Energy-Efficient Actuation in Infinite Lattice Structures

Aleksandar Donev*

Salvatore Torquato‡

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Abstract

Buildup of internal self-stresses in hyperstatic adaptive structures resists actuation. A recent paper by Guest and Hutchinson shows that periodic infinite truss structures cannot be both statically and kinematically determinate structures; therefore, a rigid infinite lattice bar framework must be hyperstatic. This paper shows that it is possible to design adaptive periodic infinite truss structures that can achieve any state of uniform strain without energy cost by actuating only a subset of the bars in a coordinated fashion. We show that actuation of only 3 bars in two dimensions or 6 bars in three dimensions per unit cell is required. A mathematical apparatus is developed and an example of such a bitriangular lattice structure is given, along with accompanying illustrations. Supporting animations can be found at the authors' website.

***Program in Applied and Computational Mathematics* and *Princeton Materials Institute*, *Princeton University*, Princeton NJ 08540

†*Department of Chemistry* and *Princeton Materials Institute*, *Princeton University*, Princeton NJ 08540

‡Corresponding author. Permanent address: 125B Frick Laboratory, Princeton University, Princeton NJ 08544

Tel.: (609) 258-3341. E-mail address: `torquato@electron.princeton.edu`

1 Introduction

The mechanical performance of pin-jointed bar frameworks, referred to simply as trusses in this paper, is a useful guide to the performance of the same framework but with welded joints (Pellegrino and Calladine, 1986). Materials with lattice-like structures find numerious applications due to their excellent mechanical properties, and can be effectively modeled as infinite periodic trusses. New advances in manufacturing techniques have enabled engineers to create lattice materials with lattice parameters on the order of 0.5mm; also, truss structures with strut diameters of only 50μm have been manufactured [see (Deshpande *et al.*, 2001) and references therein]. One novel application of such materials is in adaptive structures, where certain bars act as actuators and are used to precisely control the global shape of the structure (Hutchinson *et al.*, 2002).

Infinite bar frameworks offer many open mathematical questions. Much is known about the rigidity of finite bar frameworks in Euclidean space, particularly in the plane. Infinite systems are difficult to deal with mathematically, but are relevant to the study of large lattice truss structures, and therefore deserve special attention. We believe that repetitive bar frameworks on a flat torus (i.e., with periodic boundary conditions), in a suitably-defined limit of an infinite torus, can be used as a basis for simplified but rigorous models of lattice materials. In this work we study actuation in such periodic frameworks.

In a recent paper, Guest and Hutchinson (2002) discuss the prospect of designing an infinite lattice truss structure that is both statically and kinematically determinate. In the aformentioned paper, the authors conclude that it is impossible to design such a structure based on some counting arguments. We will revisit this problem from a different mathematical perspective to further illucidate this important fact. The importance of this investigation comes from the fact that statically and kinematically determinate structures, called (*generically*) *isostatic structures* in this report, can be used as "ideal" adaptive structures, since the length of any bar can be changed (actuated) independently of other bars. By combining actuation in a number of strategically placed bars, one can achieve useful deformations of the global structure, while still preserving mechanical stability (i.e., rigidity or stiffness). See Hutchinson *et al.* (2002) for details.

The negative result of the above paper should *not*, however, be taken as an indication that it is impossible to design a lattice truss that can be used to build "ideal" large adaptive structures. Indeed, in this paper we propose a method to design infinite lattice (i.e., periodic or repetitive) truss (i.e., pin-jointed bar framework) structures in which *any* global deformation can be achieved without energy cost by repetitively (periodically) actuating $d\,(d+1)\,/2$ (3 in two, or 6 in three dimensions) bars *per unit cell*. By a global deformation, we mean a state of uniform strain, which is modelled as a deformation of the underlying lattice vectors of the repetitive structure.

The full derivation of this relatively simple theoretical result is given here. Some of the mathematical apparatus is presented in higher generality than needed in order to enable extensions in the future, and also to point to some results interesting from a mathematical perspective along the way. Some of these are not needed in order to understand this report and can simply be skipped (sections 2.1.1, 5 and 5.1).

Our expectation is that the basic idea of using periodic actuation of $d\,(d+1)\,/2$ per unit cell can be used to design real adaptive lattice trusses. Such adaptive structures would be able to achieve any global deformation in which the strain gradient is small (that is, the strain does not change appreciably over the lengthscale of a unit cell) with very small energy cost (internal resistance).

Further discussion and animations of all the figures given in this paper can be found at our website (Donev, 2002).

2 Mechanical Equilibrium

Consider a large d-dimensional pin-jointed bar framework in Euclidean space (and unspecified boundaries) which has a repetitive structure, i.e., it is created by periodically repeating a basic building block. We model such a framework as an idealized infinite periodic network with a (reference) unit cell specified via the lattice vectors (generators) $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_d\}$. Denote the positions of the N joints (nodes) within a unit cell with $\mathbf{r}$, with node i being at $\mathbf{r}_i$. Let the number of bars per unit cell be M and the lattice vectors be columns in a d-by-d lattice matrix $\boldsymbol{\Lambda}$. The choice of unit cell and lattice vectors is not unique; however, there is a primitive unit cell with N_p nodes and M_p arcs per unit cell and lattice $\mathbf{\Lambda}_p$ which is repeated several times along each coordinate dimension to obtain the reference unit cell. If we form a diagonal matrix $\mathbf{N}_c$ from the number of repetitions of the primitive cell along each dimension, then $\mathbf{\Lambda} = \mathbf{\Lambda}_p \mathbf{N}_c$, $N = N_c N_p$ and $M = N_c M_p$, where N_c is the total number of primitive cells contained within the unit cell, $N_c = |\mathbf{N}_c|$. We use $||$ to denote a matrix determinant.

So far we described an infinite network in Eucledian space with no boundary conditions. In this work we impose periodic boundary conditions, that is to say, we focus on deformations of the network which are periodic with periodicity determined by the lattice $\mathbf{\Lambda}$. Mathematically, we wrap the network around a (flat) topological *torus* defined with the choice of unit cell. The reader should keep in mind the important distinction between the infinite structure which is obtained by periodically repeating the unit cell in Eucledian space (this is a "universal cover" of the torus in topological jargon), in which deformations need not be periodic, and the finite network on a torus, which is used to model periodic deformations of the infinite network. The remainder of this paper discusses the network on a torus, unless otherwise indicated.

Initially we focus on small (infinitesimal) deformations of the network from its original configuration (e.g., $\Delta\mathbf{\Lambda}$), however, the results are also relevant to large deformations, as discussed in section 4.2. One can either consider a fictional evolution (time) parameter t on which all quantities depend and consider directions of deformation (i.e., infinitesimal deformations) of the network (for example $d\mathbf{\Lambda}/dt$), or consider small but finite displacements (e.g., $\Delta\mathbf{\Lambda}$) up to first order. We chose the latter simply because the notation is simpler and the presentation clearer, and because we wish to avoid references to dynamics of the system.

2.1 Macroscopic Strain

The macroscopic strain $\boldsymbol{\varepsilon}$ in a periodic network is related to the deformation of the lattice $\Delta\mathbf{\Lambda}$ by the relation

$$\boldsymbol{\varepsilon} = (\Delta\mathbf{\Lambda})\,\mathbf{\Lambda}^{-1}. \tag{1}$$

To see this, note that the deformation of the lattice causes a displacement of the lattice point $\mathbf{P}$ (this is a vector of integer lattice coordinates) positioned at $\mathbf{r_P} = \mathbf{\Lambda P}$ of

$$\Delta \mathbf{r_P} = (\Delta \mathbf{\Lambda}) \mathbf{P} = \left[(\Delta \mathbf{\Lambda}) \left(\mathbf{\Lambda}^{-1} \right) \right] \mathbf{r_P},$$

which gives the strain (tensor)

$$\boldsymbol{\varepsilon} = \boldsymbol{\nabla}_{\mathbf{r}} (\Delta \mathbf{r}) = (\Delta \mathbf{\Lambda}) \mathbf{\Lambda}^{-1}.$$

Of course, the strain needs to be symmetric, $\boldsymbol{\varepsilon}^T = \boldsymbol{\varepsilon}$. It turns out that this condition eliminates rotations of the lattice, since rotations of the lattice produce skewsymmetric strains. Because rotations of the lattice belong to the category of trivial motions, which we will try to eliminate from the onset in order to simplify later counting, we will use the strain $\boldsymbol{\varepsilon}$ as a variable instead of the deformed lattice $(\mathcal{I} + \boldsymbol{\varepsilon}) \mathbf{\Lambda}$. This is only strictly valid for infinitesimal lattice deformations; finite lattice deformations in this model are to be considered as an integral of infinitesimal deformations with symmetric strain.

In order to simplify matrix algebra later on, we will need to represent the strain as a vector $\widehat{\boldsymbol{\varepsilon}}$ with $d(d+1)/2$ coordinates containing only the lower or only the upper triangle of the strain components. How we order the triangle into the vector is immaterial and a matter of convention (e.g., ordering by diagonals starting from the main diagonal or ordering by columns). This ordering establishes a correspondence $s \equiv (p, q)$ between component $\widehat{\boldsymbol{\varepsilon}}_s$ and component $\boldsymbol{\varepsilon}_{p,q} = \boldsymbol{\varepsilon}_{q,p} = \widehat{\boldsymbol{\varepsilon}}_s$. The usual convention (in three dimensions) is to use the column vector of strains

$$\widehat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_{1,1} \\ \varepsilon_{2,2} \\ \varepsilon_{3,3} \\ 2\varepsilon_{2,3} \\ 2\varepsilon_{1,3} \\ 2\varepsilon_{1,2} \end{bmatrix},$$

which contains additional factors of 2 that we omit (see also section 2.5).

2.1.1 Invariance of the Macroscopic Strain

The unit cell of a periodic system is not uniquely defined. For example, one may take a larger unit cell as the reference cell, i.e., take the lattice to be a sublattice of the original lattice:

$$\mathbf{\Lambda}' = \mathbf{\Lambda}\mathbf{N}_c,$$

where $\mathbf{N}_c$ is a diagonal matrix with positive integer entries. Now consider a lattice deformation with periodicity determined by $\mathbf{\Lambda}$ in the primed notation, where $\Delta\mathbf{\Lambda}' = (\Delta\mathbf{\Lambda})\,\mathbf{N}_c$. The macroscopic strain is

$$\boldsymbol{\varepsilon}' = (\Delta\mathbf{\Lambda})\,\mathbf{\Lambda}^{-1} = (\Delta\mathbf{\Lambda})\left(\mathbf{N}_c\mathbf{N}_c^{-1}\right)\mathbf{\Lambda}^{-1} = \boldsymbol{\varepsilon},$$

i.e., the strain is independent of the exact choice of the unit cell. This is a very important invariance property which makes our results more physical. We will give an expression for the macroscopic stress in the network later, which also possesses this kind of invariance.

2.2 Elastic Energy

We denote by $\{i,j\}$ the bar (arc) connecting joints (nodes) i and j and append the subscript ij to all quantities associated with this bar. The elastic energy stored in the structure is a sum over the energies stored in each bar:

$$E(\mathbf{r},\boldsymbol{\varepsilon},\boldsymbol{\mu}) = \sum_{\{i,j\}} E_{ij}\left[\mu_{ij}, l_{ij}(\mathbf{r}_i,\mathbf{r}_j,\boldsymbol{\varepsilon})\right].$$

Here we assume a central-force network in which the energy stored in a bar only depends on the length of the bar

$$l_{ij} = \|\mathbf{r}_{ij}\| = \|\mathbf{r}_i - \mathbf{r}_j + \mathbf{\Lambda}\mathbf{n}_{ij}\|$$

and on the activation bar parameter μ_{ij} (such as temperature, applied voltage, etc.).

The quantity $\mathbf{n}_{ij}$ appears because of the periodic nature of the structure and is a vector giving the number of unit cells that the bar $\{i,j\}$ "crosses" over. If we think of the periodic network as a graph G embedded in a flat torus defined by the lattice $\mathbf{\Lambda}$, the integer data

$\mathbf{n}$ is now to be considered part of the combinatorial part of the network, which we will denote with $\mathcal{G} = (G, \mathbf{n})$, and specifies how the network wraps around the torus, i.e., is part of the network *connectivity* information. The embedding (geometry) part of the problem specification on the other hand is characterized by the *configuration* $\mathbf{p} = (\mathbf{r}, \mathbf{\Lambda})$. Therefore, a periodic network is specified with $\mathcal{N} = (\mathcal{G}, \mathbf{p}) = [(G, \mathbf{n}), (\mathbf{r}, \mathbf{\Lambda})]$. This is to be compared to the usual specification of a network embedded in Euclidean space, $\mathcal{N} = (G, \mathbf{r})$, which lacks the periodicity information.

We can rewrite the length of bar $\{i, j\}$ as

$$l_{ij} = \|\mathbf{r}_{ij} + (\Delta\mathbf{r}_i - \Delta\mathbf{r}_j) + \boldsymbol{\varepsilon}\mathbf{\Lambda}\mathbf{n}_{ij}\| = \|\mathbf{r}_{ij} + \mathbf{T}_{ij}\Delta\mathbf{r} + \mathbf{S}_{ij}\widehat{\boldsymbol{\varepsilon}}\|,$$

where $\mathbf{T}_{ij}$ is a $[d \times Nd]$ matrix (with simple structure) and $\mathbf{S}_{ij}$ is a $\left[d \times \frac{d(d+1)}{2}\right]$ matrix, in order to emphasize the linearity of the expression inside the norm. We will denote by

$$\mathbf{u}_{ij} = \frac{\mathbf{r}_{ij}}{l_{ij}}$$

the unit vector along the current position of the bar $\{i, j\}$.

It is easy to see that uniform translations are also trivial (i.e., length-preserving) motions of the periodic network. To eliminate these from consideration, we will freeze (pin) joint 1 (i.e., $\Delta\mathbf{r}_1 = 0$ will not be included in $\Delta\mathbf{r}$), leaving the number of degrees of freedom at

$$N_f = d(N-1) + \frac{d(d+1)}{2}. \tag{2}$$

Therefore, we will take the N_f-dimensional vector

$$\Delta\mathbf{p} = (\Delta\mathbf{r}, \widehat{\boldsymbol{\varepsilon}}) = (\Delta\mathbf{r}_2, \ldots, \Delta\mathbf{r}_n, \widehat{\boldsymbol{\varepsilon}})$$

as the characterization of the deformation. Any nonzero infinitesimal $\Delta\mathbf{p}$ that does not change the bar lengths is a *mechanism* of the periodic structure. Notice that there are no trivial mechanisms in this new notation.

2.3 Force Equilibrium

We will use the notation $\nabla_{\mathbf{p}}$ and $\nabla_{\mathbf{r}}$ instead of the more appropriate $\nabla_{\Delta\mathbf{p}}$ and $\nabla_{\Delta\mathbf{r}}$ to avoid symbol havoc.

At equilibrium we have energy stationarity, i.e., there are no energy-beneficial deformations to first order, so that

$$\nabla_{\mathbf{p}}E = (\nabla_{\mathbf{p}}\mathbf{l})\,(\nabla_{\mathbf{l}}E) = \mathbf{R}\mathbf{f} = 0.$$

This is just a mechanical equilibrium condition. Here $\mathbf{f} = \nabla_{\mathbf{l}}E = \left\{\frac{\partial E_{ij}(\mu_{ij},l_{ij})}{\partial l_{ij}}\right\}$ are the elastic forces (tension or compression) in the bars, and $\mathbf{R} = \nabla_{\mathbf{p}}\mathbf{l}$ is the *rigidity matrix* of the periodic network. Note that in rigidity theory literature $\mathbf{R}^T$ is usually called the rigidity matrix (in engineering literature, $\mathbf{R}$ is sometimes called the *compatibility* matrix, while $\mathbf{R}^T$ is called the *equilibrium* matrix). It is closely related to the usual rigidity matrix, but with $d\,(d+1)\,/2$ rows appended corresponding to equilibrium with respect to the macroscopic strains, i.e., to equilibrium of the macroscopic stresses.

Our first task is to derive the form of this rigidity matrix (since we have a new non-standard piece appended to it). The column of $\mathbf{R}$ corresponding to the bar $\{i,j\}$ is

$$\mathbf{R}_{ij} = \nabla_{\mathbf{p}}\left(l_{ij}\right) = \begin{bmatrix} \mathbf{A}_{ij} \\ - \\ \mathbf{L}_{ij} \end{bmatrix}$$

The first piece of this is the corresponding column of the usual rigidity matrix:

$$\mathbf{A}_{ij} = \nabla_{\mathbf{r}}\left(l_{ij}\right) = \begin{matrix} \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{matrix} \begin{bmatrix} \vdots \\ \mathbf{u}_{ij} \\ \vdots \\ -\mathbf{u}_{ij} \\ \vdots \end{bmatrix},$$

and the second piece is due to the periodicity of the network:

$$\mathbf{L}_{ij} = \boldsymbol{\nabla}_{\widehat{\boldsymbol{\varepsilon}}}\left(l_{ij}\right) = \left\{\boldsymbol{\nabla}_{\widehat{\boldsymbol{\varepsilon}}}\left[\boldsymbol{\varepsilon}\left(\widehat{\boldsymbol{\varepsilon}}\right)\boldsymbol{\Lambda}\mathbf{n}\right]\right\}\mathbf{u}_{ij},$$

which in matrix form is

$$\mathbf{L}_{ij} = \begin{bmatrix} \left(\mathbf{S}_1\boldsymbol{\Lambda}\mathbf{n}_{ij}\right)^T\mathbf{u}_{ij} \\ \vdots \\ \left(\mathbf{S}_{d(d+1)/2}\boldsymbol{\Lambda}\mathbf{n}_{ij}\right)^T\mathbf{u}_{ij} \end{bmatrix}. \tag{3}$$

Here $\mathbf{S}_s = \boldsymbol{\nabla}_{\widehat{\boldsymbol{\varepsilon}}_s}\left[\boldsymbol{\varepsilon}\left(\widehat{\boldsymbol{\varepsilon}}\right)\right]$ has nonzero entries only at positions (p,q) and (q,p) (recall that $s \equiv (p,q)$ determined how the vectorization of the upper/lower triangle of $\boldsymbol{\varepsilon}$ was done to obtain $\widehat{\boldsymbol{\varepsilon}}$). We can also write this in indicial form suitable for computational use as

$$\left(\mathbf{L}_{ij}\right)_s = \left\{ \begin{array}{c} \left(\boldsymbol{\Lambda}\mathbf{n}_{ij}\right)_p\left(\mathbf{u}_{ij}\right)_q + \left(\boldsymbol{\Lambda}\mathbf{n}_{ij}\right)_q\left(\mathbf{u}_{ij}\right)_p \\ \left(\boldsymbol{\Lambda}\mathbf{n}_{ij}\right)_p\left(\mathbf{u}_{ij}\right)_q \text{ if } p = q \end{array} \right\}. \tag{4}$$

2.4 Adaptive Networks

A network is perfectly adaptive if the lengths of all its bars can be changed (actuated) independently of one another. Actuation of the bars will induce a commensurate deformation of the structure. It is also desirable that there be a unique deformation corresponding to every actuation. It is easy to see that in order for this to be true the rigidity matrix $\mathbf{R}$ must be invertible, since the change of the bar lengths $\Delta\mathbf{l}$ (alternatively $\Delta\mathbf{l}$ can be thought of as the rate of bar elongation/contraction) during a small deformation $\Delta\mathbf{p}$ (alternatively joint velocities) is to first order

$$\Delta\mathbf{l} = \left(\boldsymbol{\nabla}_{\mathbf{p}}\mathbf{l}\right)^T\Delta\mathbf{p} = \mathbf{R}^T\Delta\mathbf{p}.$$

This relation is bijective only when $\mathbf{R}$ is invertible. We will come to the same conclusion but in a much more general setting later on.

2.5 Macroscopic Stress

The condition of mechanical equilibrium

$$\mathbf{R}\mathbf{f} = 0$$

reduces to the Nd microscopic force balances at each node

$$\sum_{\{i,j\}} f_{ij}\mathbf{A}_{ij} = 0,$$

as well as the $d\,(d+1)\,/2$ conditions that there be no macroscopic stresses:

$$\widehat{\boldsymbol{\sigma}} = \frac{1}{|\boldsymbol{\Lambda}|}\sum_{\{i,j\}} f_{ij}\mathbf{L}_{ij} = 0. \tag{5}$$

Here $\widehat{\boldsymbol{\sigma}}$ is the vectorized version of the upper or lower triangle of the symmetrized macroscopic stress (tensor) $\boldsymbol{\sigma}$, and we normalized with the reciprocal unit cell volume $|\boldsymbol{\Lambda}|$ in order to get the correct units of stress. This is expected since stress is the strain gradient of the energy *density*, and not of energy. To be in agreement with standard convention (which adds factors of 2 to the off-diagonal strains), one should add a factor of 1/2 in Eq. (4) for the off-diagonal stresses, to obtain:

$$(\mathbf{L}_{ij})_s = \frac{1}{2}\left[(\boldsymbol{\Lambda}\mathbf{n}_{ij})_p\,(\mathbf{u}_{ij})_q + (\boldsymbol{\Lambda}\mathbf{n}_{ij})_q\,(\mathbf{u}_{ij})_p\right],$$

or consider a matrix form of the (unsymmetrized) stress tensor

$$\boldsymbol{\sigma} = \frac{1}{|\boldsymbol{\Lambda}|}\sum_{\{i,j\}} f_{ij}\left[\mathbf{u}_{ij}^T\,(\boldsymbol{\Lambda}\mathbf{n}_{ij})\right], \tag{6}$$

which more clearly displays the tensor character through the use of the diadic product $\mathbf{u}_{ij}^T\,(\boldsymbol{\Lambda}\mathbf{n}_{ij})$.

We note that the expression for the macroscopic stress (5) is invariant with respect to choosing a different unit cell as the reference cell, as it should on physical grounds. However, this is difficult to show in general as $\mathbf{n}$ depends non-trivially on the choice of the cell, and we do not give such a proof here.

It is important to point out that equivalent results for the macroscopic stress in a force network have appeared elsewhere. Compare (5) to the expressions found in Latzel *et al.* (2000) (and references therein) for the macroscopic stress in a disordered network (recast into a form more suitable for our presentation):

$$\boldsymbol{\sigma} = \frac{1}{V} \sum_{\{i,j\} \in V} f_{ij} l_{ij} \left(\mathbf{u}_{ij} \mathbf{u}_{ij}^T \right) = \frac{1}{V} \sum_{i \in V,\ j \notin V} f_{ij} \left(\mathbf{u}_{ij} \mathbf{r}_i^T \right), \tag{7}$$

The second expression in Eq. (7) only involves microscopic forces crossing the boundary of a given reference (averaging) volume V, i.e. only the bars $\{i, j\} \in \partial V$. For a periodic system it is natural to take the unit cell as the averaging volume. Consider a bar $\{i, j\}$ with nonzero $\mathbf{n}_{ij}$. It will appear twice in the sum in Eq. (7), once as the "original" bar with direction $\mathbf{u}_{ij}$, and once as an "image" bar $\{i', j'\}$ with $\mathbf{u}_{i'j'} = -\mathbf{u}_{ij}$ and $\mathbf{r}_{i'} = \mathbf{r}_i - \boldsymbol{\Lambda} \mathbf{n}_{ij} + l_{ij} \mathbf{u}_{ij}$. Therefore the contribution from this bar to the averaged macroscopic stress in Eq. (7) is

$$\frac{1}{|\boldsymbol{\Lambda}|} f_{ij} \left[\mathbf{u}_{ij}^T \left(\boldsymbol{\Lambda} \mathbf{n}_{ij} \right) \right] - \frac{1}{|\boldsymbol{\Lambda}|} f_{ij} l_{ij} \left(\mathbf{u}_{ij} \mathbf{u}_{ij}^T \right).$$

The first term in this expression is identical to the one in Eq. (6). If we take a large unit cell, in the spirit of the averaging in Eq. (7), the second term will become negligible.

2.6 Stiffness Matrix

Another important matrix describing the given network is the stiffness matrix, which is the Hessian of the energy with respect to deformations:

$$\mathbf{H} = \boldsymbol{\nabla}_{\mathbf{pp}}^2 E = \left(\boldsymbol{\nabla}_{\mathbf{p}} \mathbf{R} \right) \mathbf{f} + \mathbf{R} \left(\boldsymbol{\nabla}_{\mathbf{p}}^T \mathbf{f} \right).$$

We now take a crucial simplifying step valid for the rest of this paper: *The periodic structure is unloaded and is in equilibrium*, i.e.,

$$\mathbf{f} = 0,$$

so that we get

$$\mathbf{H} = \mathbf{R} \left(\boldsymbol{\nabla}_{\mathbf{p}}^T \mathbf{f} \right) = \mathbf{R} \left[\left(\boldsymbol{\nabla}_{\mathbf{p}} \mathbf{l} \right) \left(\boldsymbol{\nabla}_{\mathbf{l}} \mathbf{f} \right) \right]^T = \mathbf{R} \mathbf{C} \mathbf{R}^T,$$

where $\mathbf{C} = \boldsymbol{\nabla}^2_{\mathbf{ll}} E = \text{Diag}\left\{\frac{\partial^2 E_{ij}(\mu_{ij}, l_{ij})}{\partial l_{ij}^2}\right\}$ is a diagonal matrix containing the individual bar stiffnesses. We call $\mathbf{H}$ the *stiffness* matrix of the network.

3 Actuation

We now consider activating an unstressed network in equilibrium by actuating some of its bars, i.e., by changing $\boldsymbol{\mu}$. Taking the equilibrium condition

$$\boldsymbol{\nabla}_{\mathbf{p}} E = 0,$$

and differentiating with respect to $\boldsymbol{\mu}$, we get for small actuations

$$\left(\boldsymbol{\nabla}^2_{\mathbf{p}\boldsymbol{\mu}} E\right) \Delta\boldsymbol{\mu} + \left(\boldsymbol{\nabla}^2_{\mathbf{pp}} E\right) \Delta\mathbf{p} = 0,$$

which gives the deformation $\Delta\mathbf{p}$ induced by the actuation $\Delta\boldsymbol{\mu}$. If we further simplify

$$\mathbf{G} = \boldsymbol{\nabla}^2_{\mathbf{p}\boldsymbol{\mu}} E = (\boldsymbol{\nabla}_{\mathbf{p}} \mathbf{l}) \left[\boldsymbol{\nabla}^2_{\mathbf{l}\boldsymbol{\mu}} E\right] = \mathbf{R}\widetilde{\mathbf{C}},$$

where $\widetilde{\mathbf{C}} = \boldsymbol{\nabla}^2_{\mathbf{l}\boldsymbol{\mu}} E = \text{Diag}\left\{\frac{\partial^2 E_{ij}(\mu_{ij}, l_{ij})}{\partial l_{ij} \partial \mu_{ij}}\right\}$ is a diagonal matrix, we get

$$\Delta\mathbf{p} = -\mathbf{H}^{-1}\mathbf{R}\widetilde{\mathbf{C}}\Delta\boldsymbol{\mu}. \tag{8}$$

Equation (8) gives the sought-after relation between the actuation and the induced deformation, and assumes that $\mathbf{H}$ is invertible (see section 5).

3.1 Actuation Energy

Some activations will not cost any energy beyond that needed to induce the actuation $\Delta\boldsymbol{\mu}$, but others will induce self-stresses in the structure and therefore cost energy. The elastic energy stored in the network due to the stresses induced by the actuation is of second order in $\Delta\boldsymbol{\mu}$, and is given by

$$\Delta E = \frac{1}{2}\left[\Delta\boldsymbol{\mu}^T\widehat{\mathbf{C}}\Delta\boldsymbol{\mu} + \Delta\mathbf{r}^T\mathbf{H}\Delta\mathbf{r}\right] + \Delta\mathbf{r}^T\mathbf{G}\Delta\boldsymbol{\mu},$$

where $\widehat{\mathbf{C}} = \boldsymbol{\nabla}^2_{\boldsymbol{\mu}\boldsymbol{\mu}}E = \mathrm{Diag}\left\{\frac{\partial^2 E_{ij}(\mu_{ij},l_{ij})}{\partial^2\mu_{ij}}\right\}$. Using relation (8) this simplifies to

$$\Delta E = \frac{1}{2}\Delta\boldsymbol{\mu}^T\mathbf{K}\Delta\boldsymbol{\mu},$$

where

$$\mathbf{K} = \widehat{\mathbf{C}} - \mathbf{G}^T\mathbf{H}^{-1}\mathbf{G} = \widehat{\mathbf{C}} - \widetilde{\mathbf{C}}\mathbf{R}^T\mathbf{H}^{-1}\mathbf{R}\widetilde{\mathbf{C}}.$$

Therefore, any activations $\Delta\boldsymbol{\mu}$ that lie in the null eigenspace of the matrix $\mathbf{K}$ will cost no energy up to second order, i.e., they will induce no self-stresses in the network.

In this work we focus on the simplest type of actuation: One in which the actuation is achieved by changing the equilibrium lengths $\bar{\mathbf{l}}$ of the bars (say by heating/cooling them or applying a voltage), i.e.,

$$\boldsymbol{\mu} \equiv \bar{\mathbf{l}},$$

where the elastic energy is some strictly convex function of the length mismatch:

$$E_{ij} = E_{ij}(l_{ij} - \bar{l}_{ij}).$$

Furthermore, we assume that only a subset of the arcs can be actuated, and we take the $[M \times M]$ diagonal matrix $\mathbf{D} = \mathrm{Diag}\{0 \text{ or } 1\}$ to be the indicator of which arcs can be activated: a 1 on the diagonal indicating the arc is *active*, and a 0 indicating it is *inactive* (i.e., its length is fixed). We denote with M_a the number of active arcs. With these simplifications we have

$$\widetilde{\mathbf{C}} = -\mathbf{C}\mathbf{D} \text{ and } \widehat{\mathbf{C}} = \mathbf{C},$$

which gives

$$\mathbf{K} = \mathbf{C}^{1/2}\left(\mathcal{I} - \mathbf{Q}\right)\mathbf{C}^{1/2}, \tag{9}$$

where

$$\mathbf{Q} = \mathbf{D}\mathbf{C}^{1/2}\mathbf{R}^T \left(\mathbf{R}\mathbf{C}\mathbf{R}^T\right)^{-1} \mathbf{R}\mathbf{C}^{1/2}\mathbf{D}. \tag{10}$$

The meaning of the multiplications with $\mathbf{D}$ here is that we are extracting an $[M_A \times M_A]$ submatrix $\mathbf{Q}_a$ corresponding to the rows and columns of the active arcs from $\mathbf{C}^{1/2}\mathbf{R}^T \left(\mathbf{R}\mathbf{C}\mathbf{R}^T\right)^{-1} \mathbf{R}\mathbf{C}^{1/2}$. We can also take the case when all arcs are identical in the sense of their stiffness being the same, $\mathbf{C} = c\mathcal{I}$, to get the simpler expression

$$\mathbf{Q} = \mathbf{D}\mathbf{R}^T \left(\mathbf{R}\mathbf{R}^T\right)^{-1} \mathbf{R}\mathbf{D}. \tag{11}$$

3.2 Ideal Activations

Now we are in a position to clearly state the condition that there exist *ideal* actuations $\Delta\boldsymbol{\mu}$, i.e., actuations that induce no self-stresses and cost no energy: $\mathbf{Q}$ *must have a nonempty eigenspace of eigenvalue* 1. To every independent eigenvector of $\mathbf{Q}$ with eigenvalue 1 corresponds an independent ideal actuation.

It may not be obvious that $\mathbf{Q}$ will ever have eigenvalues 1. However, notice that if $\mathbf{D} = \mathcal{I}$, i.e., if all arcs are active, then all eigenvalues of $\mathbf{Q}$ are all 1 or 0. The "bad" eigenvalues 0 correspond to actuations in the null-space of $\mathbf{R}$, i.e., to *self-stresses* of the network. All the other eigenvalues of unity are "good" eigenvalues. This is a very intuitive result: If the lengths of the bars are changed along a direction that is a self-stress of the network, then this will induce no useful deformation, $\Delta\mathbf{p} = 0$, but it will induce the corresponding self-stress and store elastic energy in the network. Otherwise, the actuation of the bar lengths will produce a deformation and cost no energy.

The main point to get across is that $\mathbf{Q}$ usually has eigenvalues 1, even when *not all* arcs are active. This means that in most networks it is possible to change the lengths of only a *small* subset of the bars, in a *coordinated manner* (i.e., not independent of one another), while not changing the length of the other bars. To our knowledge, this crucial observation has heretofore not been made.

3.3 Isostatic Unit Cells

The best case, i.e., the most adaptive network, is obtained when $\mathbf{R}$ *is invertible.* In other words, the unit cell is *isostatic*, or kinematically and statically determinate. However, note that this *does not* mean that the infinite lattice network is also isostatic (this important point will be discussed later). When $\mathbf{R}$ is invertible we have that

$$\mathbf{Q} = \mathbf{D}^2 \text{ or equivalently } \mathbf{Q}_a = \mathcal{I},$$

which means that the lengths of *all* of the active arcs can be changed periodically *independently* without affecting the lengths of the other bars, i.e., without inducing stresses.

This occurs in the case of finite isostatic structures. However, here we are considering infinite periodic networks in which the actuation is also periodic, i.e., *the lengths of all the image arcs of a given active arc are changed in unison.* This is the main difference from having an infinite network in Euclidean space that is isostatic, in which case the length of *any* of the arcs can be changed independently of all other arcs. Therefore, we will say isostatic unit cell and not isostatic structure. The real structure we have in mind is an infinite structure made by repeating the unit cell periodically.

The expressions given in the previous sections simplify considerably when $\mathbf{R}$ is invertible. In particular, the relation between actuation (bar elongations) and deformation (induced strain and joint displacements) is unique and invertible and given by

$$\mathbf{R}^T \Delta\mathbf{p} = \mathbf{D}\Delta\boldsymbol{\mu}. \tag{12}$$

The rest of this paper assumes $\mathbf{R}$ is invertible, and also that isostaticity is a *generic* property, i.e., it is determined primarily by $\mathcal{G}$ [see Graver *et al.* (1991) for details], and not by the particular configuration $\mathbf{p}$. That isostaticity is a generic property for networks on a deforming torus has not yet been rigorously proven to our knowledge. Our assumption is that the unit cell of the adaptive periodic framework under consideration is *generically isostatic.*

4 Adaptive Periodic Networks

The main goal of this work is to find an infinite repetitive network which can be deformed uniformly in an arbitrary manner just by actuating a small subset of the bars in each unit cell. Since there are $d\,(d+1)\,/2$ independent strains, we need at least this many active bars. The only requirement is that actuating each active arc induces a nonzero strain, and that the strains induced by actuating different active arcs be linearly independent. If this is the case then *we can achieve any strain by combining the individual actuations accordingly.*

In mathematical terms, what we need is the submatrix formed from the last $d\,(d+1)\,/2$ rows of $\mathbf{R}^{-T}$, and a basis $\mathbf{B}\,(\mathbf{R})$ for it (i.e., $d\,(d+1)\,/2$ columns which are linearly independent):

$$\mathbf{B}\,(\mathbf{R}) = \mathbf{R}^{-T}\left[\text{last }\frac{d\,(d+1)}{2}\text{ rows, active arcs colums}\right]. \tag{13}$$

Choosing the arcs corresponding to these columns as the active arcs gives us an infinite perfectly adaptive network. Any desired strain $\widehat{\boldsymbol{\varepsilon}}$ can be achieved by using the actuation

$$\Delta\boldsymbol{\mu}_a = \Delta\boldsymbol{\mu}_{\text{active arcs}} = [\mathbf{B}\,(\mathbf{R})]^{-1}\,\widehat{\boldsymbol{\varepsilon}}.$$

4.1 The Bitriangular Lattice

We have constructed a simple example of a rigid (defined in the context of infinite structures more precisely later) two-dimensional lattice whose unit cell is isostatic as defined above, and identified 3 arcs suitable to be used for actuation. In doing so, we looked for periodic subnetworks of the triangular lattice whose unit cell consists of $2 \times 2 = 4$ unit cells of the triangular lattice. Since there are $n = 4$ joints per unit cell in such a unit cell, there need to be 9 bars in an isostatic unit cell, and so 3 bars need to be removed from the 12 bars present in the original triangular lattice. We found that removing three bars forming a (small) triangle produces a lattice which is rigid and whose unit cell is isostatic, and we call this the *bitriangular lattice*, since it is composed of two kinds of triangles (small and large).

For this lattice, it turns out that actuating any of the 3 bars forming the (remaining) small triangle does not produce any lattice deformation (global strain). Therefore, one should actuate 3 of the 6 bars bounding the larger triangle. We chose to use the 3 odd (or even) arcs as actuators, as shown in Figure 1. Therefore, in this example lattice one third of the bars are active.

Actuating each of the 3 active arcs produces an independent global uniform strain. Figure 2 shows one of these (equivalent) independent actuation modes. By combining these 3 deformations one can achieve any uniform strain in the infinite lattice. Animations illustrating how to uniformly shrink or expand the structure, i.e., achieve

$$\boldsymbol{\varepsilon} = \pm \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

are shown on the authors' website.

4.2 Large Deformations

The mathematics above was concerned with infinitesimal deformations. However, it should be stressed that an adaptive periodic network (with a generically isostatic unit cell and appropriately chosen $d\,(d+1)\,/2$ active arcs) can be finitely uniformly deformed without storing energy. To do this, an ordinary differential equation (ODE) system needs to be solved. Assume we want to achieve a time-dependent rate of strain $d\boldsymbol{\varepsilon}\,(t)\,/dt$, which integrated over time gives the desired deformation. This can be done by employing the (coordinated) time-dependent actuation $\boldsymbol{\mu}_a\,(t)$, which can be found as a solution to the ODE system (with the appropriate initial conditions)

$$\begin{aligned} \frac{d\boldsymbol{\mu}_a\,(t)}{dt} &= \{\mathbf{B}\,[\mathbf{R}\,(t)]\}^{-1}\,\frac{d\widehat{\boldsymbol{\varepsilon}}\,(t)}{dt}, && (14) \\ \frac{d\boldsymbol{\Lambda}\,(t)}{dt} &= \frac{d\boldsymbol{\varepsilon}\,(t)}{dt}\boldsymbol{\Lambda}\,(t)\,, && (15) \\ \frac{d\mathbf{r}\,(t)}{dt} &= \widetilde{\mathbf{B}}\,[\mathbf{R}\,(t)]\,\frac{d\boldsymbol{\mu}_a\,(t)}{dt}. && (16) \end{aligned}$$

Here $\widetilde{\mathbf{B}}\,(\mathbf{R})$ denotes the submatrix of $\mathbf{R}^{-T}$corresponding to the joint degrees of freedom and the active arcs:

$$\widetilde{\mathbf{B}}\,(\mathbf{R}) = \mathbf{R}^{-T}\,[\text{first }\ (N-1)\,d\text{ rows, }\ \text{active arcs colums}]\,.$$

We stress the fact that in Eqs.(14)-(16) the rigidity matrix $\mathbf{R}(t)$ is also time-dependent, since it depends on the current configuration. Solving this ODE system tells us both how to actuate the active arcs and how the network deforms in time.

We illustrate such an ideal finite actuation with the bitriangular lattice by solving the above ODE to achieve a large deformation in which we shrink the unit cell by 25% also make it into a square (from the original rhomboidal unit cell, as illustrated in Figure 3) and showing the result in Figure 4.

5 Rigidity of the Adaptive Network

This paper is concerned with the deformability of infinite periodic networks. One important assumption made throughout this is that the structure has no mechanisms (flexes), i.e., deformations $\Delta\mathbf{p}$ which change no bar lengths. This is a very important property for an adaptive structure, since it provides for uniqueness of the relationship between actuation and induced deformation. It is customary in rigidity theory to simply call such a flex-free structure a *rigid framework*. It may be better in the context of real applications to use the term *stiff framework*. A stiff structure for us is one which can support a given set of loads without too large of a deformation, defined in an application specific context.

In the above analysis, the assumption that $\mathbf{H}$ is invertible was based on the rigidity of the unit cell. However, here we are really considering rigidity of the *infinite* network. What exactly does rigidity mean in the context of infinite structures? It appears that this has not been carefully investigated. It is not necessary that the same concept of rigidity be extended from finite to infinite structures, but rather, to understand what are the relevant mathematical idealizations for modeling real large periodic trusses. Note that for the bitriangular lattice truss we can show analytically that it is (infinitesimally) rigid for all choices of unit cell. We do not reproduce this argument due to Robert Connelly here.

The mathematical framework which we believe is suitable for analysing infinite (i.e., very large) repetitive networks is the following:

Take a primitive lattice $\mathbf{\Lambda}_p$ for the periodic network and the flat torus that this lattice defines. Now consider rigidity of the framework on this torus. This means looking at deformations which have a repetition (period) of one unit cell plus deformations of the lattice. Notice that this is only a very small subset of all the possible deformations of the infinite network, which need not be periodic at all. Then take a sublattice of the primitive lattice, $\mathbf{\Lambda} = \mathbf{\Lambda}_p \mathbf{N}_c$, $\mathbf{N}_c$ integer, and the larger torus that it defines and consider rigidity of the network on this torus. It should be obvious that many of the properties depend on $\mathbf{N}_c$. Here $\mathbf{N}_c$ can in a sense be viewed as the "wavelength" at which the repetitive framework is analysed. A related novel systematic analysis is briefly explained in Hutchinson *et al.*, and consists of looking for "canonical" flexes, which can be thought of as the Fourier components of the flexes of a repetitive structure. A similar procedure can be applied when the lattice is allowed to deform and also when the self-stresses of a structure are considered, but further discussion is postponed for future work.

5.1 Determinacy of Infinite Periodic Networks

In this subsection, we revisit the subject of isostaticity of infinite periodic networks, from the perspective of the above model of rigidity on an enlarging torus. We arrive to the same conclusion as Guest and Hutchinson (2002): It is not possible to make an isostatic infinite periodic structure. However, our arguments use periodic boundary conditions.

For simplicity, we will focus on two dimensions, but the results apply as well to arbitrary space dimensions. Referring to Eq. (2), in order for a network on a torus with lattice $\mathbf{\Lambda}$ to be isostatic it must be that

$$M = 2(N - 1) + 3 = 2N + 1, \tag{17}$$

where as before $N > 1$ is the number of nodes in the unit cell and M is the number of bars per unit cell. It is the extra +1 that is of great importance in Eq.(17). Now consider a larger torus, with a lattice $\mathbf{\Lambda}' = \mathbf{\Lambda}\mathbf{N}_c$. When wrapped around this torus the network has

$$M' = N_c M = 2N_c N + N_c > 2N_c N + 1 = 2N' + 1,$$

where $N_c = |\mathbf{N}_c|$ is the number of unit cells fitting in the larger torus. Therefore, on the larger torus the network necessarily is overbraced, i.e., it must have self-stresses. This also means that it is possible for the infinite network to be sufficiently constrained and have no periodic mechanisms if its primitive cell is isostatic, as with the bitriangular lattice.

The essence of this argument is that counting equilibrium is not maintained as different tori are considered. If the network has no small-period (short-wavelength) mechanisms (i.e., it is rigid on a small torus), then it must have self-stresses of larger period (long-wavelength). If the network has no self-stresses on a large torus, it must have shorter-period mechanisms. It should be evident that any mechanism/self-stress can be replicated infinitely many times to produce a mechanism/self-stress of the infinite structure, i.e., of the periodic network wrapped around an infinite torus.

6 Future Directions

There are many directions along which future research can be based. The basic question to consider is how applicable this work is to achieving arbitrary deformations in infinite adaptive structures. It is clear that when the strain is non-uniform there will be some self-stresses induced during actuation and therefore zero energy storage is not possible. However, an expansion analysis is needed to determine how the expanded actuation energy depends on the (small) strain gradient.

Moreover, the simple analysis given in this work considered infinite structures. How does finiteness affect the deformability of repetitive structures? If the corrections induced by finite size are too large, they may compound together to completely overwhelm the first-order terms and thus make the proposed actuation mechanism unusable. Both numerical and analytical studies of non-uniformly deformed finite, but large, structures would thus be an obvious next step.

Another line of research to be pursued is to find the "best" isostatic unit cells. Based on the analysis we have given, there is no criterion beyond isostaticity and stiffness to consider when choosing among different lattices. Some of the higher-order corrections discussed above may be the guiding principle in choosing between candidate lattices. Additionally, we did assume an unloaded structure. In a real application an adaptive structure would be used to move loads. Are there higher-order corrections under global loading which differentiate betweeen different lattices? If yes, is this load-specific or are some lattices universally better? Hutchinson *et al.* (2002) point to other desirable qualities of the lattice structure, such as isotropic stiffness and high buckling and isotropic in-plane yield strength, and show that a structure like the Kagome lattice, which does not have an isostatic unit cell, is very effective in the context of adaptive structures. A comparison between this lattice and the bitriangular lattice in a practical setting might be a useful future project.

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Figure 1: *The unit cell of the bitriangular framework.* The 4 joints in the unit cell are shown as circles, while the 9 bars in the unit cell are shown with a solid line. The 3 active arcs are shown with thicker lines, and the *periodic images* of the arcs with non-zero $\mathbf{n}_c$ are also shown with dashed lines.

Figure 2: *An activation mode of the bitriangular framework.* The figure shows the deformation induced in the framework as one of the active bars is elongated by $\Delta l = \alpha t$ as a sequence of time frames with $t = 0, \Delta t, 2\Delta t, 3\Delta t$ for some arbitrarily scaled Δt and α, in the sequence: upper left, upper right, lower left, lower right. The corresponding deformations when each of the remaining 2 active arcs are actuated can be predicted from symmetry considerations. Note that we assume infinitesimal deformations but show a larger deformation for visualization purposes, which explains why some of the non-actuated bars also change their length (to second order).

Figure 3: *A finite deformation of the bitriangular lattice.* A large deformation of the the unit cell of the bitriangular lattice is shown using the lattice vectors. The original vectors are shown with a solid line, while the final ones are shown with a dashed line. During this deformation, the unit cell shrinks and becomes a square. We ensure that the lattice does not rotate so that it is possible to achieve this deformation by integrating a symmetric (time-dependent) strain rate.

Figure 4: *Achieving the deformation from Figure 3.* This sequence of time frames (as in Fig. 2) shows how one can achieve a global uniform deformation of the bitriangular structure during which the unit cell shrinks and becomes square by only actuating the 3 active arcs. Notice that the inactive arcs do not change length and therefore this actuation does not store any elastic energy. The mathematics used to produce this illustration is given in the ODE system of Eqs.(14)-(16).

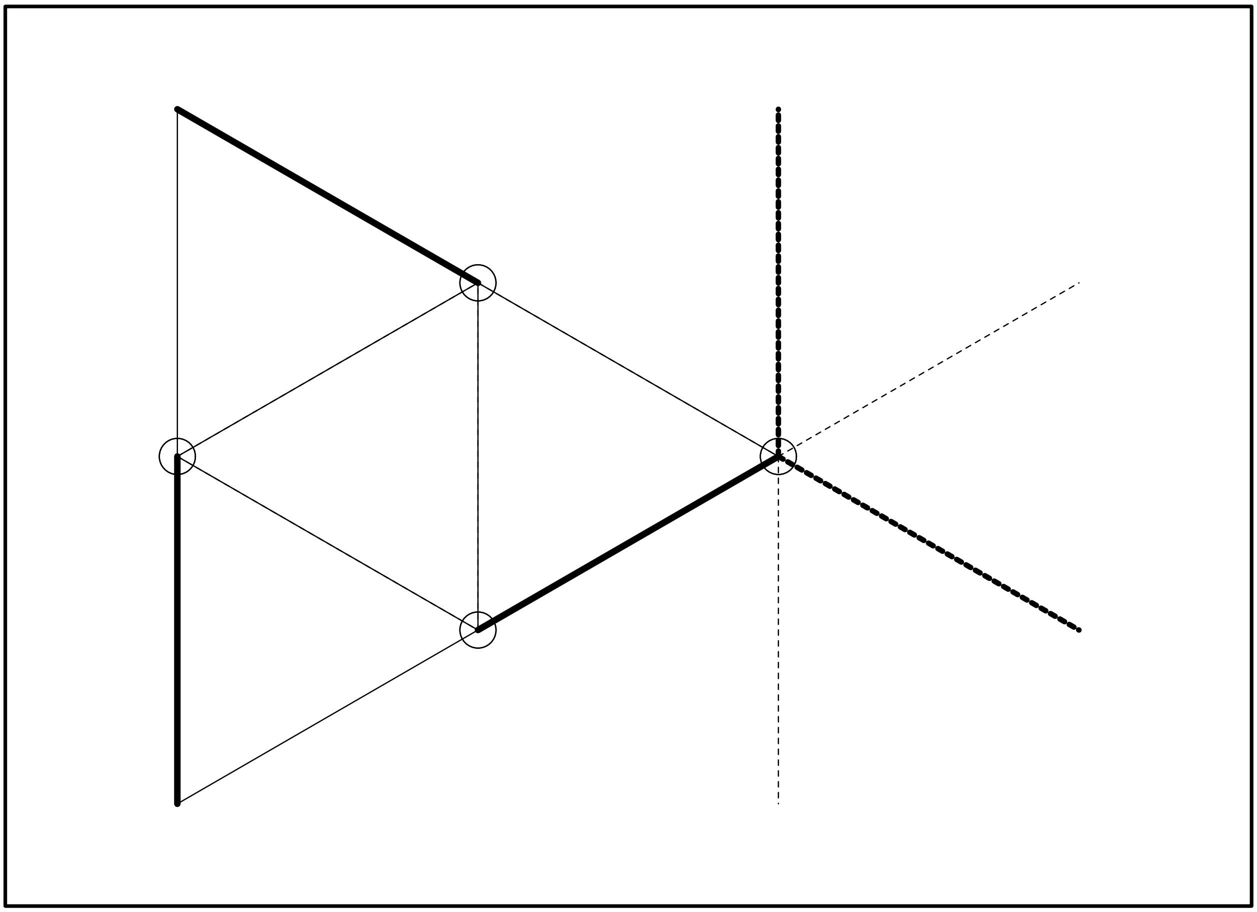

Figure 1: Donev, Torquato

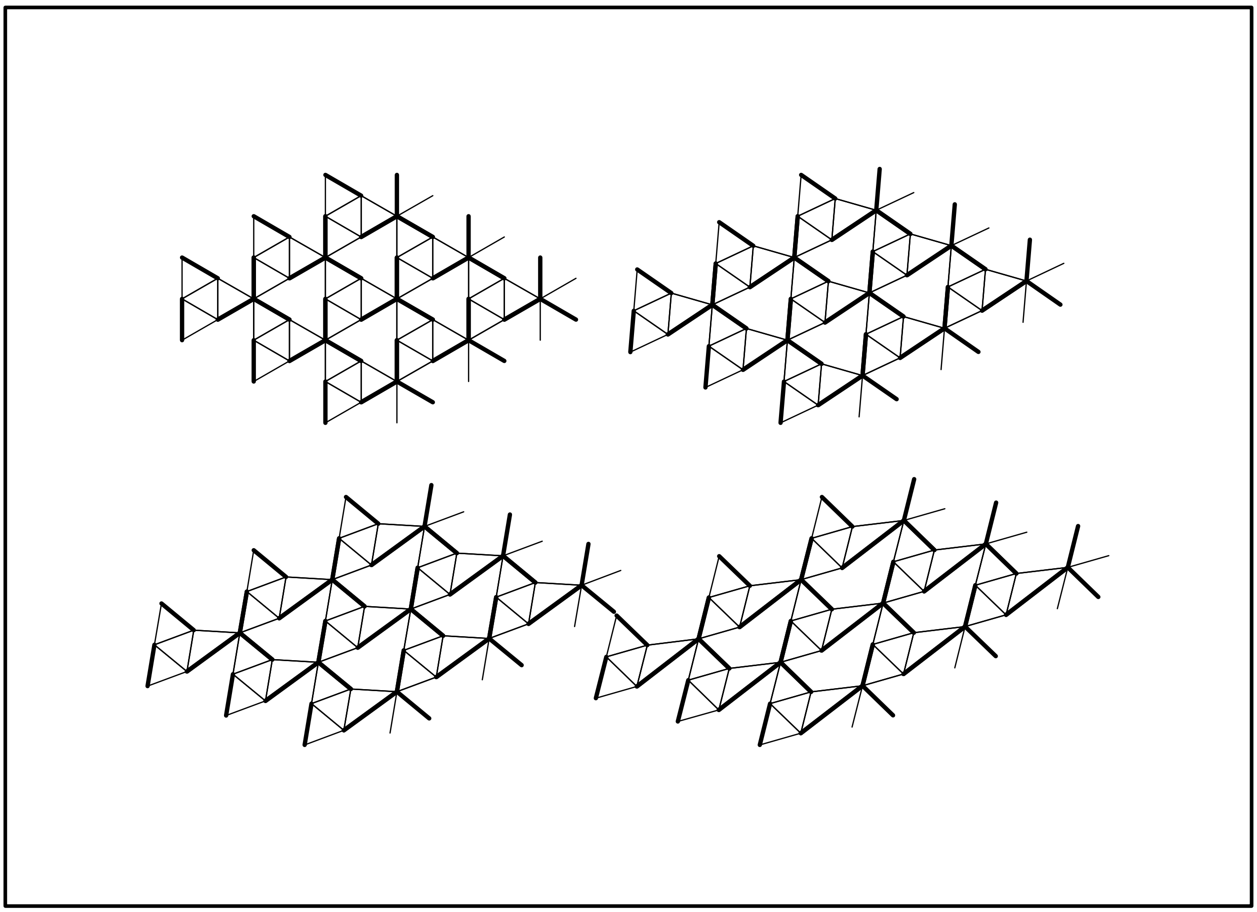

Figure 2: Donev, Torquato

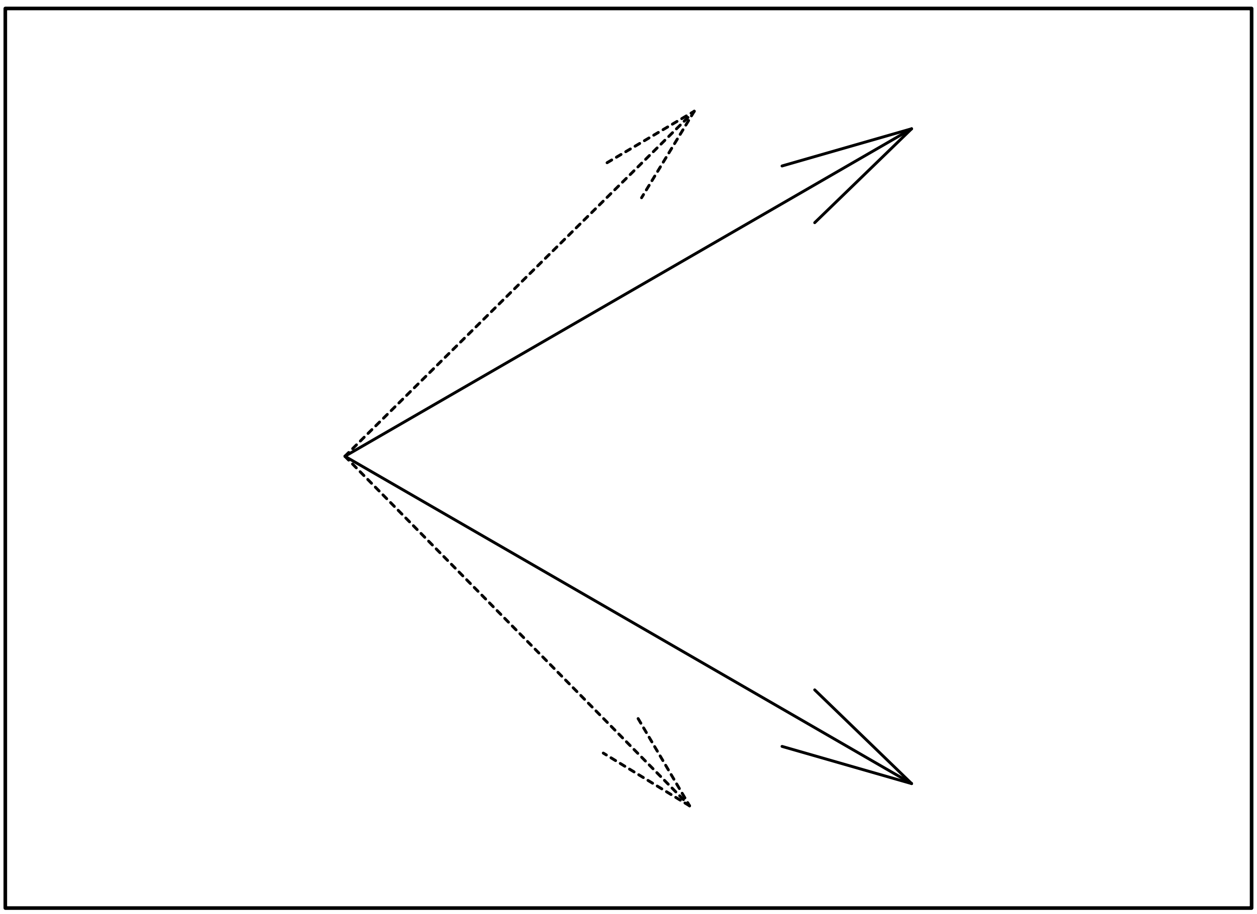

Figure 3: Donev, Torquato

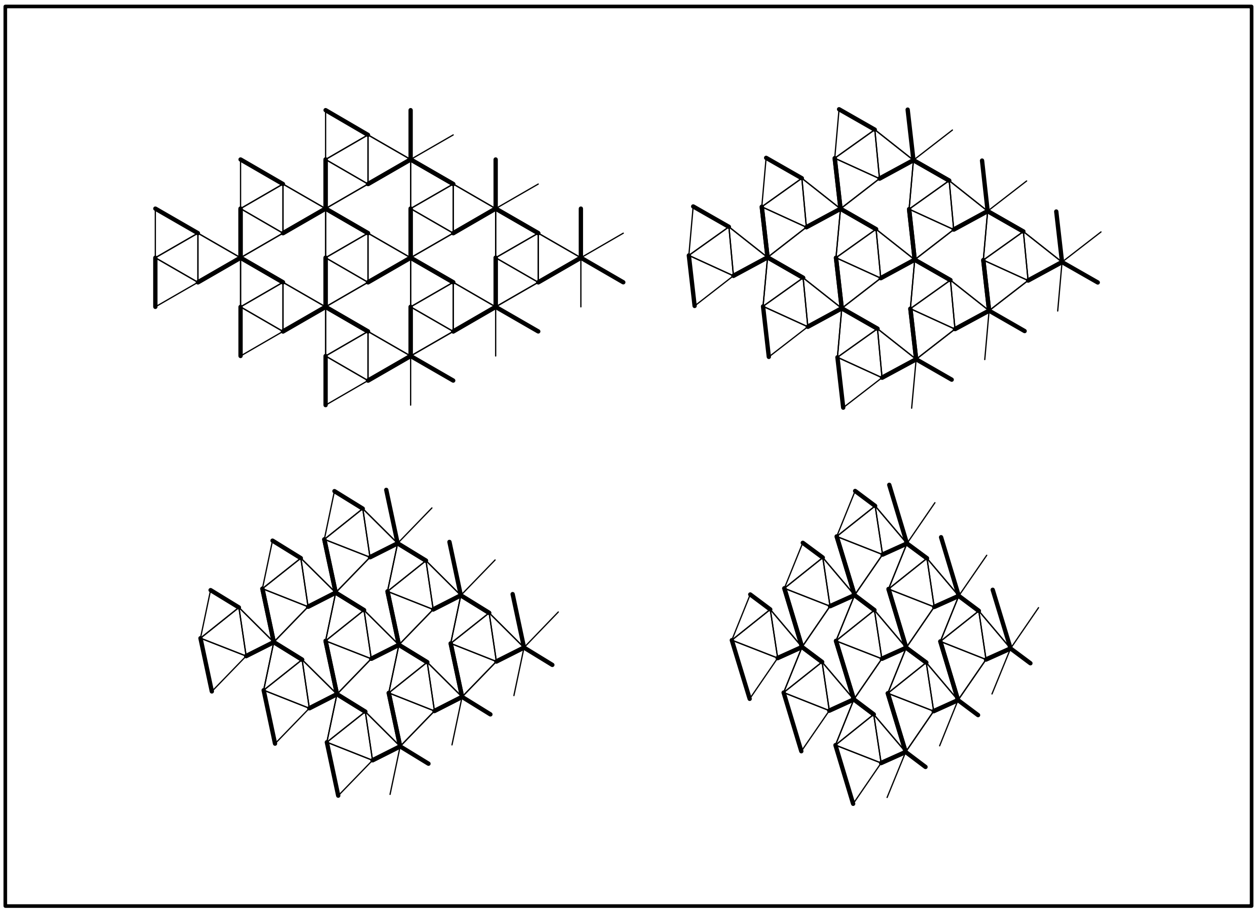

Figure 4: Donev, Torquato