Numerical Methods I
Orthogonal Polynomials

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Outline

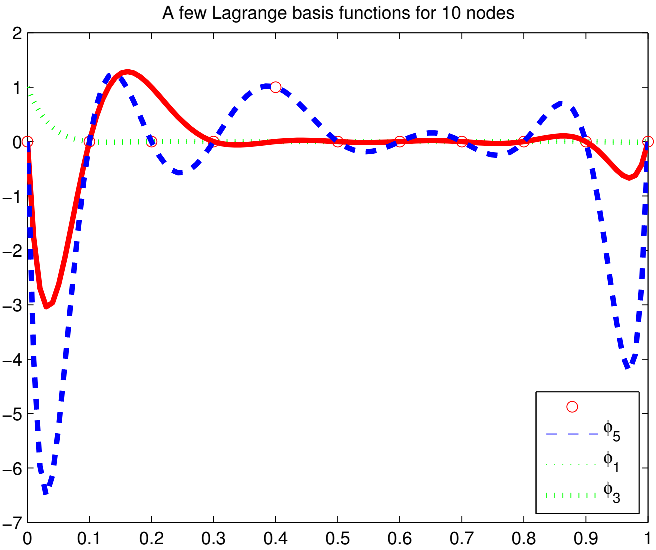
Final Project Presentations

The final presentations will take place on the following three dates (**tentative** list!):

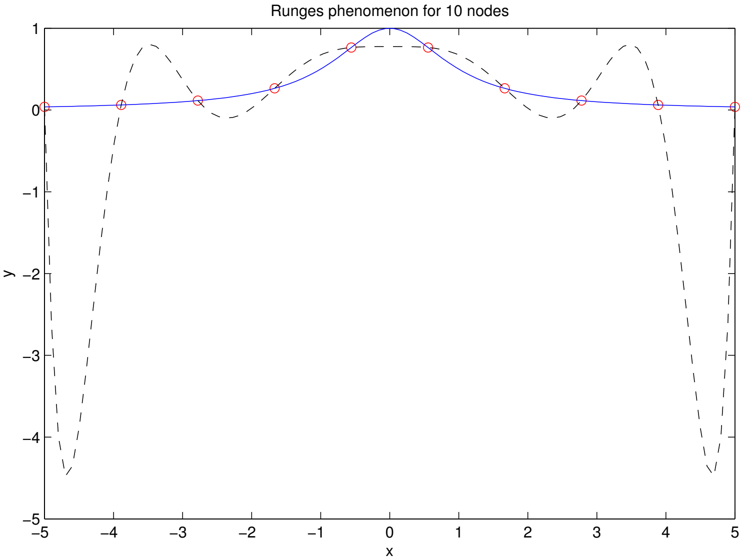
1. Thursday **Dec. 16th** 5-7pm (there will be no class on Legislative Day, Tuesday Dec. 14th)
2. Tuesday **Dec. 21st** 5-7pm
3. Thursday **Dec. 23rd 4-7pm** (note the earlier start time!)

There will be **no homework this week**: Start thinking about the project until next week's homework.

Lagrange basis on 10 nodes



Runge's phenomenon $f(x) = (1 + x^2)^{-1}$



Function Spaces

- **Function spaces** are the equivalent of finite vector spaces for functions (space of polynomial functions $\mathcal{P}$, space of smoothly twice-differentiable functions $\mathcal{C}^2$, etc.).
- Consider a one-dimensional interval $I = [a, b]$. Standard norms for functions similar to the usual vector norms:
  - **Maximum norm**: $\|f(x)\|_\infty = \max_{x \in I} |f(x)|$
  - L_1 **norm**: $\|f(x)\|_1 = \int_a^b |f(x)|\, dx$
  - **Euclidian** L_2 **norm**: $\|f(x)\|_2 = \left[\int_a^b |f(x)|^2\, dx\right]^{1/2}$
  - **Weighted norm**: $\|f(x)\|_w = \left[\int_a^b |f(x)|^2\, w(x) dx\right]^{1/2}$
- An **inner or scalar product** (equivalent of dot product for vectors):

$$(f, g) = \int_a^b f(x) g^\star(x) dx$$

Finite-Dimensional Function Spaces

- Formally, function spaces are **infinite-dimensional linear spaces**. Numerically we always **truncate and use a finite basis**.
- Consider a set of $m+1$ **nodes** $x_i \in \mathcal{X} \subset I$, $i = 0, \ldots, m$, and define:

$$\|f(x)\|_2^{\mathcal{X}} = \left[\sum_{i=0}^{m} |f(x_i)|^2 \right]^{1/2},$$

  which is equivalent to thinking of the function as being the vector $\mathbf{f}_{\mathcal{X}} = \mathbf{y} = \{f(x_0), f(x_1), \cdots, f(x_m)\}$.
- **Finite representations** lead to **semi-norms**, but this is not that important.
- A **discrete dot product** can be just the vector product:

$$(f, g)^{\mathcal{X}} = \mathbf{f}_{\mathcal{X}} \cdot \mathbf{g}_{\mathcal{X}} = \sum_{i=0}^{m} f(x_i) g^{\star}(x_i)$$

Function Space Basis

- Think of a function as a vector of coefficients in terms of a set of n **basis functions**:

$$\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\},$$

for example, the monomial basis $\phi_k(x) = x^k$ for polynomials.

- A finite-dimensional approximation to a given function $f(x)$:

$$\tilde{f}(x) = \sum_{i=1}^{n} c_i \phi_i(x)$$

- **Least-squares approximation** for $m > n$ (usually $m \gg n$):

$$\mathbf{c}^\star = \arg\min_{\mathbf{c}} \left\| f(x) - \tilde{f}(x) \right\|_2,$$

which gives the **orthogonal projection** of $f(x)$ onto the finite-dimensional basis.

Least-Squares Approximation

- Discrete case: Think of **fitting** a straight line or quadratic through experimental data points.
- The function becomes the vector $\mathbf{y} = \mathbf{f}_{\mathcal{X}}$, and the approximation is

$$y_i = \sum_{j=1}^{n} c_j \phi_j(x_i) \quad \Rightarrow \quad \mathbf{y} = \mathbf{\Phi c},$$

$$\mathbf{\Phi}_{ij} = \phi_j(x_i).$$

- This means that finding the approximation consists of solving an **overdetermined linear system**

$$\mathbf{\Phi c} = \mathbf{y}$$

- Note that for $m = n$ this is equivalent to interpolation. MATLAB's *polyfit* works for $m \geq n$.

Normal Equations

- Recall that one way to solve this is via the normal equations:

$$\left(\boldsymbol{\Phi}^{\star}\boldsymbol{\Phi}\right)\mathbf{c}^{\star}=\boldsymbol{\Phi}^{\star}\mathbf{y}$$

- A basis set is an **orthonormal basis** if

$$(\phi_i,\phi_j)=\sum_{k=0}^{m}\phi_i(x_k)\phi_j(x_k)=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$\boldsymbol{\Phi}^{\star}\boldsymbol{\Phi}=\mathbf{I}\text{ (unitary or orthogonal matrix)}\quad\Rightarrow$$

$$\mathbf{c}^{\star}=\boldsymbol{\Phi}^{\star}\mathbf{y}\quad\Rightarrow\quad c_i=\phi_i^{\mathcal{X}}\cdot\mathbf{f}_{\mathcal{X}}=\sum_{k=0}^{m}f(x_k)\phi_i(x_k)$$

Orthogonal Polynomials

- Consider a function on the interval $I = [a, b]$.
  Any finite interval can be transformed to $I = [-1, 1]$ by a simple transformation.
- Using a **weight function** $w(x)$, define a **function dot product** as:

$$(f, g) = \int_a^b w(x) \left[f(x) g(x) \right] dx$$

- For different choices of the weight $w(x)$, one can explicitly construct **basis of orthogonal polynomials** where $\phi_k(x)$ is a polynomial of degree k (**triangular basis**):

$$(\phi_i, \phi_j) = \int_a^b w(x) \left[\phi_i(x) \phi_j(x) \right] dx = \delta_{ij} \left\| \phi_i \right\|^2 .$$

Legendre Polynomials

- For equal weighting $w(x) = 1$, the resulting triangular family of of polynomials are called **Legendre polynomials**:

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= x \\
\phi_2(x) &= \frac{1}{2}(3x^2 - 1) \\
\phi_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
\phi_{k+1}(x) &= \frac{2k+1}{k+1} x\phi_k(x) - \frac{k}{k+1}\phi_{k-1}(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]
\end{aligned}$$

- These are orthogonal on $I = [-1, 1]$:

$$\int_{-1}^{-1} \phi_i(x)\phi_j(x)dx = \delta_{ij} \cdot \frac{2}{2i+1}.$$

Interpolation using Orthogonal Polynomials

- Let's look at the **interpolating polynomial** $\phi(x)$ of a function $f(x)$ on a set of $m+1$ **nodes** $\{x_0, \ldots, x_m\} \in I$, expressed in an orthogonal basis:

$$\phi(x) = \sum_{i=0}^{m} a_i \phi_i(x)$$

- Due to orthogonality, taking a dot product with ϕ_j (**weak formulation**):

$$(\phi, \phi_j) = \sum_{i=0}^{m} a_i (\phi_i, \phi_j) = \sum_{i=0}^{m} a_i \delta_{ij} \|\phi_i\|^2 = a_j \|\phi_j\|^2$$

- This is **equivalent to normal equations** if we use the right dot product:

$$(\mathbf{\Phi}^\star \mathbf{\Phi})_{ij} = (\phi_i, \phi_j) = \delta_{ij} \|\phi_i\|^2 \text{ and } \mathbf{\Phi}^\star \mathbf{y} = (\phi, \phi_j)$$

Gauss Integration

$$a_j \left\|\phi_j\right\|^2 = (\phi, \phi_j) \quad \Rightarrow \quad a_j = \left(\left\|\phi_j\right\|^2\right)^{-1} (\phi, \phi_j)$$

- Question: Can we easily compute

$$a_j \left\|\phi_j\right\|^2 = (\phi, \phi_j) = \int_a^b w(x) \left[\phi(x)\phi_j(x)\right] dx = \int_a^b w(x) p_{2m}(x) dx$$

  for a polynomial $p_{2m}(x) = \phi(x)\phi_j(x)$ of degree at most $2m$?
- Let's first consider polynomials of degree at most m

$$\int_a^b w(x) p_m(x) dx = ?$$

Gauss Weights

- Now consider the **Lagrange basis** $\{\varphi_0(x), \varphi_1(x), \ldots, \varphi_m(x)\}$, where you recall that

$$\varphi_i(x_j) = \delta_{ij}.$$

- Any polynomial $p_m(x)$ of degree at most m can be expressed in the Lagrange basis:

$$p_m(x) = \sum_{i=0}^{m} p_m(x_i)\varphi_i(x),$$

$$\int_a^b w(x)p_m(x)dx = \sum_{i=0}^{m} p_m(x_i)\left[\int_a^b w(x)\varphi_i(x)dx\right] = \sum_{i=0}^{m} w_i p_m(x_i),$$

where the **Gauss weights w** are given by

$$w_i = \int_a^b w(x)\varphi_i(x)dx.$$

Back to Interpolation

- For any polynomial $p_{2m}(x)$ there exists a polynomial quotient q_{m-1} and a remainder r_m such that:

$$p_{2m}(x) = \phi_{m+1}(x) q_{m-1}(x) + r_m(x)$$

$$\int_a^b w(x) p_{2m}(x) dx = \int_a^b \left[w(x) \phi_{m+1}(x) q_{m-1}(x) + w(x) r_m(x) \right] dx$$
$$= (\phi_{m+1}, q_{m-1}) + \int_a^b w(x) r_m(x) dx$$

- But, since $\phi_{m+1}(x)$ is orthogonal to any polynomial of degree at most m, $(\phi_{m+1}, q_{m-1}) = 0$ and we thus get:

$$\int_a^b w(x) p_{2m}(x) dx = \sum_{i=0}^{m} w_i r_m(x_i)$$

Gauss nodes

- Finally, if we choose the **nodes to be zeros of** $\phi_{m+1}(x)$, then

$$r_m(x_i) = p_{2m}(x_i) - \phi_{m+1}(x_i)q_{m-1}(x_i) = p_{2m}(x_i)$$

$$\int_a^b w(x)p_{2m}(x)dx = \sum_{i=0}^{m} w_i p_{2m}(x_i)$$

and thus we have found a way to **quickly project any polynomial** onto the basis of orthogonal polynomials:

$$(p_m, \phi_j) = \sum_{i=0}^{m} w_i p_m(x_i)\phi_j(x_i)$$

$$(\phi, \phi_j) = \sum_{i=0}^{m} w_i \phi(x_i)\phi_j(x_i) = \sum_{i=0}^{m} w_i f(x_i)\phi_j(x_i)$$

Gauss-Legendre polynomials

- For any weighting function the polynomial $\phi_k(x)$ has k simple zeros all of which are in $(-1, 1)$, called the (order k) **Gauss nodes**, $\phi_{m+1}(x_i) = 0$.
- The interpolating polynomial $\phi(x_i) = f(x_i)$ on the Gauss nodes is the **Gauss-Legendre interpolant** $\phi_{GL}(x)$.
- The orthogonality relation can be expressed as a **sum instead of integral**:

$$(\phi_i, \phi_j) = \sum_{i=0}^{m} w_i \phi_i(x_i) \phi_j(x_i) = \delta_{ij} \left\| \phi_i \right\|^2$$

- We can thus define a new weighted **discrete dot product**

$$\mathbf{f} \cdot \mathbf{g} = \sum_{i=0}^{m} w_i f_i g_i$$

Discrete Orthogonality of Polynomials

- The orthogonal polynomial basis is **discretely-orthogonal** in the new dot product,

$$\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j = (\phi_i, \phi_j) = \delta_{ij} (\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i)$$

- This means that the matrix in the normal equations is diagonal:

$$\boldsymbol{\Phi}^{\star}\boldsymbol{\Phi} = \text{Diag}\left\{ \|\phi_0\|^2, \ldots, \|\phi_m\|^2 \right\} \quad \Rightarrow \quad a_i = \frac{\mathbf{f} \cdot \boldsymbol{\phi}_i}{\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i}.$$

- The Gauss-Legendre interpolant is thus easy to compute:

$$\phi_{GL}(x) = \sum_{i=0}^{m} \frac{\mathbf{f} \cdot \boldsymbol{\phi}_i}{\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i} \phi_i(x).$$

Hilbert Space L^2_w

- Consider the **Hilbert space** L^2_w of **square-integrable functions** on $[-1, 1]$:

$$\forall f \in L^2_w : \quad (f, f) = \|f\|^2 = \int_{-1}^{1} w(x) \left[f(x) \right]^2 dx < \infty.$$

- **Legendre polynomials** form a **complete orthogonal basis** for L^2_w:

$$\forall f \in L^2_w : \quad f(x) = \sum_{i=0}^{\infty} f_i \phi_i(x)$$

$$f_i = \frac{(f, \phi_i)}{(\phi_i, \phi_i)}.$$

- The least-squares approximation of f is a **spectral approximation** and is obtained by simply truncating the infinite series:

$$\phi_{sp}(x) = \sum_{i=0}^{m} f_i \phi_i(x).$$

Spectral approximation

$$\text{Continuous (spectral approximation): } \phi_{sp}(x) = \sum_{i=0}^{m} \frac{(f, \phi_i)}{(\phi_i, \phi_i)} \phi_i(x).$$

$$\text{Discrete (interpolating polynomial): } \phi_{GL}(x) = \sum_{i=0}^{m} \frac{\mathbf{f} \cdot \boldsymbol{\phi}_i}{\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i} \phi_i(x).$$

- If we **approximate** the function dot-products with the discrete weighted products

$$(f, \phi_i) \approx \sum_{j=0}^{m} w_j f(x_j) \phi_i(x_j) = \mathbf{f} \cdot \boldsymbol{\phi}_i,$$

we see that the Gauss-Legendre interpolant is a **discrete spectral approximation**:

$$\phi_{GL}(x) \approx \phi_{sp}(x).$$

Discrete spectral approximation

- Using a spectral representation has many advantages for function approximation: **stability**, **rapid convergence**, easy to **add more basis functions**.
- The convergence, for sufficiently smooth (nice) functions, is **more rapid than any power law**

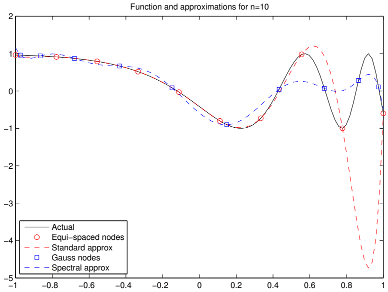
$$\|f(x) - \phi_{GL}(x)\| \leq \frac{C}{N^d} \left(\sum_{k=0}^{d} \left\| f^{(k)} \right\|^2 \right)^{1/2},$$

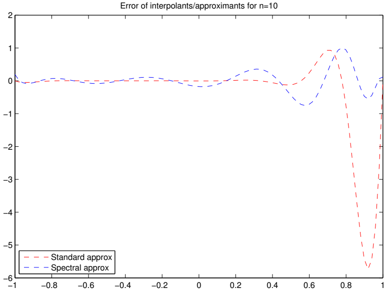
where the multiplier is related to the **Sobolev norm** of $f(x)$.
- For $f(x) \in \mathcal{C}^1$, the convergence is also **pointwise** with similar accuracy ($N^{d-1/2}$ in the denominator).
- This so-called **spectral accuracy** (limited by smoothness only) cannot be achived by piecewise, i.e., local, approximations (limited by order of local approximation).

Regular grids

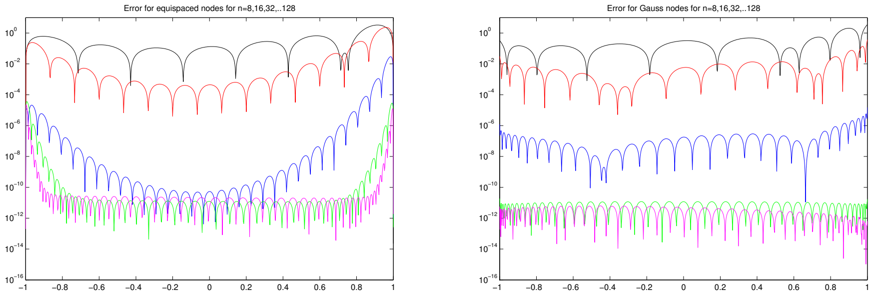
```
a=2;
f = @(x) cos(2*exp(a*x));

x_fine=linspace(-1,1,100);
y_fine=f(x_fine);

% Equi-spaced nodes:
n=10;
x=linspace(-1,1,n);
y=f(x);
c=polyfit(x,y,n);
y_interp=polyval(c,x_fine);

% Gauss nodes:
[x,w]=GLNodeWt(n); % See webpage for code
y=f(x);
c=polyfit(x,y,n);
y_interp=polyval(c,x_fine);
```

Gauss-Legendre Interpolation

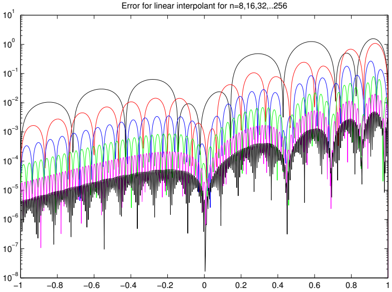


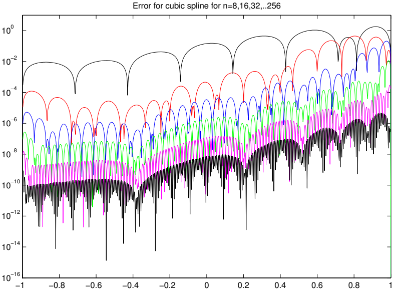


Global polynomial interpolation error



Local polynomial interpolation error





Periodic Functions

- Consider now interpolating / approximating **periodic functions** defined on the interval $I = [0, 2\pi]$:

$$\forall x \quad f(x + 2\pi) = f(x),$$

  as appear in practice when analyzing signals (e.g., sound/image processing).
- Also consider only the space of complex-valued **square-integrable functions** $L^2_{2\pi}$,

$$\forall f \in L^2_w : \quad (f, f) = \|f\|^2 = \int_0^{2\pi} |f(x)|^2 \, dx < \infty.$$

- Polynomial functions are not periodic and thus basis sets based on orthogonal polynomials are not appropriate.
- Instead, consider sines and cosines as a basis function, combined together into **complex exponential functions**

$$\phi_k(x) = e^{ikx} = \cos(kx) + i\sin(kx), \quad k = 0, \pm1, \pm2, \ldots$$

Fourier Basis

$$\phi_k(x) = e^{ikx}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- It is easy to see that these are **orhogonal** with respect to the continuous dot product

$$(\phi_j, \phi_k) = \int_{x=0}^{2\pi} \phi_j(x)\phi_k^\star(x)dx = \int_0^{2\pi} \exp\left[i(j-k)x\right] dx = 2\pi\delta_{ij}$$

- The complex exponentials can be shown to form a complete **trigonometric polynomial basis** for the space $L^2_{2\pi}$, i.e.,

$$\forall f \in L^2_{2\pi} : \quad f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx},$$

  where the **Fourier coefficients** can be computed for any **frequency or wavenumber** k using:

$$\hat{f}_k = \frac{(f, \phi_k)}{2\pi} = \frac{1}{2\pi}. \int_0^{2\pi} f(x)e^{-ikx}dx.$$

Discrete Fourier Basis

- For a general interval $[0, X]$ the **discrete frequencies** are

$$k = \frac{2\pi}{X}\kappa \quad \kappa = 0, \pm 1, \pm 2, \ldots$$

- For non-periodic functions one can take the limit $X \to \infty$ in which case we get **continuous frequencies**.
- Now consider a **discrete Fourier basis** that only includes the first N basis functions, i.e.,

$$\begin{cases} k = -(N-1)/2, \ldots, 0, \ldots, (N-1)/2 & \text{if } N \text{ is odd} \\ k = -N/2, \ldots, 0, \ldots, N/2 - 1 & \text{if } N \text{ is even,} \end{cases}$$

and for simplicity we focus on N odd.

- The least-squares **spectral approximation** for this basis is:

$$f(x) \approx \phi(x) = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k e^{ikx}.$$

Discrete Dot Product

- Now also discretize the functions on a set of N **equi-spaced nodes**

$$x_j = jh \text{ where } h = \frac{2\pi}{N}$$

  where $j = N$ is the same node as $j = 0$ due to periodicity so we only consider N instead of $N+1$ nodes.

- We also have the **discrete dot product** between two discrete functions (vectors) $\mathbf{f}_j = f(x_j)$:

$$\mathbf{f} \cdot \mathbf{g} = h \sum_{j=0}^{N-1} f_i g_i^{\star}$$

- The discrete Fourier basis is **discretely orthogonal**

$$\phi_k \cdot \phi_{k'} = 2\pi \delta_{k,k'}$$

Proof of Discrete Orthogonality

The case $k = k'$ is trivial, so focus on

$$\phi_k \cdot \phi_{k'} = 0 \text{ for } k \neq k'$$

$$\sum_j \exp\left(ikx_j\right) \exp\left(-ik'x_j\right) = \sum_j \exp\left[i\left(\Delta k\right) x_j\right] = \sum_{j=0}^{N-1} \left[\exp\left(ih\left(\Delta k\right)\right)\right]^j$$

where $\Delta k = k - k'$. This is a geometric series sum:

$$\phi_k \cdot \phi_{k'} = \frac{1 - z^N}{1 - z} = 0 \text{ if } k \neq k'$$

since $z = \exp\left(ih\left(\Delta k\right)\right) \neq 1$ and
$z^N = \exp\left(ihN\left(\Delta k\right)\right) = \exp\left(2\pi i\left(\Delta k\right)\right) = 1.$

Discrete Fourier Transform

- The **Fourier interpolating polynomial** is thus easy to construct

$$\phi_N(x) = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k^{(N)} e^{ikx}$$

where the **discrete Fourier coefficients** are given by

$$\hat{f}_k^{(N)} = \frac{\mathbf{f} \cdot \phi_k}{2\pi} = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \exp(-ikx_j)$$

- Simplifying the notation and recalling $x_j = jh$, we define the the **Discrete Fourier Transform** (DFT):

$$\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N}\right)$$

Fourier Spectral Approximation

$$\text{Forward } \mathbf{f} \rightarrow \hat{\mathbf{f}}: \quad \hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N}\right)$$

$$\text{Inverse } \hat{\mathbf{f}} \rightarrow f: \quad f(x) \approx \phi(x) = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k e^{ikx}$$

- There is a very fast algorithm for performing the **forward and backward DFTs** called the **Fast Fourier Transform (FFT)**, which we will discuss next time.
- The Fourier interpolating polynomial $\phi(x)$ has **spectral accuracy**, i.e., exponential in the number of nodes N

$$\|f(x) - \phi(x)\| \sim e^{-N}$$

for **sufficiently smooth functions** (sufficiently rapid decay of the Fourier coefficients with k, e.g., $\hat{f}_k \sim e^{-|k|}$).

Discrete spectrum

- The set of discrete Fourier coefficients $\hat{\mathbf{f}}$ is called the **discrete spectrum**, and in particular,

$$S_k = \left|\hat{f}_k\right|^2 = \hat{f}_k \hat{f}_k^\star,$$

  is the **power spectrum** which measures the frequency content of a signal.
- If f is real, then $\hat{f}$ satisfies the **conjugacy property**

$$\hat{f}_{-k} = \hat{f}_k^\star,$$

  so that half of the spectrum is redundant and $\hat{f}_0$ is real.
- For an even number of points N the largest frequency $k = -N/2$ does not have a conjugate partner.

In MATLAB

- The forward transform is performed by the function $\hat{f} = \mathit{fft}(f)$ and the inverse by $f = \mathit{fft}(\hat{f})$. Note that $\mathit{ifft}(\mathit{fft}(f)) = f$ and f and $\hat{f}$ may be complex.
- In MATLAB, and other software, the frequencies are not ordered in the "normal" way $-(N-1)/2$ to $+(N-1)/2$, but rather, the nonnegative frequencies come first, then the positive ones, so the "funny" ordering is

$$0, 1, \ldots, (N-1)/2, \quad -\frac{N-1}{2}, -\frac{N-1}{2}+1, \ldots, -1.$$

  This is because such ordering (shift) makes the forward and inverse transforms symmetric.
- The function *fftshift* can be used to order the frequencies in the normal way, and *ifftshift* does the reverse:

$$\hat{f} = \mathit{fftshift}(\mathit{fft}(f)) \text{ (normal ordering).}$$

FFT-based noise filtering (1)

```
Fs = 1000;                    % Sampling frequency
dt = 1/Fs;                    % Sampling interval
L = 1000;                     % Length of signal
t = (0:L-1)*dt;               % Time vector
T=L*dt;                       % Total time interval

% Sum of a 50 Hz sinusoid and a 120 Hz sinusoid
x = 0.7*sin(2*pi*50*t) + sin(2*pi*120*t);
y = x + 2*randn(size(t));     % Sinusoids plus noise

figure(1); clf; plot(t(1:100),y(1:100),'b--'); hold on
title('Signal Corrupted with Zero-Mean Random Noise')
xlabel('time')
```

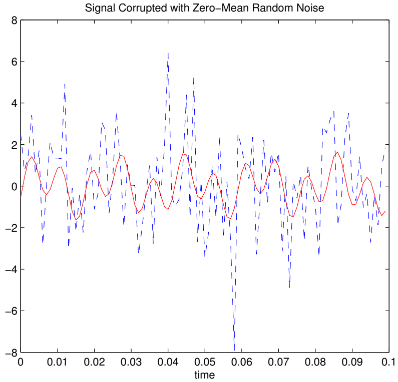
FFT-based noise filtering (2)

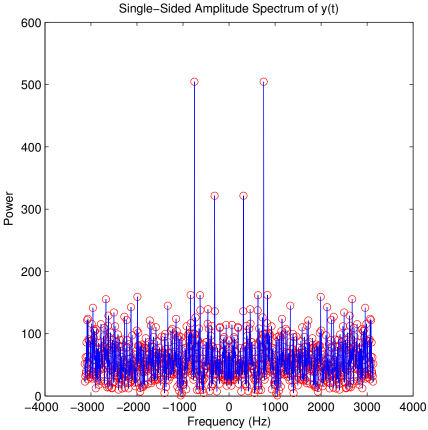
```
if(0)
   N=(L/2)*2; % Even N
   y_hat = fft(y(1:N));
   % Frequencies ordered in a funny way:
   f_funny = 2*pi/T* [0:N/2-1, -N/2:-1];
   % Normal ordering:
   f_normal = 2*pi/T* [-N/2 : N/2-1];
else
   N=(L/2)*2-1; % Odd N
   y_hat = fft(y(1:N));
   % Frequencies ordered in a funny way:
   f_funny = 2*pi/T* [0:(N-1)/2, -(N-1)/2:-1];
   % Normal ordering:
   f_normal = 2*pi/T* [-(N-1)/2 : (N-1)/2];
end
```

FFT-based noise filtering (3)

```
figure(2); clf; plot(f_funny, abs(y_hat), 'ro'); hold on;

y_hat=fftshift(y_hat);
figure(2); plot(f_normal, abs(y_hat), 'b-');

title('Single-Sided␣Amplitude␣Spectrum␣of␣y(t)')
xlabel('Frequency␣(Hz)')
ylabel('Power')

y_hat(abs(y_hat)<250)=0; % Filter out noise
y_filtered = ifft(ifftshift(y_hat));
figure(1); plot(t(1:100),y_filtered(1:100),'r-')
```

FFT results


Signal Corrupted with Zero−Mean Random Noise
8
6
4
2
0
−2
−4
−6
−8
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
time


Single−Sided Amplitude Spectrum of y(t)
600
500
400
300
200
100
0
Power
−4000
−3000
−2000
−1000
0
1000
2000
3000
4000
Frequency (Hz)

Conclusions/Summary

- Once a function dot product is defined, one can construct **orthogonal basis** for the space of functions of finite 2−norm.
- For functions on the interval $[-1, 1]$, **triangular families of orthogonal polynomials** $\phi_i(x)$ provide such a basis, e.g., **Legendre** or **Chebyshev** polynomials.
- If one discretizes at the **Gauss nodes**, i.e., the roots of the polynomial $\phi_{m+1}(x)$, and defines a suitable **discrete Gauss-weighted dot product**, one obtains **discretely-orthogonal** basis suitable for numerical computations.
- The interpolating polynomial on the Gauss nodes is closely related to the **spectral approximation** of a function.
- **Spectral convergence** is faster than any power law of the number of nodes and is only limited by the **global** smoothness of the function, unlike piecewise polynomial approximations limited by the choice of **local** basis functions.
- One can also consider **piecewise-spectral approximations**.