# ON THE LIMITING LAW OF LINE ENSEMBLES OF BROWNIAN POLYMERS WITH GEOMETRIC AREA TILTS

AMIR DEMBO, EYAL LUBETZKY, AND OFER ZEITOUNI

*Dedicated to the memory of Dima Ioffe*

Abstract. We study the line ensembles of non-crossing Brownian bridges above a hard wall, each tilted by the area of the region below it with geometrically growing pre-factors. This model, which mimics the level lines of the $(2+1)$D SOS model above a hard wall, was studied in two works from 2019 by Caputo, Ioffe and Wachtel. In those works, the tightness of the law of the top $k$ paths, for any fixed $k$, was established under either zero or free boundary conditions, which in the former setting implied the existence of a limit via a monotonicity argument. Here we address the open problem of a limit under free boundary conditions: we prove that as the interval length, followed by the number of paths, go to $\infty$, the top $k$ paths converge to the same limit as in the free boundary case, as conjectured by Caputo, Ioffe and Wachtel.

## 1. Introduction

Entropic repulsion in low temperature $(2+1)$D crystals above a hard wall has been the subject of extensive study in statistical physics. Whereas in the absence of a wall, the surface of the crystal would typically be rigid at height 0, in the presence of a wall, the surface is propelled in order to increase its entropy (i.e., to allow thermal fluctuations going downward), and becomes rigid at some height level which diverges with the side length $L$ of the box.

A rigorous study of this phenomenon in the $(2+1)$D Solid-On-Solid (SOS) model—a low temperature approximation of the 3D Ising model—dates back to Bricmont, El Mellouki and Fröhlich [1] in 1986, where it was shown that, in the presence of a hard wall at height 0, the typical height of a site in the bulk is propelled to order $\log L$. Thereafter, a detailed description of the shape of this random surface was obtained by Caputo et al. [4–6], showing that it typically becomes rigid at a height which is one of two consecutive (explicit) integers, through a sequence of nested level lines each encompassing a $(1-\varepsilon)$-fraction of the sites (analogous behavior was later established [16] for the more general family of $|\nabla\phi|^p$-random surface model, where the SOS model is the case $p=1$). The level lines near the center sides of the box behave as random walks—a ubiquitous feature of interfaces in low temperature spin systems—albeit with cube-root fluctuations, as their laws are tilted by the entropic repulsion effect. The lower the level line, the higher the reward is for generating spikes going downward, and as such, the tilting effect of the level lines increases exponentially as the height decreases.

Whereas the 2D Ising model with a pinning potential is known [11] to have an interface converging to a Ferrari–Spohn diffusion, the behavior in the SOS model—where there are $H \asymp \log L$ interacting level lines, each constrained not to cross its neighbors and inducing a tilt which is a function of the area it encompasses and its height—is far from being understood (see the review [13] for more information).

In this work, we investigate the limiting law of a line ensemble that was studied by Caputo, Ioffe and Wachtel [2, 3] to model the level lines of the SOS model in the presence of a hard wall: each level line, $X_1, X_2, \ldots$, where $X_1$ is the top one, is tilted by the area below it, with the coefficients of these area tilts increasing geometrically.

For more perspective on this model in the context of other models of Brownian polymers constrained above a barrier, starting from the influential work of Ferrari and Spohn [8] (the model there being equivalent by Girsanov's transformation—cf. [17]—to a Brownian excursion with an area tilt), see, e.g., [2, 12, 14] and the references therein.

Define

$$\mathbb{A}_n^+ = \{\underline{x} \in \mathbb{R}^n \,:\, x_1 > x_2 > \ldots > x_n > 0\}\,,$$

its closure $\bar{\mathbb{A}}_n^+$ and, for a designated interval

$$I = [\ell, r] \qquad (\ell < r \in \mathbb{R})\,,$$

let

$$\Omega_n^I = \left\{X \in \mathsf{C}(I; \mathbb{R}^n) \,:\, X(t) \in \mathbb{A}_n^+ \text{ for all } t \in I\right\}\,.$$

(Here, for $T \subset \mathbb{R}$ and $\mathcal{X}$ a topological space, we denote by $\mathsf{C}(T; \mathcal{X})$ the space of continuous functions from $T$ to $\mathcal{X}$.) Further define the area tilt of $Y \in \mathsf{C}(I; \mathbb{R})$ to be

$$\mathcal{A}_I(Y) = \int_I Y(t)\,\mathrm{d}t\,,$$

and, for given tilt parameters $\mathfrak{a} > 0$ and $\mathfrak{b} > 1$ and endpoints $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{A}_n^+$ and $\underline{y} = (y_1, \ldots, y_n) \in \mathbb{A}_n^+$, the partition function

$$Z_n^{\underline{x},\underline{y},I} = \mathbf{E}_n^{\underline{x},\underline{y},I}\left[\mathbb{1}_{\Omega_n^I}\, e^{-\mathfrak{a}\sum_{i=1}^n \mathfrak{b}^{i-1}\mathcal{A}_I(X_i(\cdot))}\right]\,, \tag{1.1}$$

in which $\mathbf{E}_n^{\underline{x},\underline{y},I} = \bigotimes_{i=1}^n \mathbf{E}_1^{x_i,y_i,I}$ and the expectation $\mathbf{E}_1^{x,y,I}$ for $I = [\ell, r]$ is w.r.t. the (unnormalized) path measures of the Brownian bridge which starts at $x$ at time $\ell$ and ends at $y$ at time $r$; that is, the total mass of $\mathbf{E}_1^{x_i,y_i,I}$ is $\phi_{r-\ell}(y_i - x_i)$, where

$$\phi_v(\underline{x}) := (2\pi v)^{-k/2} e^{-\|\underline{x}\|_2^2/(2v)} \tag{1.2}$$

denotes hereafter the density of a centered Gaussian vector of independent coordinates of variance $v$, whose dimension $k$ is implicitly given by the argument we use. (At no point in our analysis will we need to adjust the tilt parameters $(\mathfrak{a}, \mathfrak{b})$, and as such we do not include them in the notation for brevity.)

Let $\mathcal{B}_n = \mathcal{B}_{n,I}$ be the Borel $\sigma$-field on $\mathsf{C}(I, \mathbb{R}^n)$. (We omit $I$ from the notation when no confusion occurs.) For $\Gamma \in \mathcal{B}_{n,I}$ define

$$\mathbb{P}_n^{\underline{x},\underline{y},I}(\Gamma) := \frac{1}{Z_n^{\underline{x},\underline{y},I}} \mathbf{E}_n^{\underline{x},\underline{y},I} \left[ \mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^I} \, e^{-\mathfrak{a} \sum_{i=1}^n \mathfrak{b}^{i-1} \mathcal{A}_I(X_i(\cdot))} \right] . \tag{1.3}$$

Consider $I_T = [-T, T]$. Two classes of boundary conditions that are of interest are:

(a) *Zero boundary conditions*: fixing both $\underline{x}$ and $\underline{y}$ to be zero:

$$\mu_{n,T}^{\mathfrak{o}} = \mathbb{P}_n^{\underline{0},\underline{0},I_T}$$

(more precisely, this is the limit of $\mathbb{P}_n^{\varepsilon\underline{x},\varepsilon\underline{y},I_T}$ as $\varepsilon \downarrow 0$, which by stochastic domination exists and is independent of the fixed $\underline{x}$, $\underline{y}$ in $\mathbb{A}_n^+$ which one uses).

(b) *Free boundary conditions*: averaging $\mathbf{E}_n^{\underline{x},\underline{y},I_T}[\cdot]$ over $\underline{x}, \underline{y}$ w.r.t. Lebesgue measure on $\mathbb{R}^n$:

$$\mu_{n,T}^{\mathfrak{f}}(\Gamma) = \frac{1}{\mathcal{Z}_{n,T}^{\mathfrak{f}}} \int_{\mathbb{A}_n^+} \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},I_T} \left[ \mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^{I_T}} \, e^{-\mathfrak{a} \sum_{i=1}^n \mathfrak{b}^{i-1} \mathcal{A}_{I_T}(X_i(\cdot))} \right] \mathrm{d}\underline{x} \mathrm{d}\underline{y} \,,$$

where

$$\mathcal{Z}_{n,T}^{\mathfrak{f}} := \int_{\mathbb{A}_n^+} \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},I_T} \left[ \mathbb{1}_{\Omega_n^{I_T}} \, e^{-\mathfrak{a} \sum_{i=1}^n \mathfrak{b}^{i-1} \mathcal{A}_{I_T}(X_i(\cdot))} \right] \mathrm{d}\underline{x} \mathrm{d}\underline{y} \,.$$

With these definitions, Caputo, Ioffe and Wachtel [2,3] showed that $\mu_{n,T}^{\mathfrak{o}}$ converges to a limit $\mu_\infty^{\mathfrak{o}}$ as $n, T \to \infty$ (moreover, they proved that for any fixed $n$, the measures $\mu_{n,T}^{\mathfrak{o}}$ converge as $T \to \infty$ to a limit $\mu_n^{\mathfrak{o}}$, which then converges to $\mu_\infty^{\mathfrak{o}}$ as $n \to \infty$), and that for any $c > 0$, the family of distributions $\{\mu_{n,T}^{\mathfrak{f}}\}_{n \geq 1, T > c}$ is tight. In this and subsequent statements, the measure $\mu_\infty^{\mathfrak{o}}$ is defined on $\mathsf{C}(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$, and the convergence is in the sense that for any compact set $\mathcal{K} \subset \mathbb{R}$, integer $k \in \mathbb{N}$ and fixed function $f : \mathsf{C}(\mathcal{K}; \mathbb{R}^k) \mapsto \mathbb{R}$, we have that

$$\lim_{n,T \to \infty} \int f(X_1, \ldots, X_k) \mu_{n,T}^{\mathfrak{o}}(\mathrm{d}X) = \int f(X_1, \ldots, X_k) \mu_\infty^{\mathfrak{o}}(\mathrm{d}X) \,. \tag{1.4}$$

Caputo, Ioffe and Wachtel conjectured that $\mu_{n,T}^{\mathfrak{f}}$ converges as well, and to the same limit $\mu_\infty^{\mathfrak{o}}$ as $n, T \to \infty$. Our main result confirms this when $T \to \infty$ followed by $n \to \infty$.

**Theorem 1.1.** *For any fixed tilt parameters $\mathfrak{a} > 0$ and $\mathfrak{b} > 1$ and any fixed integer $n$, the measures $\mu_{n,T}^{\mathfrak{f}}$ and $\mu_{n,T}^{\mathfrak{o}}$ have the same weak limit as $T \to \infty$. In particular, if we denote by $\mu_\infty^{\mathfrak{o}}$ the limit of $\mu_{n,T}^{\mathfrak{o}}$ as $n, T \to \infty$, then*

$$\exists \lim_{n \to \infty} \lim_{T \to \infty} \mu_{n,T}^{\mathfrak{f}} = \mu_\infty^{\mathfrak{o}} \,.$$

**Remark 1.2.** *Our proof shows that, not only does the limit as $T \to \infty$ of $\mu_{n,T}^{\mathfrak{f}}$ coincide with that of $\mu_{n,T}^{\mathfrak{o}}$, but (a) the same holds if one replaces the endpoints $\underline{x} = \underline{y} = \underline{0}^+$ by any fixed points $\underline{x}, \underline{y}$ in $\mathbb{A}_n^+$; and (b) one may also replace the average over $\underline{x}, \underline{y}$ w.r.t. Lebesgue yielding $\mu_{n,T}^{\mathfrak{f}}$ by any measure stochastically dominated by Lebesgue.*

Our proof of Theorem 1.1 employs the Markovian structure of the problem. In a first step we introduce a (sub-)Markovian Kernel $K_t$, see (2.1). The key part of the proof is Lemma 2.1, where we prove that $K_1$ is compact in the appropriate $L^2$ space; the proof of the lemma involves probabilistic arguments. With the lemma, standard contraction arguments, detailed in Section 2.1, yield the exponential decay (in $T$) of the dependence in the boundary conditions. We note that some care is needed here due to the non-compactness of the set of possible boundary conditions, but that non-compactness was already handled in [2].

Many interesting open questions remain, chief among which, perhaps, is describing the precise limit of the limiting process $X_\infty(\cdot)$ (on, say, the interval $[0,1]$). We refer to [2,3] for a list of such problems.

## 2. Proof of main result

Fix the tilt parameters $\mathfrak{a} > 0$ and $\mathfrak{b} > 1$, and let $n \geq 1$ be an integer. Throughout this proof, for $X \in \Omega_n^I$, we use the abbreviated notation

$$\mathcal{A}_I(X(\cdot)) := \mathfrak{a} \sum_{i=1}^{n} \mathfrak{b}^{i-1} \mathcal{A}_I(X_i(\cdot))\,.$$

Let $\Gamma \in \mathcal{B}_{n,[0,1]}$, and define

$$K_1^\Gamma(\underline{x}, \underline{y}) = \mathbf{E}_n^{\underline{x},\underline{y},[0,1]} \left[ \mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^{[0,1]}} \, e^{-\mathcal{A}_{[0,1]}(X(\cdot))} \right] \,,$$

which we view as a linear operator on $L^2(\mathbb{A}_n^+) = L^2(\mathbb{A}_n^+, \mathrm{Leb})$:

$$(K_1^\Gamma f)(\underline{x}) = \int_{\mathbb{A}_n^+} K_1^\Gamma(\underline{x}, \underline{y}) f(\underline{y}) \, \mathrm{d}\underline{y}\,.$$

With a slight abuse of notation, we continue to write $K_1^\Gamma$ also when $\Gamma \in \mathcal{B}_{n,\mathbb{R}}$, in which case we understand that $\Gamma$ was replaced by its restriction to the interval $[0,1]$. With this convention in mind, we will further be interested in the semigroup

$$K_t^\Gamma(\underline{x}, \underline{y}) = \mathbf{E}_n^{\underline{x},\underline{y},[0,t]} \left[ \mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^{[0,t]}} \, e^{-\mathcal{A}_{[0,t]}(X(\cdot))} \right] \,. \tag{2.1}$$

When referring to the case $\Gamma = \Omega_n^{\mathbb{R}}$ (i.e., the indicator $\mathbb{1}_\Gamma$ within the expectation in the definition of $K_1^\Gamma$ is omitted), we simply write $K_t$ (with no superscript) in lieu of $K_t^{\Omega_n^{\mathbb{R}}}$, noting that $K_t(\underline{x}, \underline{y})$ is precisely the partition function $Z_n^{\underline{x},\underline{y},[0,t]}$ from (1.1).

Observe that $K_1$ is symmetric, in that $K_1(\underline{x}, \underline{y}) = K_1(\underline{y}, \underline{x})$, as well as positivity preserving:

$$(K_1 f)(\underline{x}) = \int K_1(\underline{x}, \underline{y}) f(\underline{y}) \mathrm{d}\underline{y} \geq 0 \qquad \text{whenever} \qquad f \geq 0\,.$$

As $K_1$ is symmetric, and given by a continuous Markov process with killing, it is positive definite (this follows, e.g., by [9, Theorems 1.3.1, Lemma 1.3.2 and Theorem 6.1.1], all applied to Example 1.2.3 there). A key ingredient in the proof will be that it is furthermore relatively compact:

**Lemma 2.1.** *The symmetric positive definite operator $K_1$ given by*

$$(K_1 f)(\underline{x}) = \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},[0,1]} \left[ \mathbb{1}_{\Omega_n^{[0,1]}} e^{-\mathcal{A}_{[0,1]}(X(\cdot))} \right] f(\underline{y})\, \mathrm{d}\underline{y}$$

*is compact w.r.t. $L^2(\mathbb{A}_n^+)$.*

2.1. **Proof of Theorem 1.1 modulo Lemma 2.1.** We consider throughout the convergence over the interval $[0,1]$, the changes needed for considering other compact sets (as the set $\mathcal{K}$ in (1.4)) are minimal. Expressing the measures $\mu_{n,T}^{\mathfrak{o}}$ and $\mu_{n,T}^{\mathfrak{f}}$ in terms of the operator $K_t$, we wish to show that for every $\Gamma \in \mathcal{B}_n$, the limit of

$$\mu_{n,T}^{\mathfrak{f}}(\Gamma) = \frac{\iint K_T(\underline{x},\underline{u}) K_1^{\Gamma}(\underline{u},\underline{v}) K_{T-1}(\underline{y},\underline{v})\, \mathrm{d}\underline{u}\mathrm{d}\underline{v}\, \mathrm{d}\underline{x}\mathrm{d}\underline{y}}{\iint K_{2T}(\underline{x},\underline{y})\, \mathrm{d}\underline{x}\mathrm{d}\underline{y}} \tag{2.2}$$

as $T \to \infty$ exists and coincides with that of

$$\mu_{n,T}^{\mathfrak{o}}(\Gamma) = \lim_{\varepsilon \downarrow 0} \frac{\iint K_T(\varepsilon\underline{x},\underline{u}) K_1^{\Gamma}(\underline{u},\underline{v}) K_{T-1}(\varepsilon\underline{y},\underline{v})\, \mathrm{d}\underline{u}\mathrm{d}\underline{v}}{K_{2T}(\varepsilon\underline{x},\varepsilon\underline{y})}\,, \tag{2.3}$$

where the limit exists by [3, Lemma 2.2] and is independent of the choice of $\underline{x},\underline{y}$ in $\mathbb{A}_n^+$.

By the spectral decomposition theorem for the compact positive definite operator $K_1$ (see, e.g., [18, Thms. VI.15 and VI.16] and [19, Thm. XIII.43]), we can write it as

$$K_1(\underline{x},\underline{y}) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(\underline{x})\varphi_i(\underline{y}) \quad \text{for a complete basis } \{\varphi_i\}_{i\geq 1} \text{ with } \langle \varphi_i, \varphi_j \rangle_{L^2(\mathbb{A}_n^+)} = \delta_{ij}\,,$$

such that

$$\varphi_1 > 0 \qquad \text{and} \qquad \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq 0\,.$$

Define

$$\psi(\underline{u}) = \int K_1(\underline{u},\underline{x})\, \mathrm{d}\underline{x}\,,$$

noting that $\psi(\underline{u})$ is well-defined (albeit a-priori possibly infinite) as $K_1$ is non-negative. For any $j \geq 1$ and $\underline{x},\underline{y} \in \mathbb{A}_n^+$,

$$\begin{aligned} K_j(\underline{x},\underline{y}) = \mathbf{E}_n^{\underline{x},\underline{y},[0,j]} \left[ \mathbb{1}_{\Omega_n^{[0,j]}} e^{-\mathcal{A}_{[0,j]}(X(\cdot))} \right] &\leq \mathbf{E}_n^{\underline{x},\underline{y},[0,j]} \left[ \mathbb{1}_{\Omega_n^{[0,j]}} e^{-\mathfrak{a}\mathcal{A}_{[0,j]}(X_1(\cdot))} \right] \\ &\leq \mathbf{E}_1^{x_1,y_1,[0,j]} \left[ \mathbb{1}_{\Omega_1^{[0,j]}} e^{-\mathfrak{a}\mathcal{A}_{[0,j]}(X_1(\cdot))} \right] \\ &\leq \phi_j(x_1 - y_1) e^{-\mathfrak{a} j \frac{x_1+y_1}{2}} \mathbb{E}\Big[ e^{-\mathfrak{a}\int_0^j B_s\, \mathrm{d}s} \Big] \\ &= C_j e^{-\frac{(x_1-y_1)^2}{2j}} e^{-\mathfrak{a} j \frac{x_1+y_1}{2}}\,, \end{aligned} \tag{2.4}$$

where $C_j$ are finite constants and $B_s, s \in [0,j]$ denotes the standard Brownian bridge over $[0,j]$ starting and ending at 0 (using in the second line also that the total mass of

$\mathbf{E}_1^{x_i,y_i,[0,j]}$, $i \geq 2$, is at most one). Since $K_j(\underline{x},\underline{y})$ vanishes if either $\underline{x} \notin \mathbb{A}_n^+$ or $\underline{y} \notin \mathbb{A}_n^+$, it follows that

$$\iint K_j(\underline{x},\underline{y})\,\mathrm{d}\underline{x}\mathrm{d}\underline{y} \leq C_j \int_0^\infty \int_0^\infty x_1^{n-1} y_1^{n-1} e^{-\frac{(x_1-y_1)^2}{2j}} e^{-\mathfrak{a}j\frac{x_1+y_1}{2}}\,\mathrm{d}x_1\mathrm{d}y_1 < \infty\,, \tag{2.5}$$

and

$$\int K_j(\underline{x},\underline{x})\,\mathrm{d}\underline{x} \leq C_j \int_0^\infty x_1^{n-1} e^{-\mathfrak{a}jx_1}\,\mathrm{d}x_1 < \infty\,. \tag{2.6}$$

In particular, we have from (2.5) that

$$\iint K_2(\underline{x},\underline{y})\,\mathrm{d}\underline{x}\mathrm{d}\underline{y} < \infty\,, \tag{2.7}$$

implying that $\psi \in L^2(\mathbb{A}_n^+)$ since, by the symmetry of $K_1$ and the semigroup property,

$$\int \psi(\underline{u})^2\,\mathrm{d}\underline{u} = \iint K_1(\underline{x},\underline{u})K_1(\underline{u},\underline{y})\,\mathrm{d}\underline{x}\mathrm{d}\underline{y}\mathrm{d}\underline{u} = \iint K_2(\underline{x},\underline{y})\,\mathrm{d}\underline{x}\mathrm{d}\underline{y}\,.$$

This allows us to decompose

$$\psi = \sum_{i=1}^\infty \alpha_i\varphi_i \qquad \text{where} \qquad \alpha_i := \langle \psi, \varphi_i\rangle_{L^2(\mathbb{A}_n^+)}\,, \tag{2.8}$$

so that $\|\psi\|_2^2 = \sum_i \alpha_i^2 < \infty$.

For $t \geq 2$, by the definition of $\psi$, the symmetry of $K_{t-1}$ and the semigroup property,

$$\int K_{t-1}(\underline{x},\underline{u})\,\mathrm{d}\underline{x} = (K_{t-2}\psi)(\underline{u}) = \int K_{t-2}(\underline{u},\underline{y})\psi(\underline{y})\,\mathrm{d}\underline{y} = \sum_{i=1}^\infty \lambda_i^{t-2}\alpha_i\varphi_i(\underline{u})\,,$$

where the last step used the decomposition (2.8). Similarly, for $t \geq 3$,

$$\begin{aligned}\iint K_t(\underline{x},\underline{y})\,\mathrm{d}\underline{x}\mathrm{d}\underline{y} &= \iint K_{t-1}(\underline{x},\underline{u})\psi(\underline{u})\,\mathrm{d}\underline{x}\mathrm{d}\underline{u}\\ &= \iint K_{t-2}(\underline{u},\underline{v})\psi(\underline{u})\psi(\underline{v})\,\mathrm{d}\underline{u}\mathrm{d}\underline{v} = \sum_{i=1}^\infty \lambda_i^{t-2}\alpha_i^2 := c_t\,.\end{aligned}$$

Hence, (2.2) translates into

$$\mu^{\mathfrak{f}}_{n,T}(\Gamma) = \frac{1}{c_{2T}} \iint \sum_{i=1}^\infty\sum_{j=1}^\infty \alpha_i\alpha_j\lambda_i^{T-1}\lambda_j^{T-2}\varphi_i(\underline{u})\,K_1^\Gamma(\underline{u},\underline{v})\,\varphi_j(\underline{v})\,\mathrm{d}\underline{u}\mathrm{d}\underline{v}\,. \tag{2.9}$$

Looking at $K_1^\Gamma$ and arguing as we did for $K_1$, we see that for any $\underline{u} \in \mathbb{R}^n$,

$$\int K_1^\Gamma(\underline{u},\underline{v})^2\,\mathrm{d}\underline{v} \leq \int K_1(\underline{u},\underline{v})^2\,\mathrm{d}\underline{v} = K_2(\underline{u},\underline{u}) < \infty\,,$$

where the equality holds by the symmetry of $K_1$ and the definition of $K_t$ and the last inequality by (2.4). In other words, $K_1^\Gamma(\underline{u},\cdot) \in L^2(\mathbb{A}_n^+)$ for every $\underline{u} \in \mathbb{R}^n$. Moreover, by (2.6) we have that

$$\iint K_1^\Gamma(\underline{u},\underline{v})^2\,\mathrm{d}\underline{u}\mathrm{d}\underline{v} \leq \iint K_1(\underline{u},\underline{v})^2\,\mathrm{d}\underline{u}\mathrm{d}\underline{v} = \int K_2(\underline{u},\underline{u})\mathrm{d}\underline{u} < \infty$$

and it follows that

$$K_1^\Gamma \in L^2(\mathbb{A}_n^+)^{\otimes 2}\,.$$

A complete orthonormal system $\{\varphi_i\}$ w.r.t. $L^2(\mathbb{A}_n^+)$ induces a complete orthonormal system $\{\varphi_i \otimes \varphi_j\}_{i,j\geq 1}$ w.r.t. $L^2(\mathbb{A}_n^+)^{\otimes 2}$; hence, we may decompose $K_1^\Gamma$ into

$$K_1^\Gamma(\underline{u},\underline{v}) = \sum_{i,j} \gamma_{i,j}\varphi_i(\underline{u})\varphi_j(\underline{v})$$

where

$$\gamma_{i,j} := \iint K_1^\Gamma(\underline{u},\underline{v})\varphi_i(\underline{u})\varphi_j(\underline{v})\,\mathrm{d}\underline{u}\mathrm{d}\underline{v}\,, \qquad \sum_{i,j\geq 1}\gamma_{i,j}^2 = \|K_1^\Gamma\|^2_{L^2(\mathbb{A}_n^+)^{\otimes 2}} < \infty\,.$$

This reduces (2.9) into $\mu^{\mathfrak{f}}_{n,T}(\Gamma) = \Xi^{(1)}_{n,T}/\Xi^{(2)}_{n,T}$ where

$$\Xi^{(1)}_{n,T} := \sum_{i,j\geq 1}\gamma_{i,j}\alpha_i\alpha_j\widehat{\lambda}_i^{T-1}\widehat{\lambda}_j^{T-2}\,, \qquad \Xi^{(2)}_{n,T} := \lambda_1\sum_{i=1}^\infty \widehat{\lambda}_i^{2T-2}\alpha_i^2\,, \tag{2.10}$$

and the rescaled eigenvalues $\widehat{\lambda}_i := \lambda_i/\lambda_1 \in [0,1]$ $(i=1,2,\ldots)$ satisfy

$$\widehat{\lambda}_i = 1 \qquad\text{and}\qquad \sup_{i>1}\widehat{\lambda}_i \leq 1-\delta \quad \text{for } \delta = (\lambda_1-\lambda_2)/\lambda_1 > 0\,.$$

We immediately see that $\Xi^{(2)}_{n,T}$ of (2.10) satisfies

$$\lambda_1\alpha_1^2 \leq \Xi^{(2)}_{n,T} \leq \lambda_1\alpha_1^2 + \lambda_1(1-\delta)^{2T-2}\|\psi\|_2^2\,, \tag{2.11}$$

whence

$$\lim_{T\to\infty}\Xi^{(2)}_{n,T} = \lambda_1\alpha_1^2\,.$$

To treat $\Xi^{(1)}_{n,T}$ of (2.10), note that by Cauchy–Schwarz and having $\sup_{i\geq 2}|\widehat{\lambda}_i| \leq 1-\delta$,

$$\begin{aligned}\Bigg|\sum_{\substack{i,j\geq 1\\ i+j>2}}\gamma_{i,j}\alpha_i\alpha_j\widehat{\lambda}_i^{T-1}\widehat{\lambda}_j^{T-2}\Bigg| &\leq (1-\delta)^{T-2}\sum_{i,j}|\gamma_{i,j}\alpha_i\alpha_j| \\ &\leq (1-\delta)^{T-2}\sqrt{\sum_{i,j}\gamma_{i,j}^2}\sqrt{\sum_{i,j}\alpha_i^2\alpha_j^2} \\ &= (1-\delta)^{T-2}\|K_1^\Gamma\|_{L^2(\mathbb{A}_n^+)^{\otimes 2}}\|\psi\|_2^2\,. \end{aligned}\tag{2.12}$$

Taking $T\to\infty$, we see that

$$\lim_{T\to\infty}\Xi^{(1)}_{n,T} = \gamma_{1,1}\alpha_1^2\,.$$

Altogether, we have established that

$$\lim_{T\to\infty}\mu^{\mathfrak{f}}_{n,T}(\Gamma) = \frac{\gamma_{1,1}\alpha_1^2}{\lambda_1\alpha_1^2} = \frac{\gamma_{1,1}}{\lambda_1}\,, \tag{2.13}$$

with the last equality using that $\alpha_1 \neq 0$ since $\varphi_1 > 0$ and so

$$\alpha_1 = \langle \psi, \varphi_1 \rangle_{L^2(\mathbb{A}_n^+)} = \iint K_1(\underline{x}, \underline{u}) \varphi_1(\underline{u}) \, \mathrm{d}\underline{u} \mathrm{d}\underline{x} > 0 \,.$$

We now repeat the same analysis for $\mu^{\circ}_{n,T}$, where for simplicity we opt to take $\underline{y} = \underline{x}$ and let $\psi^{(\varepsilon)}(\underline{u}) := K_1(\varepsilon\underline{x}, \underline{u})$. Inferring that $\psi^{(\varepsilon)} \in L^2(\mathbb{A}_n^+)$ (because $K_2(\varepsilon\underline{x}, \varepsilon\underline{x}) < \infty$), we can write

$$\psi^{(\varepsilon)} = \sum_{i=1}^{\infty} \alpha_i^{(\varepsilon)} \varphi_i$$

where

$$\alpha_i^{(\varepsilon)} := \big\langle \psi^{(\varepsilon)}, \varphi_i \big\rangle_{L^2(\mathbb{A}_n^+)} \,, \qquad \|\psi^{(\varepsilon)}\|^2_{L^2(\mathbb{A}_n^+)} = K_2(\varepsilon\underline{x}, \varepsilon\underline{x}) = \sum_{i=1}^{\infty} (\alpha_i^{(\varepsilon)})^2 < \infty \,.$$

The exact same argument then shows that $\mu^{\circ}_{n,T}(\Gamma)$ is the limit at $\varepsilon \to 0$ of $\Xi^{(1,\varepsilon)}_{n,T} / \Xi^{(2,\varepsilon)}_{n,T}$ where

$$\Xi^{(1,\varepsilon)}_{n,T} := \sum_{i,j \geq 1} \gamma_{i,j} \alpha_i^{(\varepsilon)} \alpha_j^{(\varepsilon)} \widehat{\lambda}_i^{T-1} \widehat{\lambda}_j^{T-2} \,, \qquad \Xi^{(2,\varepsilon)}_{n,T} := \lambda_1 \sum_{i=1}^{\infty} \widehat{\lambda}_i^{2T-2} (\alpha_i^{(\varepsilon)})^2 \,. \tag{2.14}$$

With $\psi^{(\varepsilon)} > 0$ and $\varphi_1 > 0$, we have as before that $\alpha_1^{(\varepsilon)} > 0$. Moreover, setting

$$\kappa_\varepsilon := \frac{\|\psi^{(\varepsilon)}\|^2_{L^2(\mathbb{A}_n^+)}}{(\alpha_1^{(\varepsilon)})^2} \,,$$

we have analogously to (2.11) and (2.12) that

$$0 \leq \frac{\Xi^{(2,\varepsilon)}_{n,T}}{(\alpha_1^{(\varepsilon)})^2} - \lambda_1 \leq \lambda_1 (1-\delta)^{2T-2} \kappa_\varepsilon \,,$$
$$\Big| \frac{\Xi^{(1,\varepsilon)}_{n,T}}{(\alpha_1^{(\varepsilon)})^2} - \gamma_{1,1} \Big| \leq (1-\delta)^{T-2} \|K_1^\Gamma\|_{L^2(\mathbb{A}_n^+)^{\otimes 2}} \, \kappa_\varepsilon \,.$$

We shall employ the following asymptotic as $\varepsilon \to 0$, the proof of which we defer to Section 2.3.

**Lemma 2.2.** *Setting* $\underline{n} := (2n-1, 2n-3, \ldots, 1)$, *we have that*

$$\limsup_{\varepsilon \to 0} \frac{K_2(\varepsilon\underline{n}, \varepsilon\underline{n})}{\Big( \int_{u_1 \leq 1} K_1(\varepsilon\underline{n}, \underline{u}) \varphi_1(\underline{u}) d\underline{u} \Big)^2} < \infty \,. \tag{2.15}$$

In view of Lemma 2.2, the fact that $K_1$ and $\varphi_1$ are both positive, and our freedom to choose $\underline{x} = \underline{n}$, we have that $\kappa_\varepsilon$ is uniformly bounded (as $\varepsilon \to 0$). Hence,

$$\lim_{T \to \infty} \mu^{\circ}_{n,T}(\Gamma) = \frac{\gamma_{1,1}}{\lambda_1} \,, \tag{2.16}$$

which in light of (2.13) concludes our proof. ■

2.2. **Proof of Lemma 2.1.** Letting

$$\mathfrak{B}_0 = \left\{ f \,:\, \|f\|_{L^2(\mathbb{A}_n^+)} \le 1 \right\} \qquad \text{and}$$
$$\mathfrak{B}_1 = \left\{ (K_1 f) \,:\, f \in \mathfrak{B}_0 \right\},$$

we will establish compactness by verifying the Fréchet–Kolmogorov criteria (see [21, p. 275], as well as [20]).

First, with $\mathbb{P}$ denoting the law of Brownian motion $\{W(t)\}_{t\in[0,1]}$ in $\mathbb{R}^n$ started at the origin and $\mathbb{E}$ its corresponding expectation, note that

$$(K_1 f)(\underline{x}) = \mathbb{E}\left[ \mathbb{1}_{\Omega_n^{[0,1]}}(\underline{x} + W(\cdot))\, e^{-\mathcal{A}_{[0,1]}(\underline{x}+W(\cdot))} f(\underline{x} + W(1)) \right] . \tag{2.17}$$

Now, setting for $f$ supported on $\mathbb{A}_n^+$,

$$M(f) := \sup_{\underline{x}\in\mathbb{A}_n^+} \mathbb{E}[|f(\underline{x} + W(1))|] , \tag{2.18}$$

note that by Cauchy–Schwarz,

$$M(f)^2 \le \sup_{\underline{x}\in\mathbb{A}_n^+} \mathbb{E}[f(\underline{x} + W(1))^2] \le \|f\|^2_{L^2(\mathbb{A}_n^+)} \sup_{\underline{x},\underline{y}\in\mathbb{A}_n^+} \{\phi_1(\underline{y} - \underline{x})\} \le 1 , \tag{2.19}$$

where $\phi_v(\cdot)$ denotes the density in (1.2) and the last inequality holds for all $f \in \mathfrak{B}_0$.

This readily implies the following uniform bound on $g = K_1 f \in \mathfrak{B}_1$, where by a computation similar to the third line of (2.4), for any $\underline{x} \in \mathbb{A}_n^+$,

$$|g(\underline{x})| \le \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},[0,1]}\left[ e^{-\mathfrak{a}\int_0^1 X_1(s)ds} \right] |f(\underline{y})|\, \mathrm{d}\underline{y} \le c e^{-\frac{\mathfrak{a}}{2}x_1} M(f) \le c e^{-\frac{\mathfrak{a}}{2}x_1} , \tag{2.20}$$

for some finite $c = c(\mathfrak{a})$, independent of $\underline{x}$ and $f \in \mathfrak{B}_0$. We deduce in particular that

$$\limsup_{R\to\infty} \sup_{g\in\mathfrak{B}_1} \int_{\substack{\underline{x}\in\mathbb{A}_n^+ \\ x_1 > R}} |g(\underline{x})|^2 \, d\underline{x} = 0 , \tag{2.21}$$

establishing equitightness (and, due to (2.20), also uniform boundedness, although it is not needed in view of [20]).

It remains to establish equicontinuity for $\mathfrak{B}_1$, where in view of (2.21) and the compactness of $\bar{\mathbb{A}}_n^+ \cap \{x_1 \le R\}$ it suffices to bound, in terms of $\|\underline{h}\|$, the value of

$$\sup_{g\in\mathfrak{B}_1, \underline{x}\in\mathbb{A}_n^+} \{|g(\underline{x} + \underline{h}) - g(\underline{x})|\} .$$

Using the representation (2.17) for $g = K_1 f$, we start by reducing to $\tilde{g}(\cdot)$ in which we extracted out the explicit dependence of the area tilt on $\underline{x}$. Specifically, let

$$\tilde{g}(\underline{x}) := \mathbb{E}\left[ \mathbb{1}_{\Omega_n^{[0,1]}}(\underline{x} + W(\cdot))\, e^{-\mathcal{A}_{[0,1]}(W(\cdot))} f(\underline{x} + W(1)) \right] .$$

By a slight abuse of notation, letting $\mathcal{A}_{[0,1]}(\underline{x})$ denote $\mathcal{A}_{[0,1]}(X(\cdot))$ for $X \equiv \underline{x}$, which is nothing but $\mathfrak{a}\,\langle \underline{\mathfrak{b}}, \underline{x}\rangle$ for $\underline{\mathfrak{b}} := (1, \mathfrak{b}, \ldots, \mathfrak{b}^{n-1})$, we see that

$$g(\underline{x}) = e^{-\mathcal{A}_{[0,1]}(\underline{x})}\, \tilde{g}(\underline{x}) ,$$

and therefore,

$$|g(\underline{x}) - g(\underline{x}+\underline{h})| = |e^{-\mathcal{A}_{[0,1]}(\underline{x})}\big(\tilde{g}(\underline{x}) - e^{-\mathcal{A}_{[0,1]}(\underline{h})}\tilde{g}(\underline{x}+\underline{h})\big)|$$
$$\leq \big|e^{-\mathcal{A}_{[0,1]}(\underline{h})} - 1\big|\,|g(\underline{x}+\underline{h})| + e^{-\mathcal{A}_{[0,1]}(\underline{x})}|\tilde{g}(\underline{x}) - \tilde{g}(\underline{x}+\underline{h})|\,.$$

For the first term note that $|\mathcal{A}_{[0,1]}(\underline{h})| = |\mathfrak{a}\,\langle \underline{\mathfrak{b}}, \underline{h}\rangle\,| \leq \mathfrak{a}\|\underline{\mathfrak{b}}\|_2\|\underline{h}\|_2$ and though $\underline{h}$ may be outside $\mathbb{A}_n^+$, we have $\underline{x}+\underline{h} \in \mathbb{A}_n^+$, yielding by Taylor expansion and (2.20) that

$$\sup_{g\in\mathfrak{B}_1, \underline{x}\in\mathbb{A}_n^+} \big|e^{-\mathcal{A}_{[0,1]}(\underline{h})} - 1\big|\,|g(\underline{x}+\underline{h})| \leq C(\mathfrak{a},\mathfrak{b},n)\|\underline{h}\|_2\,.$$

Further, with $\mathcal{A}_{[0,1]}(\underline{x}) \geq 0$ for all $\underline{x} \in \mathbb{A}_n^+$, it remains only to bound $|\tilde{g}(\underline{x}) - \tilde{g}(\underline{y})|$ uniformly over $g \in \mathfrak{B}_1$, $\underline{x} \in \mathbb{A}_n^+$ and $\underline{y} \in \mathbb{A}_n^+$ such that $\|\underline{y} - \underline{x}\| \leq \delta$. To this end, let

$$\tau_{\underline{x}} := \inf\big\{t \geq 0 \,:\, \underline{x} + W(t) \notin \mathbb{A}_n^+\big\}\,, \quad \text{so that} \quad \mathbb{1}_{\Omega_n^{[0,1]}}(\underline{x}+W(\cdot)) = \mathbb{1}_{\{\tau_{\underline{x}}>1\}}\,.$$

We then have in terms of

$$\Delta(\underline{x},\underline{y}) := \mathbb{1}_{\{\tau_{\underline{y}}>1\}} f(W(1)+\underline{y}) - \mathbb{1}_{\{\tau_{\underline{x}}>1\}} f(W(1)+\underline{x})$$

and $\eta \in (0,1)$, that

$$|\tilde{g}(\underline{y}) - \tilde{g}(\underline{x})| = |\mathbb{E}\big[e^{-\mathcal{A}_{[0,1]}(W(\cdot))}\Delta(\underline{x},\underline{y})\big]| \leq \mathbb{E}\,[|\Psi_1|] + |\mathbb{E}\,[\Psi_2]\,|\,,$$

where

$$\Psi_1 := e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\Big(e^{-\mathcal{A}_{[1-\eta,1]}(W(\cdot)-W(1-\eta))} - 1\Big)\,\Delta(\underline{x},\underline{y})\,,$$
$$\Psi_2 := e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\Delta(\underline{x},\underline{y})\,,$$

and

$$\mathcal{A}^*_{[0,1-\eta]}(W(\cdot)) = \mathcal{A}_{[0,1-\eta]}(W(\cdot)) + \mathcal{A}_{[1-\eta,1]}(W(1-\eta))$$

To bound $\mathbb{E}|\Psi_1|$, use the fact that $|\Delta(\underline{x},\underline{y})| \leq |f(W(1)+\underline{y})| + |f(W(1)+\underline{x})|$ together with Hölder's inequality to infer that $\mathbb{E}|\Psi_1|$ is at most

$$\mathbb{E}\Big[e^{-4\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\Big]^{\frac{1}{4}}\,\mathbb{E}\Big[\Big|e^{-\mathcal{A}_{[1-\eta,1]}(W(\cdot)-W(1-\eta))} - 1\Big|^4\Big]^{\frac{1}{4}}\Big(2\sup_{\underline{x}\in\mathbb{A}_n^+}\mathbb{E}\,\big[f(W(1)+\underline{x})^2\big]\Big)^{\frac{1}{2}}.$$

Noting that the variance of the centered Gaussian $\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))$ is at most some $v = v(\mathfrak{a},\mathfrak{b},n)$ finite, the first expectation above is uniformly bounded (namely, by $e^{8v}$). Similarly, by (2.19), the third term is at most $\sqrt{2}$, uniformly over $f \in \mathfrak{B}_0$. Finally, with $\mathcal{A}_{[1-\eta,1]}(W(\cdot) - W(1-\eta))$ a centered Gaussian of variance $c(\mathfrak{a},\mathfrak{b},n)\eta^2$ for some finite $c(\mathfrak{a},\mathfrak{b},n)$, the expectation in the second term is at most $\varepsilon_0(\eta) \to 0$ as $\eta \to 0$. Overall, we conclude that

$$\mathbb{E}|\Psi_1| \leq \varepsilon_1(\eta) \downarrow 0 \quad \text{as} \quad \eta \downarrow 0\,, \quad \text{uniformly over} \quad g \in \mathfrak{B}_1, \underline{x} \in \mathbb{A}_n^+\,. \tag{2.22}$$

Turning to $\Psi_2 = e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\Delta(\underline{x},\underline{y})$, the identity

$$\mathbb{1}_{\{\tau_{\underline{y}}>1\}} = 1 - \mathbb{1}_{\{\tau_{\underline{y}}\leq 1-\eta\,,\,\tau_{\underline{x}}\leq 1-\eta\}} - \mathbb{1}_{\{1-\eta<\tau_{\underline{y}}\leq 1\}} - \mathbb{1}_{\{\tau_{\underline{y}}\leq 1-\eta\,,\,1-\eta<\tau_{\underline{x}}\leq 1\}} - \mathbb{1}_{\{\tau_{\underline{y}}\leq 1-\eta\,,\,\tau_{\underline{x}}>1\}}\,,$$

yields the decomposition

$$\Delta(\underline{x},\underline{y}) = \Upsilon_1 - \Upsilon_2(\underline{y}) - \Upsilon_3(\underline{y},\underline{x}) - \Upsilon_4(\underline{y},\underline{x}) + \Upsilon_2(\underline{x}) + \Upsilon_3(\underline{x},\underline{y}) + \Upsilon_4(\underline{x},\underline{y}),$$

where

$$\begin{aligned}
\Upsilon_1 &:= [f(W(1)+\underline{y}) - f(W(1)+\underline{x})](1 - \mathbb{1}_{\{\tau_{\underline{x}} \le 1-\eta\,,\,\tau_{\underline{y}} \le 1-\eta\}}),\\
\Upsilon_2(\underline{y}) &:= f(W(1)+\underline{y})\mathbb{1}_{\{1-\eta<\tau_{\underline{y}}\le 1\}},\\
\Upsilon_3(\underline{y},\underline{x}) &:= f(W(1)+\underline{y})\mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta\,,\,1-\eta<\tau_{\underline{x}}\le 1\}},\\
\Upsilon_4(\underline{y},\underline{x}) &:= f(W(1)+\underline{y})\mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta\,,\,\tau_{\underline{x}}>1\}}.
\end{aligned}$$

For the contribution to $|\mathbb{E}[\Psi_2]|$ due to $\Upsilon_1$, condition on $\mathcal{F}_{1-\eta} = \sigma(\{W(s)\}_{s\le 1-\eta})$, on which the indicator in $\Upsilon_1$ is measurable, to get

$$\begin{aligned}
\Big|\mathbb{E}\big[\Upsilon_1 e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\big]\Big| \le \mathbb{E}\left[e^{-\mathcal{A}^*_{[0,1-\eta]}W(\cdot))}\right]&\\
\cdot \sup_{\underline{z}} \Big|\mathbb{E}\big[f(W(1)+\underline{y}) - f(W(1)+\underline{x}) \;\big|\; W(1-\eta) = \underline{z}\big]\Big|&.
\end{aligned}$$

While treating $\mathbb{E}|\Psi_1|$, we saw that the first term on the right-hand is some finite $C(\mathfrak{a},\mathfrak{b},n)$, independently of $\underline{x},\underline{h}$. For the second term, extending $f \in \mathfrak{B}_0$ from $\mathbb{A}_n^+$ to $\mathbb{R}^n$ via $f(\underline{x}) = 0$ for $\underline{x} \notin \mathbb{A}_n^+$, yields that $\|f\|_2 = \|f\|_{L^2(\mathbb{A}_n^+)} \le 1$. Thus, performing a change of variable $\underline{v} := W(1)+\underline{y}$ in $\mathbb{E}[f(W(1)+\underline{y}) \mid W(1-\eta) = \underline{z}]$ and $\underline{v} := W(1)+\underline{x}$ in $\mathbb{E}[f(W(1)+\underline{x}) \mid W(1-\eta) = \underline{z}]$, we get that the absolute difference between these expectations is

$$\begin{aligned}
&\Big|\int \big[\phi_\eta(\underline{v}-\underline{y}-\underline{z}) - \phi_\eta(\underline{v}-\underline{x}-\underline{z})\big] f(\underline{v})\,\mathrm{d}\underline{v}\Big|\\
&\le \|f\|_{L^2(\mathbb{A}_n^+)}\eta^{-n/4}\big\|\phi_1(\underline{w}-\eta^{-1/2}(\underline{y}-\underline{x})) - \phi_1(\underline{w})\big\|_2 \le C(n)\,\eta^{-n/4-1/2}\delta\,,
\end{aligned}$$

where the first inequality is obtained by Cauchy–Schwarz and an additional change of variable $\underline{w} = \eta^{-1/2}(\underline{v}-\underline{z}-\underline{x})$, and the second inequality by an easy computation (utilizing that $1-e^{-r} \le r$). Thus, choosing

$$\delta \le \eta^{n/4+1}, \tag{2.23}$$

makes the contribution of $\Upsilon_1$ negligible.

To deal with the contribution of $\Upsilon_2(\underline{y})$ to $|\mathbb{E}[\Psi_2]|$, observe that by Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\left[e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\mathbb{1}_{\{1-\eta<\tau_{\underline{y}}\le 1\}}|f(W(1)+\underline{y})|\right]\\
&\le \mathbb{E}\Big[e^{-4\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\Big]^{\frac14}\mathbb{P}\Big(1-\eta<\tau_{\underline{y}}\le 1\Big)^{\frac14}\Big(\sup_{\underline{y}\in\mathbb{A}_n^+}\mathbb{E}\left[f(W(1)+\underline{y})^2\right]\Big)^{\frac12}.
\end{aligned} \tag{2.24}$$

While bounding $\mathbb{E}|\Psi_1|$ we have seen that the first and third terms are at most some $c(\mathfrak{a},\mathfrak{b},n)$ finite, uniformly over $\mathfrak{B}_0$, so it suffices to show that

$$\varepsilon_2(\eta) := \sup_{\underline{y}\in\mathbb{A}_n^+}\{\mathbb{P}(1-\eta<\tau_{\underline{y}}\le 1)\} \to 0 \qquad \text{as} \quad \eta \to 0\,. \tag{2.25}$$

Indeed, taking a union bound over the $n$ different boundaries of $\mathbb{A}_n^+$ that are considered in $\tau_{\underline{y}}$, reduces, up to the factor $n$, to the bound in case $n=1$, namely for the first hitting time of level $-b<0$ by a standard Brownian motion $B_t$. The density of the latter hitting time is bounded, uniformly over $b$ and $t \ge 1/2$, thereby yielding (2.25).

The same analysis applies to the contributions from $\Upsilon_2(\underline{x})$ and from the $\Upsilon_3$ terms. Analogously to (2.24) the contribution of $\Upsilon_4(\underline{y},\underline{x})$ to $|\mathbb{E}[\Psi_2]|$ is bounded above by

$$\mathbb{E}\left[e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta\,,\,\tau_{\underline{x}}>1\}}|f(W(1)+\underline{y})|\right]$$
$$\le C\,\|f\|_{L^2(\mathbb{A}_n^+)}\mathbb{P}(\tau_{\underline{y}}\le 1-\eta\,,\ \tau_{\underline{x}}>1)^{1/4}\le C\,\varepsilon_3(\delta,\eta)^{1/4}\,,$$

for some $C=C(\mathfrak{a},\mathfrak{b},n)$, any $f\in\mathfrak{B}_0$ and

$$\varepsilon_3(\delta,\eta):=\sup_{\underline{x},\underline{y}\in\mathbb{A}_n^+,\|\underline{x}-\underline{y}\|\le\delta}\mathbb{P}(\tau_{\underline{y}}\le 1-\eta\,,\ \tau_{\underline{x}}>1)\,.$$

With the same bound applying for $\Upsilon_4(\underline{x},\underline{y})$, it remains only to show that $\varepsilon_3(\delta,\eta)\to 0$ as $\delta\to 0$ (for any fixed $\eta>0$). To this end, by a union bound over the $n$ different boundaries of $\mathbb{A}_n^+$, as done for proving (2.25), the probability in question is at most $n$ times the probability that standard Brownian motion $B(t):=\frac{1}{\sqrt{2}}(W_i(t)-W_{i+1}(t))$ reach level $-b$ by time $1-\eta$ (here $b=(y_i-y_{i+1})/\sqrt{2}$), while remaining above $-(b+\delta)$ up till time 1. With Brownian motion a strong Markov process of independent increments, we thus deduce by the reflection principle that

$$n^{-1}\varepsilon_3(\delta,\eta)\le\mathbb{P}(\inf_{s\le\eta}\{B(s)\}>-\delta)=1-2\mathbb{P}(B(\eta)\ge\delta)=\mathbb{P}(|B(\eta)|<\delta)\,,$$

which goes to zero as $\delta\to 0$ (for any fixed $\eta>0$). ■

2.3. **Proof of Lemma 2.2.** Setting $\widehat{K}_t$ for the operator $K_t$ in the case $\mathfrak{a}=0$ (no area tilt), we first establish (2.15) for $\widehat{K}_t$. Namely, we show that,

$$\limsup_{\varepsilon\to 0}\frac{\widehat{K}_2(\varepsilon\underline{n},\varepsilon\underline{n})}{\left(\int_{u_1\le 1}\widehat{K}_1(\varepsilon\underline{n},\underline{u})\varphi_1(\underline{u})d\underline{u}\right)^2}<\infty\,. \tag{2.26}$$

Our starting point for (2.26) is the following explicit formula, valid for any $\underline{y}\in\mathbb{A}_n^+$ and any $t,\varepsilon>0$,

$$\widehat{K}_t(\varepsilon\underline{n},\underline{y})=2^{n^2}\phi_t(\underline{y})e^{-\varepsilon^2\|\underline{n}\|^2/(2t)}\prod_i\sinh\big(\frac{\varepsilon y_i}{t}\big)\prod_{j<k}\Big[\sinh^2\big(\frac{\varepsilon y_j}{t}\big)-\sinh^2\big(\frac{\varepsilon y_k}{t}\big)\Big]\,. \tag{2.27}$$

Indeed, for $\varepsilon=1$ this is the explicit evaluation in [10, Display below (24)] of the Karlin–McGregor determinantal formula [15] for the transition kernel,

$$q_t(x,y)=\phi_t(y-x)-\phi_t(y+x)=2\phi_t(y)e^{-x^2/(2t)}\sinh(xy/t)\,,$$

of a scalar Brownian motion absorbed at level zero, when starting at the distinguished point $\underline{n}$. We thus get (2.27) by noting that the non-trivial factors $\sinh(x_iy_j/t)$ are invariant to changing from $(\varepsilon\underline{n},\underline{y})$ to $(\underline{n},\varepsilon\underline{y})$.

In particular, with $g(x) := \sinh(x/2)$ being zero at $x = 0$ and globally Lipschitz$(L)$ on $[0, 2n]$, we get from (2.27) that for some $c_n$, $C_n$ finite and any $\varepsilon \in [0, 1]$,

$$\widehat{K}_2(\varepsilon\underline{n}, \varepsilon\underline{n}) \le c_n \prod_i g(\varepsilon^2 n_i) \prod_{j<k} [g^2(\varepsilon^2 n_j) - g^2(\varepsilon^2 n_k)] \le c_n L^{n^2} \prod_i (\varepsilon^2 n_i) \prod_{j<k} \left[(\varepsilon^2 n_j)^2 - (\varepsilon^2 n_k)^2\right] = C_n \varepsilon^{2n^2} . \tag{2.28}$$

Next, noting that on $\mathbb{R}_+$ both $\sinh(x) \ge x$ and $\sinh^2(x) - x^2$ are non-decreasing, we deduce from (2.27) that for any $\underline{u} \in \mathbb{A}_n^+$ and $\varepsilon \in [0, 1]$,

$$\widehat{K}_1(\varepsilon\underline{n}, \underline{u}) \ge 2^{n^2} e^{-\|\underline{n}\|^2/2} \varepsilon^{n^2} \hat{\phi}(\underline{u}) , \quad \text{where} \quad \hat{\phi}(\underline{u}) := \phi_1(\underline{u}) \prod_i u_i \prod_{j<k} (u_j^2 - u_k^2) .$$

With $\hat{\phi}(\cdot)$ and $\varphi_1(\cdot)$ positive on $\mathbb{A}_n^+$, we get from the latter bound that

$$\inf_{\varepsilon \in [0,1]} \varepsilon^{-n^2} \int_{u_1 \le 1} \widehat{K}_1(\varepsilon\underline{n}, \underline{u}) \varphi_1(\underline{u}) d\underline{u} > 0 ,$$

which in combination with (2.28) establishes (2.26).

Next, recall that $K_t(\underline{x}, \underline{y})$ is point-wise decreasing in $\mathfrak{a}$ and in particular bounded from above by $\widehat{K}_t(\underline{x}, \underline{y})$; thus, the sought bound (2.15) for $K_t$ follows from (2.26) once we show that for some finite $C = C(\mathfrak{a}, \mathfrak{b}, n)$ and any $\underline{u} \in \mathbb{A}_n^+$ with $u_1 \le 1$,

$$\sup_{\varepsilon \in (0,1]} \left\{ \frac{\widehat{K}_1(\varepsilon\underline{n}, \underline{u})}{K_1(\varepsilon\underline{n}, \underline{u})} \right\} \le C . \tag{2.29}$$

Turning to the latter bound, we define for finite $M$ the event

$$\Gamma_M := \left\{ \max_{t \in [0,1]} \{X_1(t)\} \le M \right\} ,$$

noting that for $c := \mathfrak{a}\langle \underline{\mathfrak{b}}, \underline{1} \rangle$, any $u_1 \le 1$ and $\varepsilon \le 1$,

$$K_1(\varepsilon\underline{n}, \underline{u}) \ge e^{-cM} \mathbf{E}_n^{\varepsilon\underline{n}, \underline{u}, [0,1]} \left[ \mathbb{1}_{\Gamma_M} \mathbb{1}_{\Omega_n^{[0,1]}} \right] = e^{-cM} \widehat{K}_1(\varepsilon\underline{n}, \underline{u}) \widehat{\mathbb{P}}_n^{\varepsilon\underline{n}, \underline{u}, [0,1]}(\Gamma_M) \ge e^{-cM} \widehat{K}_1(\varepsilon\underline{n}, \underline{u}) \widehat{\mathbb{P}}_n^{\underline{n}, \underline{n}, [0,1]}(\Gamma_M) ,$$

where $\widehat{\mathbb{P}}_n^{\underline{x}, \underline{y}, [0,1]}$ is the measure $\mathbb{P}_n^{\underline{x}, \underline{y}, [0,1]}$ from (1.3) corresponding to $\mathfrak{a} = 0$, and with the second inequality due to [7, Lemma 2.7] (taking there $A = [0, 1]$, $f \equiv 0$, noting that $\underline{n} > \underline{u}$ and $\underline{n} > \varepsilon\underline{n}$ whenever $u_1 \le 1$ and $\varepsilon \le 1$ and that the event $\Gamma_M$ is decreasing).

Finally, moving to the unconditional space of $n$ independent bridges rooted at $\underline{n}, \underline{n}$ via a multiplicative cost of at most $1/\widehat{K}_1(\underline{n}, \underline{n})$, we see that $\widehat{\mathbb{P}}_n^{\underline{n}, \underline{n}, [0,1]}(\Gamma_M^c)$ is at most $\mathbb{P}(\sup_{s \in [0,1]} \{B(s)\} > M - 2n)/\widehat{K}_1(\underline{n}, \underline{n})$ for a one dimensional Brownian bridge from $(0, 1)$ to $(1, 1)$. By the tightness of the maximum of the latter bridge (and recalling that $\widehat{K}_1(\underline{n}, \underline{n}) > 0$), one thus has for $M$ large, depending only on $n$, that

$$\widehat{\mathbb{P}}_n^{\underline{n}, \underline{n}, [0,1]}(\Gamma_M) \ge \tfrac{1}{2} .$$

Combining the last two displays yields (2.29), thereby completing the proof. ■

**Acknowledgment.** We thank Ivan Corwin for bringing to our attention the paper [10]. A.D. was supported in part by NSF grant DMS-1954337. E.L. was supported by NSF grants DMS-1812095 and DMS-2054833. O.Z. was partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 692452). This research was further supported in part by BSF grant 2018088.

## References

[1] J. Bricmont, A. El Mellouki, and J. Fröhlich. Random surfaces in statistical mechanics: roughening, rounding, wetting,. . . . *J. Statist. Phys.*, 42(5-6):743–798, 1986.

[2] P. Caputo, D. Ioffe, and V. Wachtel. Confinement of Brownian polymers under geometric area tilts. *Electron. J. Probab.*, 24:Paper No. 37, 21, 2019.

[3] P. Caputo, D. Ioffe, and V. Wachtel. Tightness and line ensembles for Brownian polymers under geometric area tilts. In *Statistical mechanics of classical and disordered systems*, volume 293 of *Springer Proc. Math. Stat.*, pages 241–266. Springer, Cham, 2019.

[4] P. Caputo, E. Lubetzky, F. Martinelli, A. Sly, and F. L. Toninelli. The shape of the $(2+1)D$ SOS surface above a wall. *C. R. Math. Acad. Sci. Paris*, 350(13-14):703–706, 2012.

[5] P. Caputo, E. Lubetzky, F. Martinelli, A. Sly, and F. L. Toninelli. Dynamics of $(2+1)$-dimensional SOS surfaces above a wall: Slow mixing induced by entropic repulsion. *Ann. Probab.*, 42(4):1516–1589, 2014.

[6] P. Caputo, E. Lubetzky, F. Martinelli, A. Sly, and F. L. Toninelli. Scaling limit and cube-root fluctuations in SOS surfaces above a wall. *J. Eur. Math. Soc. (JEMS)*, 18(5):931–995, 2016.

[7] I. Corwin and A. Hammond. Brownian Gibbs property for Airy line ensembles. *Invent. Math.*, 195(2):441–508, 2014.

[8] P. L. Ferrari and H. Spohn. Constrained Brownian motion: fluctuations away from circular and parabolic barriers. *Ann. Probab.*, 33(4):1302–1325, 2005.

[9] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*. De Gruyter, Berlin, second edition, 2011.

[10] D. J. Grabiner. Brownian motion in a Weyl chamber, non-colliding particles, and random matrices. *Ann. Inst. H. Poincaré Probab. Statist.*, 35(2):177–204, 1999.

[11] D. Ioffe, S. Ott, S. Shlosman, and Y. Velenik. Critical prewetting in the 2D Ising model. *Ann. Probab.*, 2022. To appear.

[12] D. Ioffe, S. Shlosman, and Y. Velenik. An invariance principle to Ferrari-Spohn diffusions. *Comm. Math. Phys.*, 336(2):905–932, 2015.

[13] D. Ioffe and Y. Velenik. Low-temperature interfaces: prewetting, layering, faceting and Ferrari-Spohn diffusions. *Markov Process. Related Fields*, 24(3):487–537, 2018.

[14] D. Ioffe, Y. Velenik, and V. Wachtel. Dyson Ferrari-Spohn diffusions and ordered walks under area tilts. *Probab. Theory Related Fields*, 170(1-2):11–47, 2018.

[15] S. Karlin and J. McGregor. Coincidence probabilities. *Pacific J. Math.*, 9:1141–1164, 1959.

[16] E. Lubetzky, F. Martinelli, and A. Sly. Harmonic pinnacles in the discrete Gaussian model. *Comm. Math. Phys.*, 344(3):673–717, 2016.

[17] P. Maillard and O. Zeitouni. Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1144–1160, 2016.

[18] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

[19] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[20] V. N. Sudakov. Criteria of compactness in function spaces. *Uspekhi Mat. Nauk*, 12:221–224, 1957.
[21] K. Yosida. *Functional Analysis*. Springer-Verlag, Heidelberg, sixth edition, 1980.

Amir Dembo
Mathematics Department and Statistics Department, Stanford University, Stanford, CA 94305, USA.
*Email address*: adembo@stanford.edu

Eyal Lubetzky
Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA.
*Email address*: eyal@courant.nyu.edu

Ofer Zeitouni
Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel, and Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA.
*Email address*: ofer.zeitouni@weizmann.ac.il