Theorem 5.4 we will prove two lemmas that will be necessary for the application of Proposition 4.8. We first introduce some preliminary notation.

For any $n \leq N$, label the following edges in $\partial\Lambda_{n,n'}$:

$$e^\star_{\mathrm{S}} = (\lfloor \tfrac{n}{2} \rfloor + \tfrac12, 0)\,, \qquad \textit{and} \qquad e^\star_{\mathrm{N}} = (\lfloor \tfrac{n}{2} \rfloor + \tfrac12, n')\,.$$

Recall the definitions of the bridge modification ξ^e and the side modification ξ^s from Definitions 4.2–4.3. We will, throughout the proof of Theorem 5.4, for any boundary condition ξ on $\partial\Lambda_{n,n'}$, let the modification $\xi' \leq \xi$ be given by

$$\xi' := \xi^{e^\star_{\mathrm{S}}} \wedge \xi^{e^\star_{\mathrm{N}}} \wedge \xi^s\,, \tag{5.2}$$

i.e., the bridge modification of ξ on $e^\star_{\mathrm{S}}$ and $e^\star_{\mathrm{N}}$, combined with the side modification ξ^s.

If $\Xi_{K,N}, \Upsilon_{K,N}$ are the sets of boundary conditions defined in Definition 5.5, we let $\Xi'_{K,N}, \Upsilon'_{K,N}$ be the sets corresponding to the modification $\xi \mapsto \xi'$ of every element in the original sets. Observe that $\Upsilon'_{K,N} \subset \Upsilon_{K,N}$ and likewise, $\Xi'_{K,N} \subset \Xi_{K,N}$.

Lemma 5.6. *Let $\alpha \in (0,1]$ and consider the systematic block dynamics $\{Y_k\}_{k\in\mathbb{N}}$ on $\Lambda_{n,n'}$ with $\lfloor \alpha n \rfloor \leq n' \leq n$ and blocks given by Definition 5.5. There exist $c_Y, c_\star(\alpha, q) > 0$ such that for every two initial configurations ω_1, ω_2, and every boundary condition ξ on $\partial\Lambda_{n,n'}$, modified to ξ' by Eq. (5.2), for all $k \geq 2$,*

$$\|\mathbb{P}^{\xi'}_{\omega_1}(Y_k \in \cdot) - \mathbb{P}^{\xi'}_{\omega_2}(Y_k \in \cdot)\|_{\mathrm{TV}} \leq \exp(-c_Y k n^{-c_\star})\,.$$

In particular, for all $k \geq 2$,

$$\max_{\omega_0} \|\mathbb{P}^{\xi'}_{\omega_0}(Y_k \in \cdot) - \pi^{\xi'}_{\Lambda_{n,n'}}\|_{\mathrm{TV}} \leq \exp(-c_Y k n^{-c_\star})\,.$$

Proof. We construct a coupling between the two systematic block dynamics chains, starting from two arbitrary initial configurations ω_1, ω_2, as follows. The systematic block dynamics first samples a configuration on B^o_{E} according to $\pi^{\xi',\omega_i}_{B_{\mathrm{E}}}$ for $i = 1, 2$, where (ξ', ω_i) is the boundary condition induced by $\omega_i \restriction_{B_{\mathrm{W}} - B^o_{\mathrm{E}}} \cup \xi'$ on ∂B_{E}. By Proposition 3.1, applied to the box

$$R^* = B^*_{\mathrm{W}} \cap B^*_{\mathrm{E}}\,,$$

and monotonicity in boundary conditions,

$$\pi^1_{B_{\mathrm{E}}}(e^\star_{\mathrm{S}} \xleftrightarrow{R^*} e^\star_{\mathrm{N}}) \gtrsim n^{-c_\star}\,,$$

where $c_\star(\min\{\frac12, \alpha\}, \varepsilon = \frac14, q) > 0$ is given by that proposition.

We can condition on the west-most vertical dual-crossing between $e^\star_{\mathrm{S}}$ and $e^\star_{\mathrm{N}}$ (if such a dual-crossing exists) as follows: reveal the open components of $\partial B_{\mathrm{E}} \cap [\![0, \lfloor \frac{n}{2} \rfloor]\!] \times [\![0, n']\!]$ as in [16] or [10], so that no edges in other components are revealed. If the open components do not connect to the eastern half of $\partial\Lambda_{n,n'}$ then it must be the case that the desired open dual-crossing exists and can be exposed without revealing any information about edges east of it.

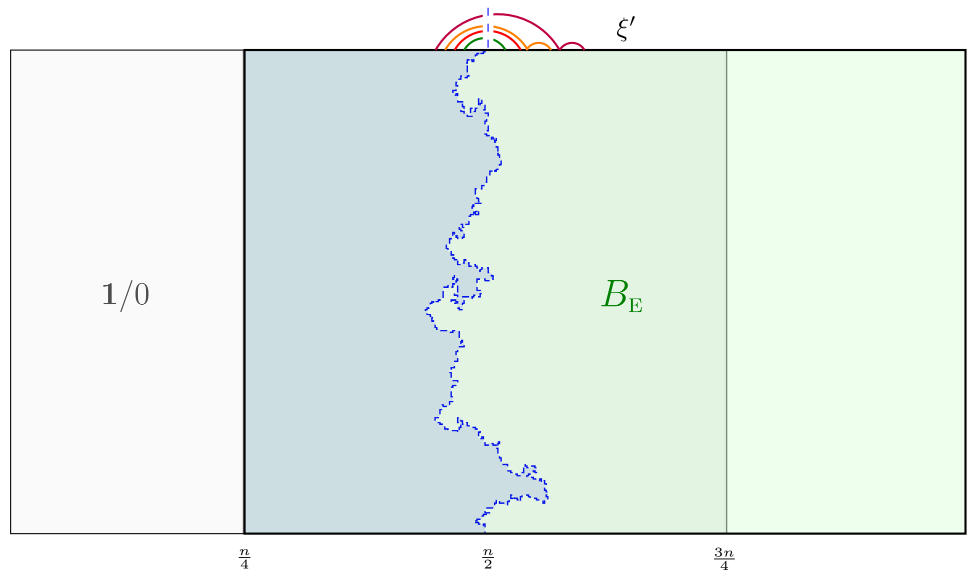

FIGURE 6. If the depicted dual-crossing exists under any (ξ', ω_i), and the bridges over $e^*_{\mathrm{S}}, e^*_{\mathrm{N}}$ are disconnected, one can couple the two chains on the green shaded region, and in particular on $B_{\mathrm{E}} - B^o_{\mathrm{W}}$.

By monotonicity in boundary conditions, if under $\pi^1_{B_{\mathrm{E}}}$ such a vertical dual-connection from $e^\star_{\mathrm{S}}$ to $e^\star_{\mathrm{N}}$ exists, the grand coupling (see §2.2) ensures that the same under $\pi^{\xi',\omega_i}_{B_{\mathrm{E}}}$ for any $\omega_i\restriction_{\Lambda_{n,n'}-B^o_{\mathrm{E}}}$. By definition of the modification ξ', there are no bridges over $e^\star_{\mathrm{S}}$, no bridges over $e^\star_{\mathrm{N}}$, and no components of ξ' with vertices in multiple sides of $\partial\Lambda_{n,n'}$; thus, conditional on this vertical dual-crossing, the following event holds:

$$\bigcap \left\{ v \overset{\xi'}{\not\longleftrightarrow} w : \begin{array}{c} v \in \partial\Lambda_{n,n'} \cap [\![0, \frac{n}{2}]\!] \times [\![0, n']\!] \\ w \in \partial\Lambda_{n,n'} \cap [\![\frac{n}{2}, n]\!] \times [\![0, n']\!] \end{array} \right\} .$$

By the domain Markov property (see Fig. 6), for any pair $\omega_1\restriction_{B_{\mathrm{W}}-B^o_{\mathrm{E}}}$ and $\omega_2\restriction_{B_{\mathrm{W}}-B^o_{\mathrm{E}}}$,

$$\pi^{\xi',\omega_1}_{B_{\mathrm{E}}} \left(\omega\restriction_{[\![\frac{3n}{4},n]\!]\times[\![0,n']\!]} \mid e^\star_{\mathrm{S}} \overset{R^*}{\longleftrightarrow} e^\star_{\mathrm{N}} \right) \overset{d}{=} \pi^{\xi',\omega_2}_{B_{\mathrm{E}}} \left(\omega\restriction_{[\![\frac{3n}{4},n]\!]\times[\![0,n']\!]} \mid e^\star_{\mathrm{S}} \overset{R^*}{\longleftrightarrow} e^\star_{\mathrm{N}} \right) ,$$

using that the boundary conditions to the east of the vertical dual-crossing are the same under both measures. (In the presence of bridges over $e^\star_{\mathrm{S}}$ or $e^\star_{\mathrm{N}}$ the above distributional equality does *not* hold; different configurations west of such a dual-crossing could still induce different boundary conditions east of the dual-crossing, preventing coupling (as illustrated in Fig. 2)—cf. the case of integer q where this problem does not arise.)

This implies that, on the event $e^\star_{\mathrm{S}} \overset{R^*}{\longleftrightarrow} e^\star_{\mathrm{N}}$, the grand coupling couples the two systematic block dynamics chains so that they agree on $\Lambda_{n,n'} - B^o_{\mathrm{W}}$ with probability 1. In this case, let η be the resulting configuration on $B_{\mathrm{E}} - B^o_{\mathrm{W}}$, so that

$$\eta = Y_1\restriction_{B_{\mathrm{E}}-B^o_{\mathrm{W}}} .$$

If the two chains were coupled on $B_{\mathrm{E}} - B_{\mathrm{W}}^o$, the boundary conditions (ξ', η) on ∂B_{W} would be the same for any pair of systematic block dynamics chains with initial configurations ω_1, ω_2; in particular the identity coupling would couple them on all of $\Lambda_{n,n'}$ in the next step when B_{W} is resampled from $\pi_{B_{\mathrm{W}}}^{\xi',\eta}$. Thus, for some $c > 0$,

$$\|\mathbb{P}_{\omega_1}^{\xi'}(Y_2 \in \cdot) - \mathbb{P}_{\omega_2}^{\xi'}(Y_2 \in \cdot)\|_{\mathrm{TV}} \leq 1 - cn^{-c_\star}\,.$$

Since the systematic block dynamics is Markovian and all of the above estimates were uniform in ω_1 and ω_2, the probability of not having coupled in time k under the grand coupling is bounded above by

$$(1 - cn^{-c_\star})^{\lfloor k/2 \rfloor} \leq \exp(-c\lfloor k/2 \rfloor n^{-c_\star})\,. \qquad \blacksquare$$

The next lemma will be key to obtaining the desired upper bound on ρ as defined in (4.4); it shows that with high probability, the boundary conditions induced by the FK measure on a segment will be in $\Xi_{K,N}$, hence the term "typical" boundary conditions.

Lemma 5.7. *Fix $q \in (1, 4]$. There exists $c_\Upsilon(q) > 0$ so that, for every $\Xi_{K,N}$ given by Definition 5.1 on $\Lambda_{n,n'}$ with $n' \leq n \leq N$ and $K > 0$, and every boundary condition ξ,*

$$\pi_{B_{\mathrm{E}}}^{\xi}(\omega\restriction_{\partial_{\mathrm{E}} B_{\mathrm{W}}} \notin \Xi_{K,N}) \lesssim N^{-c_\Upsilon K}\,,$$

where $\omega\restriction_{\partial_{\mathrm{E}} B_{\mathrm{W}}}$ denotes the boundary conditions induced on $\partial_{\mathrm{E}} B_{\mathrm{W}}$ by $\omega\restriction_{B_{\mathrm{E}} - B_{\mathrm{W}}^o} \cup \xi$. The same statement holds when exchanging E *and* W.

Proof. By symmetry, it suffices to prove the bound for the boundary conditions on $\partial_{\mathrm{E}} B_{\mathrm{W}}$. Consider the rectangle

$$R = [\![\tfrac{n}{2}, n]\!] \times [\![0, n']\!]\,.$$

By Proposition 3.9 with aspect ratio $\frac{1}{2}$, there exists $c(q) = c(\alpha = \frac{1}{2}, q) > 0$ such that, for every edge $e \in \partial_{\mathrm{E}} B_{\mathrm{W}}$ and every boundary condition η on ∂R,

$$\pi_R^\eta(|\Gamma^e| \geq K \log N) \lesssim N^{-cK}\,,$$

where, for a configuration ω_R on R, we recall that $|\Gamma^e|$ is the number of disjoint bridges in $\omega_R\restriction_{R - B_{\mathrm{W}}^o} \cup \xi_R$ over e. A union bound over all n' edges on $\partial_{\mathrm{E}} B_{\mathrm{W}}$ implies that

$$\max_\eta \pi_R^\eta(\omega\restriction_{\partial_{\mathrm{E}} B_{\mathrm{W}}} \notin \Xi_{K,N}) \lesssim n' N^{-cK} \lesssim N^{-cK+1}\,,$$

using $n \leq N$. Consequently,

$$\pi_{B_{\mathrm{E}}}^{\xi}(\omega\restriction_{\partial_{\mathrm{E}} B_{\mathrm{W}}} \notin \Xi_{K,N}) = \mathbb{E}_{\pi_{B_{\mathrm{E}}}^{\xi}}\Big[\pi_R^{\xi_R}(\omega\restriction_{\partial_{\mathrm{E}} B_{\mathrm{W}}} \notin \Xi_{K,N})\Big] \lesssim N^{-cK+1}\,,$$

where the expectation is w.r.t. $\pi_{B_{\mathrm{E}}}^{\xi}$ over the boundary conditions ξ_R induced on R by ξ and the configuration on $B_{\mathrm{E}} - R^o$. This concludes the proof of the lemma. $\blacksquare$

Corollary 5.8. *Fix $q \in (1, 4]$, and consider the systematic block dynamics on $\Lambda_{n,n'}$ for $n' \leq n \leq N$ with block choices as given in Definition 5.5. There exists $c_\Upsilon(q) > 0$ so that, for every fixed $K > 0$ and every boundary condition $\xi' \in \Upsilon'_{K,N}$ on $\partial \Lambda_{n,n'}$,*

$$\rho \lesssim N^{-c_\Upsilon K}\,,$$

where ρ is as defined as in (4.4) w.r.t. the initial configuration $\omega_0 \in \{0, 1\}$ and the permissible boundary conditions $\Upsilon_{K,N}$.

Proof. Let Y_k be the systematic block dynamics on $\Lambda_{n,n'}$ where $n \le N$. Recall the definition of ρ in Eq. (4.4), so that in the present setting,

$$\rho = \max_{\omega_0 \in \{0,1\}} \max_{k \ge 1} \max_{i \in \{\mathrm{E},\mathrm{W}\}} \mathbb{P}^{\xi'}_{\omega_0}(Y_k\restriction_{\partial B_i} \notin \Upsilon_{K,N})\,.$$

In the first time step, $\omega_0\restriction_{B_\mathrm{E}}$ induces wired or free boundary conditions on $\partial_\mathrm{W} B_\mathrm{E}$ and so, by Remark 5.3, the boundary condition on $\partial_\mathrm{W} B_\mathrm{E}$ is trivially in $\Xi_{K,N}$. Furthermore, the boundary conditions on $\partial_{\mathrm{N,E,S}} B_\mathrm{E}$ also belong to $\Xi_{K,N}$ by the hypothesis $\xi' \in \Upsilon_{K,N}$. Finally, there cannot be more than $2K\log N$ components in the boundary condition on ∂B_E consisting of vertices on multiple sides for the following reason: as a result of the side modification on ξ', such components can only arise from connections between $\partial_\mathrm{W} B_\mathrm{E}$ and the bridges in $\Gamma^{(n/4,0)}$ and $\Gamma^{(n/4,n')}$; however, there are at most $K\log N$ bridges in each set under *any* configuration on $\Lambda - B^o_\mathrm{E}$ (summing to at most $2K\log N$ components, as claimed). Altogether, $Y_1\restriction_{\partial B_\mathrm{E}} \in \Upsilon_{K,N}$ deterministically.

To address all subsequent time steps, by reflection symmetry and the definition of the systematic block dynamics, is suffices to consider $Y_2\restriction_{\partial B_\mathrm{W}}$. By Lemma 5.7, the probability that a boundary condition on $\partial_\mathrm{E} B_\mathrm{W}$ induced by the systematic dynamics will not be in $\Xi_{K,N}$ is $O(N^{-c_\Upsilon K})$, with $c_\Upsilon > 0$ from that lemma. The fact that, deterministically, the boundary conditions on $\partial_{\mathrm{N,S,W}} B_\mathrm{W}$ are in $\Xi_{K,N}$, and there are at most $2K\log N$ components of the boundary condition on ∂B_W containing vertices of multiple sides of ∂B_W, follows by the same reasoning argued for the first time step. ∎

We are now in a position to prove Theorem 5.4.

Proof of Theorem 5.4. Consider $\Lambda = \Lambda_{N,N'}$ with aspect ratio $\bar\alpha \in (0,1]$ and boundary conditions $\xi \in \Upsilon_{K,N}$ for a fixed

$$K \ge K_0 := 6(c_\star + 1)\max\{c_\Upsilon^{-1}, 1\}\,, \tag{5.3}$$

where $c_\star = c_\star(\min\{\bar\alpha, \frac12\}, \frac14, q)$ is the constant given by Proposition 3.1, and $c_\Upsilon = c_\Upsilon(q)$ is given by Corollary 5.8. It suffices to prove the proposition for all K sufficiently large, as $\Upsilon_{K,N} \subset \Upsilon_{K',N}$ for every $K \le K'$.

We prove the following inductively in $n \in [\![1,N]\!]$: for every $K > K_0$ as above, every $(\bar\alpha \wedge \frac12)n \le n' \le n$, and every $\xi \in \Upsilon_{K,N}$, if

$$t_n = N^{2(c_\star+\lambda+1)\log_{4/3} n} \qquad \text{where} \qquad \lambda := 32K\log q + 5\,,$$

then Glauber dynamics for the critical FK model on $\Lambda_{n,n'}$ has

$$\|\mathbb{P}^\xi_1(X_{t_n} \in \cdot) - \mathbb{P}^\xi_0(X_{t_n} \in \cdot)\|_{\mathrm{TV}} \le N^{-3}\,. \tag{5.4}$$

To see that Eq. (5.4) implies Theorem 5.4, note that (2.1), with the choice $n = N$, implies that $\bar d_{\mathrm{TV}}(N^{c(\bar\alpha,q)\log n}) = O(1/N) = o(1)$ for some $c(\bar\alpha,q) > 0$.

For the base case, fix a large constant M, where clearly $t_{\mathrm{MIX}} = O(1)$ for all $n \le M$. Next, let $m \in [\![M,N]\!]$, and assume (5.4) holds for all $n \in [\![1,m-1]\!]$. Consider the censored and systematic block dynamics, $(\bar X_t)_{t\ge0}$ and $\{Y_k\}_{k\ge0}$, respectively, on the blocks defined in Definition 5.5 on $\Lambda_m = \Lambda_{m,m'}$ for some $(\bar\alpha \wedge \frac12)m \le m' \le m$ and boundary conditions $\xi \in \Upsilon_{K,N}$.

Recall that $\xi \in \Upsilon_{K,N}$ has at most $K \log N$ bridges over any edge and at most $2K \log N$ components spanning multiple sides of $\partial\Lambda_m$; thus, by Fact 4.5, the boundary modification ξ' defined in (5.2) satisfies

$$d(\xi', \xi) \le 8K \log N\,.$$

By the definition of λ, we have $|E|^2 q^{4d(\xi',\xi)} = o(N^\lambda)$. Hence, by Lemma 4.6 (Eq. (4.2), where we increased the time on the right-hand to N^λ, for large enough N, by the monotonicity of $d_{\rm TV}$) and the above bound on $d(\xi', \xi)$, we have that for all $k, T \ge 0$,

$$\begin{aligned}\|\mathbb{P}_1^\xi(X_{N^\lambda kT} \in \cdot) - \mathbb{P}_0^\xi(X_{N^\lambda kT} \in \cdot)\|_{\rm TV} &\le 2 \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^\xi_{\omega_0}(X_{N^\lambda kT} \in \cdot) - \pi^\xi_{\Lambda_m}\|_{\rm TV} \\ &\le 16 \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m}\|_{\rm TV} + 2e^{-N^{\lambda/4}}\,,\end{aligned}$$

and subsequently, by Theorem 2.5,

$$\|\mathbb{P}_1^\xi(X_{N^\lambda kT} \in \cdot) - \mathbb{P}_0^\xi(X_{N^\lambda kT} \in \cdot)\|_{\rm TV} \le 16 \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(\bar X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m}\|_{\rm TV} + 2e^{-N^{\lambda/4}}\,. \tag{5.5}$$

We will next show that the first term in the right-hand above satisfies

$$\max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(\bar X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m}\|_{\rm TV} = o(N^{-3})\,, \tag{5.6}$$

which will imply (5.4) (and conclude the proof) if we choose k, T such that $N^\lambda kT \le t_m$. By the triangle inequality,

$$\begin{aligned}&\max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(\bar X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m}\|_{\rm TV} \\ &\qquad\le \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(\bar X_{kT} \in \cdot) - \mathbb{P}^{\xi'}_{\omega_0}(Y_k \in \cdot)\|_{\rm TV} + \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(Y_k \in \cdot) - \pi^{\xi'}_{\Lambda_m}\|_{\rm TV} \\ &\qquad\le \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(\bar X_{kT} \in \cdot) - \mathbb{P}^{\xi'}_{\omega_0}(Y_k \in \cdot)\|_{\rm TV} + e^{-c_Y k m^{-c_\star}}\,,\end{aligned}$$

where the last inequality is valid for every $k \ge 2$ by Lemma 5.6. Using $\Upsilon'_{K,N} \subset \Upsilon_{K,N}$ and Proposition 4.8,

$$\max_{\omega_0 \in \{0,1\}} \|\mathbb{P}^{\xi'}_{\omega_0}(\bar X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m}\|_{\rm TV} \le k(\rho + \varepsilon(T)) + e^{-c_Y k m^{-c_\star}}\,,$$

where ρ and ε were given in (4.3)–(4.4), that is, in our context,

$$\varepsilon(T) = \max_{\omega' \in \Omega} \max_{i \in \{\mathrm{E},\mathrm{W}\}} \max_{\zeta \in \Upsilon^{B_i}_{K,N}} \|\mathbb{P}^{\zeta, B_i}_{\omega'}(X_T \in \cdot) - \pi^\zeta_{B_i}\|_{\rm TV}\,,$$

$$\rho = \max_{k \ge 1} \max_{i \in \{\mathrm{E},\mathrm{W}\}} \mathbb{P}_{\omega_0}\left(Y_k \restriction_{\partial B_i} \notin \Upsilon^{B_i}_{K,N}\right)\,.$$

We will bound $\varepsilon(T)$ by the inductive assumption for the choice of

$$k := c_Y^{-1}(c_\star + 6) N^{c_\star} \log N \qquad \text{and} \qquad T := k t_{\lfloor 3m/4 \rfloor} N^\lambda K \log N\,. \tag{5.7}$$

In order to apply the induction hypothesis for a box whose side lengths are smaller by a constant factor vs. the original dimensions of $m \times m'$, we repeat the above analysis for the sub-block B_i (whose dimensions are $\lfloor\frac{3}{4}m\rfloor \times m'$), and get from Fact 2.4 and the above arguments that

$$\begin{aligned}\varepsilon(T) &\lesssim N^2 \max_{i\in\{\mathrm{E},\mathrm{W}\}} \max_{\zeta\in\Upsilon_{K,N}^{B_i}} \|\mathbb{P}_0^{\zeta,B_i}(X_T\in\cdot)-\mathbb{P}_1^{\zeta,B_i}(X_T\in\cdot)\|_{\mathrm{TV}} \\ &\lesssim N^2 k\left(\rho'+\varepsilon'(\tfrac{T}{kN^\lambda})\right)+N^2 e^{-c_Y k m^{-c_\star}}+N^2 e^{-N^{\lambda/4}}\,,\end{aligned}$$

where $\varepsilon'(T)$ and ρ' are the counterparts of $\varepsilon(T)$ and ρ w.r.t. the sub-blocks of B_i, as per Definition 5.5, rotated by $\pi/4$. This yields the following new bound on (5.6):

$$\max_{\omega_0\in\{0,1\}}\|\mathbb{P}_{\omega_0}^{\xi'}(\bar{X}_{kT}\in\cdot)-\pi_{\Lambda_m}^{\xi'}\|_{\mathrm{TV}} \lesssim N^2k^2\left(\rho+\rho'+\varepsilon'(\tfrac{T}{kN^\lambda})\right)+kN^2e^{-c_Y k m^{-c_\star}}+o(N^{-3}).$$

Note that the dimensions of the sub-blocks of B_i (those under consideration in $\varepsilon'(T)$) are $\lfloor\frac{3}{4}m\rfloor\times\lfloor\frac{3}{4}m'\rfloor$. Hence, by the inductive assumption at scale $\lfloor\frac{3}{4}m\rfloor$ and Fact 2.4,

$$\varepsilon'\left(t_{\lfloor 3m/4\rfloor}\right)=O(1/N)\,,$$

which, along with (2.1) and the submultiplicativity of $\bar{d}_{\mathrm{TV}}(t)$, yields that for T from (5.7),

$$\varepsilon'(\tfrac{T}{kN^\lambda})=\varepsilon'\left(t_{\lfloor 3m/4\rfloor}K\log N\right)\lesssim N^{-K}\le N^{-6(c_\star+1)}\,.$$

By Corollary 5.8, we have $\rho\lesssim N^{-c_\Upsilon K}\le N^{-6(c_\star+1)}$ by our choice of K_0, and similarly for ρ'. So, for $k=N^{c_\star+o(1)}$ as in (5.7), $k^2\rho\lesssim N^{-4c_\star-6+o(1)}=o(N^{-5})$, and similarly, $k^2\rho'=o(N^{-5})$. Finally, this choice of k guarantees that $kN^2\exp(-c_Y k m^{-c_\star})$ is at most $kN^{-c_\star-4}=o(N^{-3})$. Combining the last three displays with these bounds yields (5.6). The proof is concluded by noting that indeed $N^\lambda kT\le N^{2c_\star+2\lambda+o(1)}t_{\lfloor 3m/4\rfloor}\le t_m$. ∎

5.3. **Mixing on the torus.** In this section we extend Theorem 5.4 to the $n\times n$ torus, proving Theorem 1. Observe that the periodic FK boundary conditions identified with $(\mathbb{Z}/n\mathbb{Z})^2$ in fact have order n components with vertices on multiple sides of $\partial\Lambda$. We thus have to extend the bound of Theorem 5.4 to periodic boundary conditions using the topological structure of $(\mathbb{Z}/n\mathbb{Z})^2$. The proof draws from the extension of mixing time bounds in [16] and [10] from fixed boundary conditions to $(\mathbb{Z}/n\mathbb{Z})^2$. In the present setting, having to deal with a specific class of boundary conditions forces us to reapply the bridge modification and the censored and systematic block dynamics techniques.

We first bound the mixing time on a cylinder with typical boundary conditions on its non-periodic sides. In what follows, for any $\Lambda_{n,n'}$, label the following edges:

$$\begin{aligned} e^\star_{\mathrm{SW}}&=(0,\lfloor\tfrac{n'}{2}\rfloor+\tfrac12)\,, & e^\star_{\mathrm{SE}}&=(n,\lfloor\tfrac{n'}{2}\rfloor+\tfrac12)\,,\\ e^\star_{\mathrm{NW}}&=(0,\lfloor\tfrac{3n'}{4}\rfloor+\tfrac12)\,, & e^\star_{\mathrm{NE}}&=(n,\lfloor\tfrac{3n'}{4}\rfloor+\tfrac12)\,.\end{aligned}$$

Then define the modification ξ' of boundary conditions ξ by

$$\xi'=\xi^{e^\star_{\mathrm{SW}}}\wedge\xi^{e^\star_{\mathrm{SE}}}\wedge\xi^{e^\star_{\mathrm{NW}}}\wedge\xi^{e^\star_{\mathrm{NE}}}\wedge\xi^s \tag{5.8}$$

and define $\Xi'_{K,N},\Upsilon'_{K,N}$ as before, for the new modification. We say that a boundary condition on $\partial_{\mathrm{N,S}}\Lambda$ is in $\Upsilon_{K,N}$ if its restriction to each side is in $\Xi_{K,N}$ and there are fewer than $2K\log N$ distinct components with vertices in $\partial_{\mathrm{N}}\Lambda$ and $\partial_{\mathrm{S}}\Lambda$, and analogously for boundary conditions on $\partial_{\mathrm{E,W}}\Lambda$.

Theorem 5.9 (Mixing time on a cylinder)**.** *Fix $q \in (1,4]$, $\alpha \in (0,1]$ and $K > 0$. There exists some $c(\alpha,q,K) > 0$ such that the critical FK model on $\Lambda = \Lambda_{N,N'}$ with $\alpha N \le N' \le \alpha^{-1} N$ and boundary conditions, denoted by (p,ξ), that are periodic on $\partial_{\mathrm{N,S}}\Lambda$ and $\xi \in \Upsilon_{K,N}$ on $\partial_{\mathrm{E,W}}\Lambda$, satisfies $t_{\mathrm{MIX}} \lesssim N^{c\log N}$.*

Proof. We will use a similar modification and censoring approximation as in the proof of Theorem 5.4 to reduce the cylinder to rectangles with "typical" boundary conditions. As before, it suffices to prove the theorem for large enough K, since $\Upsilon_{K,N} \subset \Upsilon_{K',N}$ for $K \le K'$. We will establish it for any fixed

$$K \ge K_0 + K_0' \quad \text{where} \quad K_0 = 4(c_\star + 1)(c_\Upsilon^{-1} \vee 1) \quad \text{and} \quad K_0' = K_0(c_\Psi^{-1} \vee 1)\,,$$

in which $c_\star = c_\star(\frac{\alpha}{5}, \frac14, q) > 0$ is given by Proposition 3.1, the constant c_Υ is $c(\frac{2\alpha}{5}, q) > 0$ from Proposition 3.9, and $c_\Psi = c_\Psi(\frac{3\alpha}{5}, q) > 0$ is given by Proposition 3.14.

Define, as in Definition 4.7, the censored and systematic block dynamics on

$$B_1 := [\![0,N]\!] \times [\![0,\tfrac{N'}{5}]\!] \cup [\![0,N]\!] \times [\![\tfrac{2N'}{5}, N']\!]\,,$$
$$B_2 := [\![0,N]\!] \times [\![0,\tfrac{3N'}{5}]\!] \cup [\![0,N]\!] \times [\![\tfrac{4N'}{5}, N']\!]\,.$$

The choice of boundary class on B_i for $i = 1,2$ is $\Gamma_i = \Upsilon_{3K,N}$. (Observe that B_1 and B_2 are, by construction, $N \times \frac45 N'$ rectangles with non-periodic boundary conditions.)

As in the proof of Theorem 5.4, it suffices, by Fact 2.4, to show that there exists some $c(\alpha,q,K) > 0$ such that

$$\|\mathbb{P}_1^{p,\xi}(X_{N^{c\log N}} \in \cdot) - \mathbb{P}_0^{p,\xi}(X_{N^{c\log N}} \in \cdot)\|_{\mathrm{TV}} \le N^{-3}\,. \tag{5.9}$$

In the setting of the cylinder, the side modification (p,ξ^s) of (p,ξ) only disconnects $\partial_{\mathrm{E}}\Lambda$ from $\partial_{\mathrm{W}}\Lambda$, and so, if ξ' is as in (5.8), then $d(\xi',\xi) \le 6K\log N$. Thus, by (4.2), the triangle inequality and Theorem 2.5 (as explained in the derivation of (5.5)), if

$$t_N = N^\lambda kT \quad \text{for} \quad \lambda := 24K + 5$$

(so that $|E|^2 q^{4d(\xi',\xi)} = o(N^\lambda)$), then for every $k, T \ge 0$,

$$\|\mathbb{P}_1^{p,\xi}(X_{t_N} \in \cdot) - \mathbb{P}_0^{p,\xi}(X_{t_N} \in \cdot)\|_{\mathrm{TV}} \le 16 \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}_{\omega_0}^{p,\xi'}(\bar X_{kT} \in \cdot) - \pi_\Lambda^{p,\xi'}\|_{\mathrm{TV}} + 2e^{-N^{\lambda/4}}\,,$$

which, by Proposition 4.8, is at most

$$16 \max_{\omega_0 \in \{0,1\}} \|\mathbb{P}_{\omega_0}^{p,\xi'}(Y_k \in \cdot) - \pi_\Lambda^{p,\xi'}\|_{\mathrm{TV}} + 16k(\rho + \varepsilon(T)) + 2e^{-N^{\lambda/4}}\,, \tag{5.10}$$

where $\varepsilon(T)$ and ρ are given by (4.3) and (4.4), respectively, w.r.t. the blocks B_1, B_2, the permissible boundary conditions $\Upsilon_{3K,N}$, and the initial configuration $\omega_0 \in \{0,1\}$.

We next bound the first term in the right-hand side of Eq. (5.10) by the probability of not coupling the systematic block dynamics chains started from two arbitrary initial configurations under the grand coupling (cf. Lemma 5.6). In the first time step, we try to couple the chains started from ω_1, ω_2 on

$$R := [\![0,N]\!] \times [\![\tfrac{3N'}{5}, \tfrac{4N'}{5}]\!]$$

so that in the second step the identity coupling couples them on all of Λ. It suffices to couple the systematic chains started from $\omega_1 = 0$ and $\omega_2 = 1$ under the grand coupling. In order to couple samples from the $(0,\xi')$ and $(1,\xi')$ boundary conditions on R (induced by $\omega_1=0$ and $\omega_2=1$ resp.), define the following two sub-blocks of B_1:

$$R_{\mathrm{S}} := [\![0,N]\!] \times [\![\tfrac{2N'}{5},\tfrac{3N'}{5}]\!]\,, \qquad R_{\mathrm{N}} := [\![0,N]\!] \times [\![\tfrac{4N'}{5},N']\!]\,.$$

By Proposition 3.1, monotonicity in boundary conditions, and the FKG inequality,

$$\min_{\eta} \pi_{B_1}^{\eta,\xi'}\left(e^\star_{\mathrm{SW}} \xleftrightarrow{R^*_{\mathrm{S}}} e^\star_{\mathrm{SE}}\,,\, e^\star_{\mathrm{NW}} \xleftrightarrow{R^*_{\mathrm{N}}} e^\star_{\mathrm{NE}}\right) \gtrsim N^{-2c_\star}\,.$$

By the Domain Markov property, and the definition of the boundary modification ξ',

$$\pi_{B_1}^{1,\xi'}\left(\omega\restriction_R \mid e^\star_{\mathrm{SW}} \xleftrightarrow{R^*_{\mathrm{S}}} e^\star_{\mathrm{SE}}\,,\, e^\star_{\mathrm{NW}} \xleftrightarrow{R^*_{\mathrm{N}}} e^\star_{\mathrm{NE}}\right) \stackrel{d}{=} \pi_{B_1}^{0,\xi'}\left(\omega\restriction_R \mid e^\star_{\mathrm{SW}} \xleftrightarrow{R^*_{\mathrm{S}}} e^\star_{\mathrm{SE}}\,,\, e^\star_{\mathrm{NW}} \xleftrightarrow{R^*_{\mathrm{N}}} e^\star_{\mathrm{NE}}\right)\,.$$

As before, using the grand coupling and revealing edges from $\partial_{\mathrm{S}} R_{\mathrm{S}}$ and $\partial_{\mathrm{N}} R_{\mathrm{N}}$ until we reveal a pair of such horizontal dual-crossings, by monotonicity, we can couple $\pi_{B_1}^{\omega_1,\xi'}$ and $\pi_{B_1}^{\omega_2,\xi'}$ on R with probability at least $cN^{-2c_\star}$. On that event, the two chains are coupled in the next step (and thereafter) on all of Λ with probability 1. By the definition of the systematic block dynamics, we conclude that, for some $c_Y>0$ and every $k\geq 2$,

$$\max_{\omega_0\in\{0,1\}} \|\mathbb{P}_{\omega_0}^{p,\xi'}(Y_k\in\cdot) - \pi_\Lambda^{p,\xi'}\|_{\mathrm{TV}} \leq \exp(-c_Y k N^{-2c_\star})\,.$$

To bound ρ, first note that, for $\omega_0\in\{0,1\}$, the block B_1 has boundary conditions $(0,\xi')$ or $(1,\xi')$, both of which are in $\Upsilon_{3K,N}$ by Remark 5.3. Thereafter, the uniformity of Proposition 3.9 in boundary conditions implies that for every η,

$$\pi_{B_1}^{\eta,\xi'}(\omega\restriction_{\partial B_2} \notin \Xi_{K,N}) \lesssim N^{-c_\Upsilon K}\,,$$

and likewise when exchanging B_1 and B_2. In order to obtain a corresponding bound on ρ, we note that in addition to connections between $\partial_{\mathrm{N,S}} B_i$ and $\partial_{\mathrm{E,W}} B_i$ (for $i=1,2$), which we bound deterministically by $4K\log N$ as in the proof of Corollary 5.8, in the present setting there could also be (multiple) open components connecting $\partial_{\mathrm{S}} B_i$ to $\partial_{\mathrm{N}} B_i$ in $\Lambda - B_i$. By Proposition 3.14 and monotonicity in boundary conditions, for every η,

$$\pi_{B_1}^{\eta,\xi}(|\Psi_{\Lambda-B_2}| \geq K\log N) \lesssim N^{-c_\Psi K}\,,$$

where, as in that proposition, $|\Psi_{\Lambda-B_2}|$ is the number of distinct vertical crossings of $\Lambda - B_2$. By the choices of K_0 and K_0', a union bound yields

$$\rho \lesssim \max_\eta \pi_{B_1}^{\eta,\xi'}(\omega\restriction_{\partial B_2} \notin \Upsilon_{3K,N}) \lesssim N^{-4c_\star - 4}\,.$$

Observe that on their respective translates, B_1 and B_2 are $N\times\frac{4}{5}N'$ rectangles, so we can bound $\max_i \max_{\xi\in\Upsilon_{3K,N}} t_{\mathrm{MIX}}^{\xi,B_i}$ using Theorem 5.4; by that theorem, rotational symmetry, and the sub-multiplicativity of $\bar{d}_{\mathrm{TV}}$, we have that for some $c_B(\alpha,q,K)>0$,

$$\varepsilon(T) \lesssim \exp(-c_B^{-1} T N^{-c_B\log N})\,,$$

uniformly over $\alpha N\leq N'\leq \alpha^{-1}N$. Altogether, combining the bounds on ρ, ε, and the systematic block dynamics distance from stationarity, in Eq. (5.10), we see that for

$$k = N^{2c_\star+1} \qquad \text{and} \qquad T = N^{(c_B+1)\log N}$$

one has

$$\|\mathbb{P}_1^{p,\xi}(X_{t_N}\in\cdot)-\mathbb{P}_0^{p,\xi}(X_{t_N}\in\cdot)\|_{\mathrm{TV}}=o(N^{-3})\,,$$

implying Eq. (5.9) and concluding the proof. ■

Proof of Theorem 1. The theorem is obtained by reducing the mixing time on the torus to that on a cylinder and then applying Theorem 5.9. Fix $\bar\alpha\in(0,1]$ and consider $\Lambda=\Lambda_{n,n'}$ with $\bar\alpha n\le n'\le\bar\alpha^{-1}n$ and periodic boundary conditions, denoted by (p), identified with $(\mathbb{Z}/n\mathbb{Z})\times(\mathbb{Z}/n'\mathbb{Z})$ (the special case $n'=n$ is formulated in Theorem 1).

Let $c_\star=c_\star(\frac{\bar\alpha}{5},\frac14,q)>0$ be given by Proposition 3.1 and let c_Υ,c_Ψ, K_0 and K_0' be given as in the proof of Theorem 5.9. Define $K=K_0+K_0'$. We consider the censored and systematic block dynamics with the block choices,

$$B_1:=[\![0,n]\!]\times[\![0,\tfrac{n'}{5}]\!]\cup[\![0,n]\!]\times[\![\tfrac{2n'}{5},n']\!]\,,$$
$$B_2:=[\![0,n]\!]\times[\![0,\tfrac{3n'}{5}]\!]\cup[\![0,n]\!]\times[\![\tfrac{4n'}{5},n']\!]\,.$$

and boundary class

$$\Upsilon^p_{3K,n}:=\left\{\xi:\xi\restriction_{\partial_{\mathrm{N,S}}\Lambda}=p,\xi\restriction_{\partial_{\mathrm{E,W}}\Lambda}\in\Upsilon_{3K,n}\right\}\,.$$

By Theorem 2.5, the triangle inequality and Proposition 4.8, for every $k,T\ge0$,

$$\|\mathbb{P}_1^{p}(X_{kT}\in\cdot)-\mathbb{P}_0^{p}(X_{kT}\in\cdot)\|_{\mathrm{TV}}\le 2\max_{\omega_0\in\{0,1\}}\|\mathbb{P}^p_{\omega_0}(Y_k\in\cdot)-\pi^p_\Lambda\|_{\mathrm{TV}}+2k(\rho+\varepsilon(T))\,,$$

where ρ and $\varepsilon(T)$ are w.r.t. the class $\Upsilon^p_{3K,n}$ of permissible boundary conditions. It suffices, as before, to prove that the right-hand side is $o(n^{-3})$ and then use (2.1) and the sub-multiplicativity of $\bar d_{\mathrm{TV}}(t)$ to obtain the desired result.

Recall the edges $e^\star_{\mathrm{SW}},e^\star_{\mathrm{NW}},e^\star_{\mathrm{SE}}$ and $e^\star_{\mathrm{NE}}$ on $\Lambda_{n,n'}$. As in the proof of Theorem 5.9, if

$$R_{\mathrm{S}}:=[\![0,n]\!]\times[\![\tfrac{2n'}{5},\tfrac{3n'}{5}]\!]\,,\qquad R_{\mathrm{N}}:=[\![0,n]\!]\times[\![\tfrac{4n'}{5},n']\!]\,,$$

then by Proposition 3.1 and the FKG inequality, we have

$$\pi^{1,p}_{B_1}\left(e^\star_{\mathrm{SW}}\xleftrightarrow{R^*_{\mathrm{S}}}e^\star_{\mathrm{SE}}\,,\,e^\star_{\mathrm{NW}}\xleftrightarrow{R^*_{\mathrm{N}}}e^\star_{\mathrm{NE}}\right)\gtrsim n^{-2c_\star}\,.$$

Crucially, while no boundary modification was done in this case, the periodic sides of B_1 have no bridges over the four designated edges, and the two horizontal dual-crossings, from the event above, disconnect its non-periodic sides ($\partial_{\mathrm{N}}B_1$ and $\partial_{\mathrm{S}}B_1$) from $\partial_{\mathrm{N}}B_2$ and $\partial_{\mathrm{S}}B_2$. Therefore, if that event occurs for the systematic block dynamics chain started from $\omega_0=1$, the grand coupling carries it to the chains started from all other initial states, and yields a coupling of all these chains on $[\![\frac{3n}{5},\frac{4n}{5}]\!]\times[\![0,n']\!]\supset\partial B_2$. By definition of the systematic block dynamics and submultiplicativity of $\bar d_{\mathrm{TV}}(t)$, for $k\ge2$,

$$\max_{\omega_0\in\{0,1\}}\|\mathbb{P}^p_{\omega_0}(Y_k\in\cdot)-\pi^p_\Lambda\|_{\mathrm{TV}}\le\exp(-c_Ykn^{-2c_\star})\,.\tag{5.11}$$

Observe that at every time step of the systematic block dynamics, the block B_i $(i=1,2)$ is an $n\times\frac45 n'$ rectangle with periodic boundary conditions on $\partial_{\mathrm{E,W}}B_i$ and boundary conditions η induced by the chain on $\partial_{\mathrm{N,S}}B_i$. By Theorem 5.9, for some $c(\bar\alpha,q,K)>0$,

$$\max_i\max_{(p,\eta)\in\Upsilon^p_{3K,n}}t_{\mathrm{MIX}}^{(p,\eta),B_i}\lesssim n^{c\log n}\,,$$

and by submultiplicativity of $\bar{d}_{\mathrm{TV}}(t)$,

$$\varepsilon(T) \lesssim \exp(-c^{-1} T n^{c \log n}) .$$

As in the proof of Theorem 5.9, since the estimate on ρ was uniform in the boundary conditions, we again have $\rho \lesssim n^{-4c_\star - 4}$ (using Propositions 3.9 and 3.14). Combining the bounds on ρ and ε with (5.11), there exists some $c(\bar{\alpha}, q, K) > 0$ such that

$$\|\mathbb{P}^p_1(X_{n^{c \log n}} \in \cdot) - \mathbb{P}^p_0(X_{n^{c \log n}} \in \cdot)\|_{\mathrm{TV}} = o(n^{-3}) ,$$

as required. ■

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